CONTEMPORARY PERSPECTIVES ON HILBERT'S SECOND PROBLEM AND THE GÖDEL INCOMPLETENESS THEOREMS

AMS/ASL PANEL DISUCSSION
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January 6, 2007

1. First Incompleteness Theorem.
2. First Incompleteness Theorem in (¬,<,0,1,+,•). (!!!)
3. Second Incompleteness Theorem.
4. Is there any real logical strength?
5. Strict Reverse Mathematics.

It is not yet clear just what the most illuminating ways of rigorously stating the Incompleteness Theorems are. This is particularly true of the Second. Also I believe that there are more illuminating proofs of the Second that have yet to be uncovered.

NOTE: See “Formal Statements of Godel’s Second Incompleteness Theorem”, http://www.math.ohio-state.edu/%7Efriedman/

There is also a very interesting viscously anti foundational argument which suggests that mathematics can be developed in a way that can be proved to be free of contradiction in Peano Arithmetic, or even in weak fragments such as Exponential Function Arithmetic = EFA = $I\Sigma_0$(exp) – thereby suggesting that the Incompleteness Theorems are an irrelevant and misleading distraction.

Refutation of this mind numbing heresy is ongoing and leads to some very interesting formal work, called Strict Reverse Mathematics.

NOTE: See “The Inevitability of Logical Strength”, February, 2007, recently submitted for publication. http://www.math.ohio-state.edu/%7Efriedman/

1. FIRST INCOMPLETENESS THEOREM.

R.M. Robinson’s Q. L(Q) = 0,S,+,•, with =, and

1. Sx ≠ 0.
2. Sx = Sy □ x = y.
3. x ≠ 0 □ (□y)(x = Sy).
4. x+0 = x.
5. x+Sy = S(x+y).
6. x•0 = 0.
7. x•Sy = (x•y)+x.

THEOREM 1.1. Let T be a consistent many sorted theory with finitely many axioms, and □ be an interpretation of Q in T. Then there is a sentence □ of L(Q) such that □(□) is neither provable nor refutable in T.

Note that Theorem 1.1 is extremely clean and fully rigorously stated. There is an important extension that is not so clean.

THEOREM 1.2. Let T be a consistent many sorted theory, and □ be an interpretation of Q into T. *Assume that the set of axioms of T is recursively enumerable.* Then there is a sentence □ of L(Q) such that □(□) is neither provable nor refutable in T.

Now the statement involves Gödel numberings of syntax. Questions of robustness occur, which are nowadays considered trivial.

Here is a well known clarifying result that makes Theorem 1.2 an immediate consequence of Theorem 1.1, at least for finite languages.

THEOREM 1.3. Let T be a many sorted theory in a finite language. Then T has a recursively enumerable axiomatization if and only if T has a finitely axiomatized conservative extension in a finite language.

Q is a particularly clear and natural system of arithmetic that can be used for Theorems 1,2. What is the "simplest" system of arithmetic that can be used?

There are some very interesting alternatives that can be used, that are not arithmetics, and are considerably simpler. What is the "simplest" system that can be used?

(□x)(□y)(□y □ x).
(□z)(□w)(w □ z □ w □ x   w = y).

This system interprets Q, and so can be used.

NOTE: A proof that Q is interpetable in this system was given in A.Mancini, F. Montagna, “A minimal predicative set theory”, Notre Dame J. of Formal Logic 35 (1994) 186-203, where the authors give credit to Jan Krajicek for an earlier proof of the result (unpublished and unknown to the authors before completion of their work).

How logically simple can these independent statements be? We will stick to the cleanest formulations.

THEOREM 1.4. Let T be a consistent many sorted theory with finitely many axioms, and □ be an interpretation of Q in T. Then there is a false existentially quantified equation, □, in L(Q), such that □(□) is not refutable in T.

THEOREM 1.5. Let T be a consistent many sorted theory with finitely many axioms, and □ be an interpretation of $I□_0$ in T. Then there is a false existentially quantified equation, □, in L(Q), such that □(□) is neither provable nor refutable in T.

The above uses the MRDP solution to Hilbert’s 10^{th} problem. For MRDP, we can fix the number of variables and the degree in various ways. This means that we can do so for Theorem 1.4. What is the relationship between these two situations with regard to fixing the number of variables and the degree?

2. FIRST INCOMPLETENESS THEOREM IN (¬,<,0,1,+,•). (!!!)

I originated the study of finite forms of the Incompleteness Theorems in

H. Friedman, On the consistency, completeness, and correctness problems, Ohio State University, 1979, unpublished.

This concerns Gödel’s Second Incompleteness only: in order to prove that there is no inconsistency of length n, one needs length f(n). Lower bounds on f(n) are of the form $n^{□}$. This topic was later developed further in

P. Pudlak, On the length of proofs of finitistic consistency statements in first order theories, in Logic

Colloquium ’84, Eds. J.B. Paris, A.j. Wilkie, and G.M. Wilmers, North-Holland, 1986, pp. 165-196.

P. Pudlak, Impoved bounds to the length of proofs of finitistic consistency statements, Contemporary Mathematics, Logic and Combinatorics, volume 65, ed. S. Simpson, 1987, pp. 309-332.

A fully systematic development of these ideas would include going back to the First Incompleteness theorem.

I can’t get into this underdeveloped area here. But I do want to discuss length of proof issues in the context of the field of real numbers.

To begin with, the field of real numbers has a complete axiomatization – in fact, several kinds of very nice ones. Here are two, based on <,0,1,+,-,•.

1. Ordered field axioms, every positive element has a square root, every polynomial of odd degree with leading coefficient 1 has a root.

2. Ordered field axioms, the least upper bound principle for all first order formulas in <,0,1,+,-,•.

What can we say about the relationship between lengths of proofs in 1 and 2? It is clear that 2 is more “powerful” than 1, and should very significantly shorten proofs over 1.

It is well known that the first order theory of (¬,<,0,1,+,-,•) is nondeterministic exponential hard, and exponential space easy.

We write #(□) for the number of symbols occurring in □.
COROLLARY 2.1. For n ≥ 1, there exists infinitely many sentences □ true in (¬,<,0,1,+,-,•), such that every proof in 1 (2) above uses at least $\#(\square)^n$ symbols.

CONJECTURE 2.2. For n ≥ 1, there exists infinitely many existential closures of equations, □, true in (¬,<,0,1,+,-,•), such that every proof in 1 (2) above uses at least $\#(\square)^n$ symbols.

This cannot be proved in the same way as Corollary 2.1 without solving some notorious computer science problems.

However, using some nontrivial algebra, it might be possible to prove this.

3. SECOND INCOMPLETENESS THEOREM.

The usual statement of the Second is very clean, but far from rigorous.

THEOREM 3.1. If T is a reasonable system containing a reasonable amount of arithmetic, then T cannot prove its own consistency.

i. What is a reasonable system?
ii. What is a reasonable amount of arithmetic?
iii. What is the statement Con(T)?

Modern research on weak systems reveals that we can get away with $I\square_0$.

THEOREM 3.1′. Let T be a consistent many sorted theory with finitely many axioms, and □ be an interpretation of $I\square_0$ in T. Then T does not prove □(Con(T)).

The remaining problem is: what is Con(T)? Here are the current approaches.

a. Give a version of Con(T) explicitly. This involves giving a version of formalized predicate calculus using the Gödel numbering. This is horribly ugly and ad hoc. This situation can aruguably be improved, by adhering to set theories only, where one can treat syntactic objects as sets.

b. Define what we mean by "adequate treatment of predicate calculus", and then take Con(T) to be the canon-ical statement relative to this treatment. Feferman's, "Arithmetization of Metamathematics".

c. Avoid b (as much as possible), but instead isolate crucial properties of "provability in T" that are used in the proof. A highlight is the condition "it can be proved in T that if a sentence is provable in T then it is provable that it is provable in T". Hilbert and Bernays "derivability conditions".

There are problems with all of these approaches. It turns out that b is very complicated, and c is still too complicated and subtle to be satisfactory. I am not fully able to articulate complaints about c, but I certainly believe that there is a much better way to do this.

I am still thinking about a better way, but there is an approach that avoids this issue entirely, and is in full consonance with Hilbert’s idea that consistency and existence should be identified, at least in certain contexts.

The idea is to focus on interpretability.

THEOREM 3.2. Let T be a consistent many sorted theory with finitely many axioms. Let □ be an interpretation of I□$_0$ in T. There exists a true sentence □ in L(I□$_0$) such that T+□(□) is not interpretable in T.

The only way I know how to prove Theorem 3.2 is to use Second Incompleteness or some variant thereof.

Of course, any T is interpretable in T. But modest desirable extensions of T may not be interpretable in T.

THEOREM 3.3. Let T be a consistent many sorted theory with finitely many axioms. Let □ be an interpretation of I□$_0$ in T. Then T + IND(L(T),□) is not interpretable in T. In fact, T plus some single instance of IND(L(T),□) is not interpretable in T.

COROLLARY 3.4. ZF\P cannot be interpreted into PA. ACA$_0$ cannot be interpreted in PA. Con(PA) cannot be proved in PA.

There are quite a number of interesting variants of Theorem 3.3, asserting that T + □ is not interpretable in T. involving different kinds of □ - and also involving a sort being added to T.

4. IS THERE ANY REAL LOGICAL STRENGTH?

See “The Inevitability of Logical Strength”, February, 2007, recently submitted for publication. http://www.math.ohio-state.edu/%7Efriedman/

5. STRICT REVERSE MATHEMATICS.

We view “The Inevitability of Logical Strength” as the foundations of Strict Reverse Mathematics, where all statements, even in the base theory, must be strictly mathematical. Some initial development of this is can be found on my website under downloadable manuscripts.