Geometry Axioms
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A key lemma for the geometry axioms is the following. Let F be an ordered field in which

every polynomial in one variable assumes a maximum value over any nonempty closed interval.

Then F is a real closed field.

Of course, the hypothesis is equivalent to

every polynomial in one variable assumes a maximum and a minimum value over any nonempty closed interval.

We will actually prove the following. If a polynomial assumes a positive and a negative value then it has a zero.

To prove this, we fix P(x) to be any polynomial of degree ≥ 1 with a positive and negative value. We define a critical interval to be any nonempty open interval on which P is strictly monotone and where P is not strictly monotone on any larger open interval. Here an open interval may not have endpoints in F, and may be infinite on the left or right or both sides. Obviously, the critical intervals are pairwise disjoint.

LEMMA 1. There are finitely many critical intervals.

Proof: Suppose there are infinitely many critical intervals. In particular, let $I_1,I_2,...,I_{2p+1}$ be critical intervals from left to right, where p = deg(P). Look at $J_1,...,J_p = I_2,I_4,...,I_{2p}$, which must be finite intervals. We can stretch each of these intervals to get intervals $K_1,...,K_p$, from left to right, which are not abutting or overlapping, and where P is not strictly monotone on any of these K's.

Inside each K_i we can find a triple of elements a < b < c such that P is not strictly monotone on {a,b,c}.

Let F* be the real closure of F. In F*, the derivative P' must have a zero in [a,c] by the mean value theorem in F* applies to P on [a,b] and P on [b,c] together with the intermediate value theorem in F*. I.e., in F*, P' has a zero in each of the K's, for a total of at least deg(P) zeros. This implies that P' is constant, and so P is of degree 1. But then there is obviously just 1 critical interval. QED

LEMMA 2. Suppose P(x) > 0. Then there exists h > 0 such that P is > P(x)/2 on (x-h,x+h). Alternatively, suppose P(x) < 0. Then there exists h > 0 such that P is < P(x)/2 on (x-h,x+h).

Proof: We can write P(x+b)-P(b) = Q(x,b)b, where Q is a polynomial. We first bound Q(x,b) over $-1 \le b \le 1$. Then choose h appropriately. QED

LEMMA 3. Suppose P'(x) > 0. There exists h > 0 such that P is strictly increasing on (x-h,x+h). Alternatively, suppose P'(x) < 0. There exists h > 0 such that P is strictly decreasing on (x-h,x+h).

Proof: We handle the case P'(x) > 0, the other case being analogous. By algebra, write P(x+b)-P(x) = Q(x,b)(b), where Q is a polynomial. By algebra, Q(x,0) = P'(x). Apply Lemma 2 to the polynomial Q(x,b) of b with b set to 0. QED

LEMMA 4. If x does not lie in any critical interval then P'(x) = 0. There are finitely many points that lie outside any critical interval.

Proof: By Lemma 3. QED

LEMMA 5. Suppose P has a positive and a negative value on the critical interval I. Then P has a zero on I.

Proof: By symmetry, we can assume that P is increasing on I. Let P(a) < 0 and P(b) > 0. By hypothesis, P^2 attains a mimum value over [a,b], say at $a \le x \le b$. Assume that this minimal magnitude value is c ≠ 0. According to Lemma 2 and that P is increasing on I, we see that P^2 must take on values of smaller magnitude than c on I. QED

LEMMA 6. Suppose that I,J are two abutting critical intervals, where P is positive on I and P is negative on J. Then the point between I and J exists and P is zero at that point.

Proof: Assume I is to the left of J, the other case being handled analogously. Let a in I and b in J.

case 1. P is increasing on I. The maximum value of P on [a,b] cannot be taken on I since P is increasing on I, and cannot be taken on J since P is negative on J. Hence it must be taken at the point between I and J.

case 2. P is increasing on J. The minimum value of P on [a,b] cannot be taken on J since P is increasing on J, and cannot be taken on I since P is positive on I. Hence it must be taken at the point between I and J.

case 3. P is decreasing on I and decreasing on J. Then I union J is a critical interval, which is impossible.

Thus we have proved the existence of the point x between I and J. By Lemma 2, P(x) = 0. QED

LEMMA 7. Suppose P cannot be positive on all critical intervals, and cannot be negative on all critical intervals.

Proof: We can assume that P is positive on all critical intervals and derive a contradiction. Since P is negative at some point (by hypothesis), let P(x) < 0, where x is not on any critical interval. By Lemmas 1 and 4, let x be an endpoint of the critical interval I. Since P is positive on I, we have a contradiction by Lemma 2. QED

LEMMA 8. P has a zero.

Proof: Assume that P has no zero. By Lemma 5, on every critical interval, P is either entirely positive or entirely negative. By Lemma 7, there must be a mixture. By Lemmas 1 and 4, there are two abutting critical intervals, where P is entirely positive on one and entirely negative on the other. By Lemma 6, P has a zero. QED

THEOREM 9. Let F be an ordered field where every polynomial assumes a maximum value on every nonempty closed interval. Then F is a real closed field.

Proof: We have shown that any polynomial that assumes a positive and a negative value, has a zero. But this obviously implies that odd degree polynomials have roots and that every positive element has a square root. QED