INTEGER INNER PRODUCTS AND ALGORITHMIC UNSOLVABILITY

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Abstract. Is there a finite list of integer triples with a given partial list of integer inner products? Is there a finite list of integers with a given partial list of integer inner products between various pairs? We prove that these problems are algorithmically unsolvable by reduction from Hilbert's 10th Problem (over the integers). The status of the first problem for pairs is unresolved.

1. INTRODUCTION

This paper is a contribution to the search for yet more elementary examples of algorithmic unsolvability. We have previously developed quite a number of fundamental examples in elementary Euclidean geometry [Fr10]. Also see [Ma93], [Poxx], [Po14].

Many examples of (algorithmic) unsolvability are proved by reduction from the known unsolvable Hilbert's 10th Problem, [Da73]. We proceed in this way here as well as in [Fr10]. A considerable portion of [Poxx], [Po14] also proceeds in this way. Also see [Ma93], sections 9,10.

One problem is reducible to another if and only if there is an algorithm that converts any instance of the first problem to an equivalent instance of the second problem. So if the second problem is solvable then the first problem is

solvable. If the first problem is unsolvable then the second problem is unsolvable.

The original unsolvable problem is the Halting Problem. Given a Turing Machine initialized with a blank tape, does it eventually halt?

HILBERT'S TENTH PROBLEM/Z. H10(Z). Does a given polynomial with integer coefficients vanish in Z?

H10(Z) is unsolvable, and in fact mutually reducible with the halting problem. See [Da73], [Ma93].

Here we use the usual inner product or dot product on $\Re^k$, given by the familiar $(x_1,...,x_k)\bullet(y_1,...,y_k) = x_1y_1 + ... + x_ky_k$.

PARTIAL INNER PRODUCT SPECIFICATION/Z^k. PIPS(Z^k). Is there a finite list of integer k-tuples with a given partial list of integer inner products?

We show that PIPS(Z^k) is unsolvable in section 2 for any $k \geq 3$. PIPS(Z) is easily solvable (see section 4), and we do not know if PIPS(Z^2) is solvable.

We also consider a number of variants in section 2.

PARTIAL INNER PRODUCT SPECIFICATION/Z^k,{-1,0,1}. PIPS(Z^k,{-1,0,1}). Is there a finite list of k-tuples with a given partial list of inner products from {-1,0,1}?

PARTIAL INNER PRODUCT SPECIFICATION/Z^k,n. PIPS(Z^k,n). Is there a list of n integer k-tuples with a given partial list of integer inner products?

In section 2 we show that for all $k \geq 3$, PIPS(Z^k,{-1,0,1}) is unsolvable. and for all $k \geq 3$ there exists n such that PIPS(Z^k,n) is unsolvable. Either one of these two results immediately implies that for all $k \geq 3$, PIPS(Z^k) is unsolvable.

In section 3, we address some additional problems involving inner products of pairs, while leaving open the status of PIPS(Z^2).

NUMBER BASED INNER PRODUCT SPECIFICATION/Z. NBIPS(Z). Is there a finite list of integers with a given partial list of integer inner products between various pairs of pairs?

NUMBER BASED INNER PRODUCT SPECIFICATION/Z,{-1,0,1}. NBIPS(Z,{-1,0,1}). Is there a finite list of integers with a given partial list of integer inner products, from {-1,0,1}, between various pairs of pairs?

NUMBER BASED INNER PRODUCT SPECIFICATION/Z,n. NBIPS(Z,n). Is there a list of n integers with a given partial list of integer inner products between various pairs of pairs?

In section 3, we show that NBIPS(Z,{-1,0,1}) is unsolvable, and for some n, NBIPS(Z,n) is unsolvable. Either one of these two results immediately implies that NBIPS(Z) is unsolvable.

In section 4, we consider yet more variants based on imposing some further natural restrictions on the conditions and realizations. We also present a short list of open questions.

In the development, we use the words "condition" and "forces" in ways that are not meant to be significantly related to the Cohen forcing method in set theory.

2. PARTIAL INNER PRODUCT SPECIFICATION

We begin with a formal statement of PIPS(Z^k).

DEFINITION 2.1. We use variables v[1],v[2],... . An atom is a statement of the form $v[i] \bullet v[j] = c$, $1 \leq i,j \leq n$, where c is a constant from Z. A condition is a finite set of atoms. A Z^k realization of the condition C is a sequence $(x_1,...,x_n) \in Z^k$, where for all $v[i] \bullet v[j] = c \in C$, we have $x_i \bullet x_j = c$. An n-condition is a condition whose variables are among v[1],...,v[n]. A condition/$\neq$ is a condition with no $v[i] \bullet v[i]$. An n-condition/$\neq$ is an n-condition with no $v[i] \bullet v[i]$. A condition is with only -1,0,1 if and only if all $v[i] \bullet v[j[= c \in C$ has $c \in \{-1,0,1\}$.

PARTIAL INNER PRODUCT SPECIFICATION/Z^k. PIPS(Z^k). Does a given condition have a Z^k realization?

PARTIAL INNER PRODUCT SPECIFICATION/Z^k,n. PIPS(Z^k,n). Does a

given n-condition have a Z^k realization?

PARTIAL INNER PRODUCT SPECIFICATION/Z^k,{-1,0,1}. PIPS(Z^k,{-1,0,1}). Does a given condition with only -1,0,1 have a Z^k realization?

The two strongest unsolvability results proved in this section involve the following two problems.

PARTIAL INNER PRODUCT SPECIFICATION/Z^k,≠,{-1,0,1}. PIPS(Z^k,≠,{-1,0,1}). Does a given condition/≠ with only -1,0,1 have a Z^k realization?

PARTIAL INNER PRODUCT SPECIFICATION/Z^k,≠,n. PIPS(Z^k,≠,n). Does a given n-condition/≠ have a Z^k realization?

The unsolvability of PIPS(Z^k,≠,{-1,0,1}) immediately implies the unsolvability of PIPS(Z^k), PIPS(Z^k,{-1,0,1}). The unsolvability of PIPS(Z^k,≠,n) immediately implies the unsolvability of PIPS(Z^k), PIPS(Z^k,≠), PIPS(Z^k,n).

In this section, we prove the unsolvability of PIPS(Z^k,≠,{-1,0,1}) for $k \geq 3$. We also show that for all $k \geq 3$ there exists n such that PIPS(Z^k,≠,n) is unsolvable. For $k = 3$, we give a crude value for n that is, unfortunately, uncomfortably large.

We have no solvability or unsolvability results for dimension $k = 2$. For easy solvability in dimension $k = 1$, see section 4.

DEFINITION 2.2. In the context of k dimensions, $0^* = (0,...,0)$. $A \subseteq \Re^k$ is orthogonal if and only if for all $x,y \in A$, $x \neq 0^*$ and $x \neq y \rightarrow x \cdot y = 0$.

LEMMA 2.1. Let $A \subseteq \Re^k$ be orthogonal. A is linearly independent over $\Re$. Now suppose, in addition, that $|A| = k$. Then A is a basis for the vector space $\Re k$ over $\Re$. Furthermore, $(\forall x,y \in \Re^k)(x = y \leftrightarrow (\forall z \in A)(x \cdot z = y \cdot z))$.

Proof: Well known linear algebra. Let A be orthogonal. Write $x_i = a_1x_1 + \ldots a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + \ldots + a_nx_n$, $x_i \cdot x_i = 0$, $x_i = 0^*$, contradiction. Hence A is linearly independent over $\Re$, and we assume $|A| = k$. By linear algebra, A is a basis. Now suppose $(\forall z \in A)(x \cdot z = y \cdot z)$. Then $(\forall z \in A)(z \cdot (x-y) = 0)$. Hence for all linear combinations z of elements of A,

$z \bullet (x-y) = 0$. Hence for all $z \in \Re^k$, $z \bullet (x-y) = 0$. In particular $(x-y) \bullet (x-y) = 0$, $x-y = 0$, $x = y$. QED

DEFINITION 2.3. Let $e[i,k] = (0,...,0,1,0,...,0)$ be the i-th standard unit basis vector in k dimensions. $B[k] = \{e[k,i]: 1 \leq i \leq k\}$, $\pm B[k] = B[k] \cup -B[k]$. Let f be a partial function from Z^k into Z^k. We say that f is -1,0,1 dot product preserving if and only if for all $x,y \in dom(f)$, $x \neq y \wedge x \bullet y \in \{-1,0,1\} \rightarrow f(x) \bullet f(y) = x \bullet y$.

LEMMA 2.2. Let $k \geq 2$. The -1,0,1 dot product preserving $f:\{-1,0,1\}^k \rightarrow \{-1,0,1\}^k$ are exactly the $2^k k!$ functions given by $f(x_1,...,x_k) = (\pm x_{\pi 1},...,\pm x_{\pi k})$, where π is a permutation of $\{1,...,k\}$, each of which are dot product preserving.

Proof: Let $f:\{-1,0,1\}^k \rightarrow \{-1,0,1\}^k$ be -1,0,1 dot product preserving. We first show that f is one-one. Let $x \neq y$, where $f(x) = f(y)$. Fix $x_i \neq y_i$.

case 1. $e[k,i] \notin \{x,y\}$. Then $e[k,i] \bullet x = f(e[k,i]) \bullet f(x) = x_i$, and $e[k,i] \bullet y = f(e[k,i]) \bullet f(y) = y_i$, violating $f(x) = f(y) \wedge x_i \neq y_i$.

case 2. $-e[k,i] \notin \{x,y\}$. Argue as in case 1, with $e[k,i]$ replaced by $-e[k,i]$.

case 3. $x = e[k,i] \wedge y = -e[k,i]$. Let $1 \leq i \neq j \leq k$ (this uses $k \geq 2$). Then $(e[k,i]+e[k,j]) \bullet x = f(e[k,i]+e[k,j]) \bullet f(x) = xi$, and and $(e[k,i]+e[k,j]) \bullet y = f(e[k,i]+e[k,j]) \bullet f(y) = y_i$, violating $f(x) = f(y) \wedge x_i \neq y_i$.

case 4. $x = -e[k,i] \wedge y = e[k,i]$. Symmetric with case 3.

This establishes that f is one-one and therefore a bijection. Now every $0^* \bullet x = 0$, and therefore $x \neq 0^* \rightarrow f(0^*) \bullet x = 0$, and so $f(0^*) = 0^*$.

Note that $x \in \pm B[k] \rightarrow x \bullet y \in \{-1,0,1\}$, and so $x \in \pm B[k] \wedge x \neq y \rightarrow f(x) \bullet y \in \{-1,0,1\}$. So $f(x)$ has dot product in $\{-1,0,1\}$ for all but at most one $y \in \{-1,0,1\}^k$. Hence $f(x) \in \pm B[k] \cup \{0^*\}$. Since $f(0^*) = 0^*$ and f is a bijection, we have $f(x) \in \pm B[k]$, thus establishing that f maps $\pm B[k]$ one-one onto $\pm B[k]$.

Now since each e[k,i]•-e[k,i] = -1, we have each f(e[k,i])•f(-e[k,i]) = -1. Since f(e[k,i]),f(-e[k,i]) ∈ B[k] with dot product -1, clearly each f(-e[k,i]) = -f(e[k,i]). So f is of the form $f(x_1,...,x_k) = (\pm x_{\pi 1},...,\pm x_{\pi k})$, π a permutation of {1,...,k}, for x ∈ ±B[k] ∪ {0*}. It remains to show that this equation holds for all x ∈ $\{-1,0,1\}^k$, with the same π and choice of signs. We can write this as $f(x)_{\pi i} = \pm x_i$, where ± varies between +,- dependently on i only. We have this equation for x ∈ ±B[k] ∪ {0*}, and we want to establish it for all x ∈ $\{-1,0,1\}^s$.

We have $f(x)_{\pi i} = e[k,\pi i]\bullet f(x) = f(\pm e[k,i])\bullet f(x) = \pm e[k,i]\bullet x = \pm x_i$ as required, this time for arbitrary x ∈ $\{-1,0,1\}^k$.

Conversely, it is evident that these $2^k k!$ permutations of $\{-1,0,1\}^k$ are dot product preserving, not just -1,0,1 dot product preserving. QED

DEFINITION 2.4. Δ[k] is the enumeration of $\{-1,0,1\}^k$ which, for specificity, is listed lexicographically. $C_0[k]$ is the set of all atoms v[i]•v[j] = c such that $\Delta[k]_i\bullet\Delta[k]_j = c$, where c ∈ {-1,0,1}, and 1 ≤ i ≠ j ≤ k.

Note the conditions ∈ {-1,0,1} and i ≠ j. This allows us to use Δ[k] in the unsolvability proof of PIPS(Z^k,≠,{-1,0,1}).

Note that $C_0[k]$ is a 3^k-condition/≠ using only -1,0,1, and which uses exactly the variables v[1],...,v[k], but only certain v[i]•v[j], i ≠ j.

LEMMA 2.3. Let k ≥ 3. Let α be a Z^k realization of $C_0[k]$. α is a length 3^k sequence of elements of Z^k. α has no repetitions. Every term of α lies in $\{-1,0,1\}^k$. α is obtained from Δ[k] using one of the $2^k k!$ maps $(\pm x_{\pi i},...,\pm x_{\pi k})$, for permutations π of 1,...,k.

Proof: Let α be a Z^k realization of $C_0[k]$. The first claim is by the definition of Z^k realization. By examining $C_0[k]$, we see that for all 1 ≤ i ≠ j ≤ k there exists p ≠ i,j such that $\alpha_i\bullet\alpha_p$ and $\alpha_j\bullet\alpha_p$ are distinct elements of {-1,0,1}. This is because Δ[k] is an enumeration of $\{-1,0,1\}^k$ without repetition, which contains e[k,1],...,e[k,k], which can be

used for the α_p. Hence α has no repetitions.

Now let $\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_k,\delta_1,\ldots,\delta_k$ be terms of $\Delta[k]$ at the distinct positions $\varepsilon_1,\ldots,\varepsilon_{3k}$, where

i. For $1 \le i \ne j \le k$, $\beta_i \bullet \beta_j = 0$.
ii. For $1 \le i \le k$, $\beta_i \bullet \gamma_i = -1$.
iii. For $1 \le i \ne j \le k+1$, $\delta_i \bullet \delta_j = 0$.
iv. For $1 \le i \le k$, $\beta_i \bullet \delta_i \in \{-1,1\}$.
v. For $1 \le i,j \le k$, $\beta_i \bullet \delta_j = -\gamma_i \bullet \delta_j \in \{-1,0,1\}$.
vi. For all $1 \le i \le k$ and $1 \le j \le 3^k$, $\beta_i \bullet \Delta[k]_j, \gamma_i \bullet \Delta[k]_j \in \{-1,0,1\}$.
vii. If $\Delta[k]_p, \Delta[k]_q$ have the same dot products with $\beta_1,\ldots,\beta_k$, then $p = q$.

For the above, we use $\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_k = e[k,1],\ldots,e[k,k],-e[k,1],\ldots,-e[k,k]$. For $\delta_1,\ldots,\delta_k$, if k is even, use $e_1-e_2,-e_1+e_2,e_3-e_4,-e_3+e_4,\ldots,e_{k-1}-e_k,e_k-e_{k-1}$. If k is odd, use $e_1-e_2,-e_1+e_2,e_3-e_4,-e_3+e_4,\ldots,e_{k-2}-e_{k-1},-e_{k-2}+e_{k-1},e_k$. Then $\{\beta_1,\ldots,\beta_k\},\{\gamma_1,\ldots,\gamma_k\},\{\delta_1,\ldots,\delta_k\}$ are each orthogonal, and vii holds because $\{\beta_1,\ldots,\beta_k\} = B[k]$.

Since α is a Z^k realization of $C_0[k]$, let $\beta_1',\ldots,\beta_k',\gamma_1',\ldots,\gamma_k',\delta_1',\ldots,\delta_k'$ be the terms of α at the same distinct positions $\varepsilon_1,\ldots,\varepsilon_{3k}$, where

i. For $1 \le i \ne j \le k$, $\beta_i' \bullet \beta_j' = 0$.
ii. For $1 \le i \le k$, $\beta_i' \bullet \gamma_i' = -1$.
iii. For $1 \le i \ne j \le k+1$, $\delta_i' \bullet \delta_j' = 0$.
iv. For $1 \le i \le k$, $\beta_i' \bullet \delta_i' \in \{-1,1\}$.
v. For $1 \le i,j \le k$, $\beta_i' \bullet \delta_j' = -\gamma_i' \bullet \delta_j' \in \{-1,0,1\}$.
vi. For all $1 \le i \le k$ and $1 \le j \le 3^k$, $\beta_i' \bullet \alpha_j, \gamma_i' \bullet \alpha_j \in \{-1,0,1\}$.
vii. If α_p, α_q have the same dot products with $\beta_1',\ldots,\beta_k'$, then $p = q$.

We obtain vii since by vi, the dot products involved all lie in $\{-1,0,1\}$. Note that by vii, α is without repetition.

By ii, each $\beta_i' \ne 0^*$, and hence by i, $\{\beta_1',\ldots,\beta_k'\}$ is orthogonal. By iv, each $\delta_i' \ne 0^*$, and hence by iii, $\{\delta_1',\ldots,\delta_k'\}$ is orthogonal. By v, $\beta_i'+\gamma_i'$ has dot product 0 with each of $\delta_1',\ldots,\delta_k'$. By Lemma 2.1, each $\beta_i'+\gamma_i' = 0$, and

therefore each $\gamma_i' = -\beta_i'$. By ii, each unordered pair β_i',γ_i' is some e[k,j],-e[k,j]. By vi, every term of α lies in $\{-1,0,1\}^k$. Since the length of α is 3^k and α has no repetition, α is an enumeration without repetition of $\{-1,0,1\}^k$. Since the $\beta_1',\ldots,\beta_k',\gamma_1',\ldots,\gamma_k'$ occupy the same positions in α as do $\beta_1,\ldots,\beta_k,\gamma_1,\ldots,\gamma_k$ in $\Delta[k]$, we define the bijection $f:\{-1,0,1\}^k \to \{-1,0,1\}^k$ by $f(\Delta_i) = \alpha_i$, $1 \le i \le 3^k$.

We claim that f is -1,0,1 dot preserving. Suppose $\Delta_i \bullet \Delta_j = c \in \{-1,0,1\}$. Then $v[i] \bullet v[j] = c$ lies in $C_0[k]$. Since α is a Z^k realization of $C_0[k]$, clearly $\alpha_i \bullet \alpha_j = c$. By Lemma 2.2, f is among the $2^k k!$ maps $(\pm x_{*i},\ldots,\pm x_{*k})$. QED

LEMMA 2.4. Let C be a condition extending $C_0[k]$. C has a Z^k realization if and only if C has a Z^k realization extending $\Delta[k]$.

Proof: Let C extend $C_0[k]$ with Z^k realization α. By Lemma 2.3, $\alpha_1,\ldots,\alpha_{3\hat{}k}$ is obtained from $\Delta[k]$ using one of our $2^k k!$ maps $h = (\pm x_i\ldots,\pm x_k)$. Now apply h^{-1} to α, obtaining β. h^{-1} doesn't affect the various dot products, and so β is also a Z^k realization of C. Note that β extends $\Delta[k]$. QED

LEMMA 2.5. Let $k \ge 3$. There is a 3^k+6-condition/$\neq$ $C_1[k]$ extending $C_0[k]$, using only -1,0,1, such that the following holds. For all n the unique Z^k realization of $C_1[k]$ extending $\Delta[k],(n,1,1,0,\ldots,0)$, is:

$\Delta[k]$
$a = (n,1,1,0,\ldots,0)$
$b = (-1,n,0,\ldots,0)$.
$c = (n,1,-n^2,0,\ldots,0)$.
$d = (0,n^2,1,0,\ldots,0)$.
$e = (1,1,-n^2,0,\ldots,0)$.
$f = (n^2,1,1,0,\ldots,0)$.

Proof: First use $\Delta[k]$ in the obvious way to force all of the displayed -1,0,1 as above via the e[k,i]; the e[k,i] have been secured by $C_0[k]$). Below, we leave off the last k-3 0's for convenience.

$a \bullet b = 0$ forces $b = (-1,n,0)$.
$b \bullet c = 0$ forces $-c_1+n = 0$, $c_1 = n$.
$a \bullet c = 1$ forces $n^2+1+c_3 = 1$, $c_3 = -n^2$, $c = (n,1,-n^2)$.

$c \bullet d = 0$ forces $d_2 - n^2 = 0$, $d_2 = n^2$, $d = (0,n^2,1)$.
$d \bullet e = 0$ forces $n^2 + e_3 = 0$, $e_3 = -n^2$, $e = (1,1,-n^2)$.
$e \bullet f = 1$ forces $f_1 + 1 - n^2 = 1$, $f_1 = n^2$, $f = (n^2,1,1)$.

$\Delta[k]$,a-f is a Z^k realization of the set of specified dot products above by inspection. QED

LEMMA 2.6. Let $k \geq 3$. There is a 3^k+9-condition/$\neq C_2[k]$ extending $C_0[k]$, using only -1,0,1, such that the following holds. For all n the unique Z^k realization of $C_2[k]$ extending $\Delta[k],(n,1,1,0,...,0),(m,1,1,0,...,0)$ is
$\Delta[k]$
a = (n,1,1,0,...,0)
b = (m,1,1,0,...,0)
c = (-1,0,n,0,...,0)
d = (-1,0,m,0,...,0)
e = (m,m,1,0,...,0)
f = (0,-1,m,0,...,0)
g = (n,m,1,0,...,0)
h = (1,1,-n-m,0,...,0)
i = (n+m,1,1,0,...,0)

Proof: First use $\Delta[k]$ in the obvious way to force all of the displayed -1,0,1 above via the e[k,i]. Below, we leave off the last k-3 0's for convenience.

$a \bullet c = 0$ forces $c = (-1,0,n)$.
$b \bullet d = 0$ forces $d = (-1,0,m)$.
$(1,-1,0) \bullet e = 0$ forces $e_1 = e_2$.
$d \bullet e = 0$ forces e1 = m, $e = (m,m,1)$.
$e \bullet f = 0$ forces $f = (0,-1,m)$.
$f \bullet g = 0$ forces $g_2 = m$.
$c \bullet g = 0$ forces $g_1 = n$, $g = (n,m,1)$.
$g \bullet h = 0$ forces $h = (1,1,-n-m)$.
$h \bullet i = 0$ forces $i = (n+m,1,1)$.

$\Delta[k]$,a-i is a Z^k realization of the set of specified dot products above by inspection. QED

LEMMA 2.7. There is an algorithm that converts any polynomial P with integer coefficients and variables $x_1,...,x_k$ to a finite list P* of equations of the form $x_i+x_j = x_p$, $x_ix_j = x_p$, $x_i = 1$, such that the following holds. P has a zero over Z if and only if P* has a solution over Z.

Proof: This well known construction proceeds as follows. Let P be as given, and write P = Q-R, where Q,R have only positive coefficients. P = 0 is therefore equivalent to Q = R. For each monomial $cy_1...y_n$, $c > 0$, $n \geq 0$, in Q,R, we first associate the set of equations $z_1 = c$, $z_2 = z_1y_1$, ..., $z_{n+1} = z_ny_n$. Now replace $z_1 = c$ by breaking into $w_1 = 1,...,w_c = w_{c-1}+w_1, z_1 = w_c$. Make sure that the new variables introduced are distinct across monomials.

We now write Q = R in the form $u_1+...+u_n = v_1+...+v_m$, since Q,R are each sums of these monomials. Here the u's and v's are among the variables introduced above. Then we introduce more new variables to break up both sums, with a final equation of the form $\alpha = \beta$, where α,β are variables. Finally, we replace $\alpha = \beta$ by the two equations $\alpha = \gamma\beta$, $\gamma = 1$. QED

LEMMA 2.8. There is an algorithm that converts any finite list S of equations of the form $x_i+x_j = x_p$, $x_ix_j = x_p$, $x_i = 1$, to a finite list S' of equations of the form $x_j+x_j = x_p$, $x_i^2 = x_j$, $x_i = 1$, such that the following holds. S has a solution over Z if and only if S' has a solution over Z.

Proof: This well known construction is as follows. We replace all x = yz in favor of equations of the given form using the new variables a,b,c,d,e,f,g,h as follows. First replace any x = yz by $2x = (y+z)^2-y^2-z^2$, and then by $a = y+z$, $b = a^2$, $c = y^2$, $d = z^2$, $x+x = b-c-d$. Replace $x+x = b-c-d$ by $e = b-c$, $f = e-d$, $x+x = f$. Thus we replace x = yz by

$y+z = a$
$a^2 = b$
$y^2 = c$
$z^2 = d$
$e+c = b$
$f+d = e$
$x+x = f$

Note that for $x,y,z \in Z$, x = yz if and only if there is a solution from Z of these 7 equations. QED

We now want to apply Lemmas 2.5, 2.6 to the lists S' from Lemma 2.8. In order to do this, we first put these S' into a special list form.

DEFINITION 2.4. Let $k \geq 3$. A special k-list consists of a list of statements in unknowns $v[3^k+1],...,v[3^k+t]$, $t \geq 0$, of the following form, interpreted conjunctively:
1. $v[3^k+1],...,v[3^k+t+1]$ are k-tuples $(x_{(3^k)+1},1,1,0,...,0),...,(x_{(3^k)+t+1},1,1,0,...,0)$, respectively.
2. For certain $3^k+1 \leq i,j,p \leq 3^k+t$, $x_i+x_j = x_p$.
3. For certain $3^k+1 \leq i,j \leq 3^k+t$, $x_i^2 = x_j$.
4. For certain $3^k+1 \leq i \leq 3^k+t$, $x_i = 1$.

LEMMA 2.9. Let $k \geq 3$. There is an algorithm that converts any finite list S of equations of the form $x_i+x_j = x_p$, $x_i^2 = x_j$, $x_i = 1$ to a special list L such that the following holds. S has a solution from Z if and only if L has a solution $v[3^k+1],...,v[3^k+t+1]$ from Z^k.

Proof: Straightforward. All we have done is change notation from $x_1,...,x_t$ to $v[3^k+1],...,v[3^k+t]$ and attached $1,1,0,...,0$ in order to convert to k-tuples (only k-tuples ending with $1,1,0,...,0$). QED

LEMMA 2.10. Let $k \geq 3$. There is an algorithm that converts any special list L of statements in $v[3^k+1],...,v[3^k+t]$, $t \geq 0$, to a condition $C[k]$ extending $C_0[k]$ such that the following are equivalent.
i. L has a solution over Z.
ii. $C[k]$ has a Z^k realization extending $\Delta[k]$.
iii. $C[k]$ has a Z^k realization.

Proof: Let L,t be as given. We start with $C_0[k]$. We first use the $e[k,i]$, which have been secured by $C_0[k]$, to force that $v[3^k+1],...,v[3^k+t]$ are of the form $(x_{3^k},1,1,0,...,0),...,(x_{(3^k)+t},1,1,0,...,0)$. For each $x[i]+x[j] = x[p]$ in L, we add clauses given by changing variables in the Lemma 2.6 display that force $v[i]+v[j] = v[p]$, according to Lemma 2.6. We also force each $x[i]^2 = x[j]$ in L analogously, using Lemma 2.5. Finally, we treat each $x[i] = 1$ in L using $e[k,1]$. The first two of these three additions require new variables v[r], whereas the third of these additions does not.

The equivalence between ii,iii is given by Lemma 2.4. QED

LEMMA 2.11. There is an algorithm that converts any polynomial P with integer coefficients to a condition C such that the following holds. P has a zero from Z if and

only if C[k] has a Z^k realization.

Proof: By chaining together Lemmas 2.7, 2.8, 2.9, 2.10. QED

THEOREM 2.12. Let $k \geq 3$. The following problems are mutually reducible.
i. Halting Problem.
ii. H10(Z).
iii. PIPS(Z^k,≠,{-1,0,1}).
iv. PIPS(Z^k,≠).
v. PIPS(Z^k,{-1,0,1}).
vi. PIPS(Z^k).
All of these six problems are unsolvable, and in fact r.e. complete.

Proof: Reduction i → ii is by the MRDP theorem, [Da73], [Ma93]. Reduction ii → iii is by Lemma 2.11. Reductions iii → iv,v,vi are immediate. Reductions iv,v,vi → i is obvious by crude computer search. QED

Theorem 2.12 controls the dimension and the dot product values while necessarily leaving the number of variables uncontrolled. Now we want to control the dimension and the number of variables, while necessarily leaving the dot product values uncontrolled. I.e., we show that PIPS(Z^k,≠,n) is unsolvable, for some n depending on k.

We fix $k \geq 1$. First we need to modify Lemma 2.7 using universal polynomials.

LEMMA 2.13. There exists n a finite list K of equations using variables $x_1,...,x_n$, of the form $x_i+x_j = x_p$, $x_ix_j = x_p$, $x_i = 1$, such that $\{c \in Z: K \cup \{x_1 = c\}\}$ has a solution over Z) is complete r.e.

Proof: By [Da73], [Ma93], let $P(x_1,y_1,...,y_m)$ be an integral polynomial such that $\{x_1 \in Z: (\exists y_1,...,y_m \in Z)(P(x_1,y_1,...,y_m) = 0)\}$ is complete r.e. Then break P down as in the proof of Lemma 2.7, into a finite set K of equations of the required form such that for all $x_1 \in Z$, $P(x_1,y_1,...,y_m) = 0$ has a solution over Z if and only if K has a solution over Z, by introducing variables $x_2,...,x_n$. QED

LEMMA 2.14. There exists n and a finite list K of equations using variables $x_1,...,x_n$, of the form $x_i+x_j = x_p$, $x_i^2 = x_j$,

$x_i = 1$, such that $\{c \in Z$: $K \cup \{x_1 = c\}$ has a solution over $Z\}$ is complete r.e.

Proof: From Lemma 2.13 by the construction given in the proof of Lemma 2.8. QED

We fix K,n from Lemma 2.14. We use the notion of special k-list from Definition 2.4.

LEMMA 2.15. There is a special k-list L such that the following holds. Let $c \in Z$. $K \cup \{x_1 = c\}$ has a solution over Z if and only if $L \cup \{x_{(3\hat{}k)+1} = c\}$ has a solution over Z.

Proof: See the proof of Lemma 2.9. QED

We fix L as given by Lemma 2.15.

THEOREM 2.16. For all $k \geq 3$ there exists a finite list D[k] of unevaluated dot products, without any $v[i] \bullet v[i]$, such that the set of conditions using exactly D[k] which have a Z^k realization extending $\Delta[k]$, is complete r.e. We can drop "extending $\Delta[k]$". For all $k \geq 3$ there exists n such that $IPS(Z^k, \neq, n)$ is complete r.e.

Proof: This is proved using Lemmas 2.4,2.5,2.6 starting with L. Let L use variables $v[3^k+1],...,v[3^k+m]$. Let C* be the condition/$\neq$ with -1,0,1, corresponding to L, constructed using Lemmas 2.5,2.6, and $C_0[k]$, whose Z^k realizations correspond to the solutions to L over Z. We take D[k] to be the result of removing the evaluations of the dot products in C* and adding the unevaluated dot product $v[r] \bullet v[3^k+1]$, where $\Delta[k]_r = e[k,1]$. Then the set of conditions (evaluated dot products) using exactly D[k] which have a Z^k realization extending $\Delta[k]$ corresponds to the set of solutions over Z of the $L \cup \{x_{(3\hat{}k)+1} = c\}$. Hence by Lemma 2.15, the set of conditions using exactly D[k] which have a Z^k realization extending $\Delta[k]$ is complete r.e. The second claim is by Lemma 2.4. The third claim follows immediately. QEDs

Sets of unevaluated dot products suggest graphs.

DEFINITION 2.5. A graph is a G = (V,E) where E is an irreflexive symmetric relation on E. V is the set of vertices and E is the set of edges.

IPS(G,Z^k). Given an assignment of integers to the edges of the finite graph G, is there is an assignment of elements of Z^k to the vertices of G such that the dot products given by the edge assignment are correct?

THEOREM 2.17. For all $k \geq 3$ there exists a finite graph G such that IPS(G,Z^k) is complete r.e.

Proof: Immediate from Theorem 2.16. QED

We now give rough upper bound on the number of vertices and edges for the G in Theorem 2.17 and for the n in Theorem 2.16 using work of Jones. We restrict the estimate to the case k = 3.

THEOREM 2.18. There is a graph G with at most 70,000 vertices and edges such that IPS(G,Z^3) is complete r.e. In particular, PIPS(Z^3,≠,70,000) is complete r.e.

Proof: From [Jo82], we see that the K in Lemma 2.13 can be taken to have at most 100 variables and equations, with two caveats. Firstly, N is used instead of Z. Secondly, equations $x_i = c$ are allowed, for any $c \in Z$. The former will be rectified later, and the latter needs to be noted, but ultimately causes no difficulties.

Next, the K in Lemma 2.14 involves a multiplication of the number of variables and equations by 9, to at most 900 variables and equations, but still over N.

Next, we move from N to Z. This involves introducing equations $x = y^2+z^2+w^2+u^2$, which then has to be unraveled with extra variables and equations, and this multiples the bound by 7 to 6300 variables and equations, now over Z instead of N.

Next, we use Lemmas 5,6. These involve a multiplication by at most 11 for the variables and equations, as the displayed -1's,0's,1's need to be enforced and at most 9 equations are created for each equation. We have arrived at 69,300 as the upper bound on the number of variables and equations thus far.

Finally, there is $C_0[3]$. Since 69,300 is already so large, we simply use the crude 27(26)/2 = 351, for a total upper bound of 69,651 on the number of variables and equations, to get complete r.e. in dimension 3. The fact that [Jo82] uses equations x = c, c ∈ Z, instead of just x = 1, gets absorbed in the unsolvability. QED

CHALLENGE. Show that PIPS(Z^3,≠,n) is solvable for tiny n, and unsolvable for n considerably smaller than 70,000.

3. NUMBER BASED INNER PRODUCT SPECIFICATION

DEFINITION 3.1. We use variables v[1],v[2],... over Z. An atom* is a statement of the form (v[i],v[j])•(v[p],v[q]) = c, 1 ≤ i,j,p,q ≤ n, where c is a constant from Z. A condition* is a finite set of atoms*. A Z realization of the condition* C is a sequence $(x_1,...,x_n) \in Z$, where for all (v[i],v[j])•(v[p],v[q]) = c ∈ C, we have $(x_i,x_j)•(x_p,x_q)$ = c. An n-condition* is a condition whose variables are among v[1],...,v[n].

We will consider the following three problems.

NUMBER BASED INNER PRODUCT SPECIFICATION/Z. NBIPS(Z). Does a given condition* have a Z realization?

NUMBER BASED INNER PRODUCT SPECIFICATION/Z,n. NBIPS(Z,n). Does a given n-condition* have a Z realization?

NUMBER BASED INNER PRODUCT SPECIFICATION/Z,{-1,0,1}. NBIPS(Z,{-1,0,1}). Does a given condition* with only -1,0,1 have a Z realization?

In this section, we prove unsolvability of NBIPS(Z,{-1,0,1}), and there exists n such that NBIPS(Z,n) is unsolvable. These imply the unsolvability of NBIPS(Z).

We achieve this by first establishing a nontrivial fact about the structure (Z,•,S), where S(x) = x+1. Namely, a strong kind of definability of + in this structure.

DEFINITION 3.2. A relation $R \subseteq Z^k$ is very definable if and only if we can define $(\forall x_1,...,x_k)(R(x_1,...,x_k) \leftrightarrow (\exists y_1,...,y_n)(\varphi_1 \wedge ... \wedge \varphi_m))$, where each φ_i is an equation $w_iw_j = w_p$ or an equation $S(w_i) = w_j$, with w_i,w_j,w_p among

$x_1,...,x_k,y_1,...,y_n$. Here we take $x_1,...,x_k,y_1,...,y_n$ to be k+n distinct variables ranging over Z.

Our immediate goal is to show that + is very definable. This was established in [Ro49] for $(Z^+,\bullet,S)$, but it does seem that a new idea is needed for $(Z,\bullet,S)$.

LEMMA 3.1. Any relation defined by $(\forall x_1,...,x_k)(R(x_1,...,x_k) \leftrightarrow (\exists y_1,...,y_n)(\varphi_1 \wedge ... \wedge \varphi_m))$ where each φ_i is either a very definable relation of $x_1,...,x_k,y_1,...,y_n$, or is an equation s = t, where s,t are terms in $x_1,...,x_k,y_1,...,y_n,\bullet,S$, is very definable.
i. n = 0.
ii. n = 1.
iii. n = -1.
iv. n = m.
v. n = -m.
vi. n+m = 1.

Proof: The first claim is by obvious predicate calculus manipulations where new existential quantifiers are introduced.

For i, $n = 0 \leftrightarrow n(n+1) = n$.
For ii, $n = 1 \leftrightarrow (\exists m)(n = S(m) \wedge m = 0)$.
For iii. $n = -1 \leftrightarrow (\exists m)(m = 0 \wedge S(n) = m)$.
For iv, $n = m \leftrightarrow (\exists r)(S(n) = r \wedge S(m) = r)$.
For v, $n = -m \leftrightarrow (\exists r)(r = -1 \wedge n = rm)$.
For vi, $n+m = 1 \leftrightarrow (\exists r)(n = S(r) \wedge r = -m)$.
QED

DEFINITION 3.3. Integers n,m are relatively prime if and only if the only positive integer that divides both is 1. Note that n,0 are relatively prime if and only if |n| = 1. Note that every n,1 and n,-1 are relatively prime.

THEOREM 3.2. n,m are relatively prime if and only if there exists a,b such that an+bm = 1.

Proof: This is well known as Bezout's identity. QED

LEMMA 3.3. Let ad-bc = 1. Then ax+by = cx+dy = 1 has exactly one solution, $x = d-b \wedge y = a-c$.

Proof: Let ad-bc = 1. Obviously x = d-b ∧ y = a-c is a solution by substitution. Now let x,y be a solution. Note that a ≠ c ∨ b ≠ d.

case 1. a ≠ c. We have the following chain of implications.

ax+by = cx+dy = 1
ax+by-cx-dy = 0.
(b-d)y = (c-a)x.
x = (b-d)y/(c-a).
a(b-d)y/(c-a) + by = 1.
(ab-ad)y = (1-by)(c-a).
(ab-ad)y = c-a+(ab-bc)y.
(bc-ad)y = c-a.
y = a-c.
x = (b-d)(a-c)/c-a = d-b.

case 2. b ≠ d. We have the following chain of implications.

ax+by = cx+dy = 1
ax+by-cx-dy = 0.
(b-d)y = (c-a)x.
y = (c-a)x/(b-d).
ax + b(c-a)x/(b-d) = 1.
(bc-ab)x = (1-ax)(b-d).
(bc-ab)x = b-d-a(b-d)x.
(bc-ad)x = b-d.
x = d-b.
y = (c-a)(d-b)/b-d = a-c.

QED

LEMMA 3.4. d-b = e if and only if there exists x,y,a,b',c,d',e',f such that
i. ad'-b'c = 1.
ii. ax+b'y = cx+d'y = 1.
iii. x = e'.
iv. fb' = b ∧ fd' = d ∧ fe' = e.
The predicate d-b = e is very definable.

Proof: Let d-b = e.

case 1. d = b = 0. Then e = 0. Set a = b' = d' = 1 and c = 0. Set x = 0 and y = 1. Set e' = f = 0.

case 2. Otherwise. Set f be the greatest positive common divisor of d,b. (Since ¬(d = b = 0), f exists). Set b' =

b/f and d' = d/f. Then b',d' are relatively prime. By Bezout, let ad'-b'c = 1. Set x = d'-b' and y = a-c. Set e' = d'-b'. For ii, substitute x,y, and use ad'-b'c = 1. iii and the first two conjuncts of iv are immediate. fe' = fd'-fb' = d-b = e.

Conversely, let x,y,z,b',c,d',e',f be such that i-iv hold. By i,ii and Lemma 3.3, we have x = d'-b' and y = a-c. By iii, e' = d'-b'. Hence fe' = fd'-fb'. By iv, e = d-b = e.

To see that d-b = e is very definable, introduce new variables for ad',b'c,ax,b'y,cx,d'y, and use Lemma 3.1. QED

DEFINITION 3.3. Let D_0 be the 3-condition* consisting of all (v[i],v[j])•(v[p],v[q]) = c, where $1 \leq i,j,p,q \leq 3$, $c \in \{-1,0,1\}$, that is realized by -1,0,1.

LEMMA 3.5. α is a realization of D_0 if and only if α is (-1,0,1) or (1,0,-1). Let D be a condition* extending D_0. D has a Z realization if and only if D has a Z realization extending -1,0,1.

Proof: Let α be a realization of D_0. Since $(\alpha_2,\alpha_2)\bullet(\alpha_2,\alpha_2) = 0$, we have $\alpha_2 = 0$. Since $(\alpha_1,\alpha_2)\bullet(\alpha_1,\alpha_2) = 1$, we have $\alpha_1 = \pm 1$. Since $(\alpha_2,\alpha_3)\bullet(\alpha_2,\alpha_3) = 1$, we have $\alpha_3 = \pm 1$. At this point we have only that $\alpha_1 = \pm\alpha_3$. Since $(\alpha_1,\alpha_1)\bullet(\alpha_1,\alpha_3) = 0$, we have $\alpha_1 = -\alpha_3$. Hence α is (-1,0,1) or (1,0,-1). The converse is obvious since the minus function preserves all of the relevant dot products.

For the second claim, suppose α is a Z realization of D. Then α extends -1,0,1 or 1,0,-1. Suppose α extends 1,0,-1. Then $-\alpha$ is a Z realization of D and extends -1,0,1. QED

LEMMA 3.6. There is a condition* D_1 extending D_0, using only -1,0,1, such that the following holds. For all n the unique Z realization of D_1 extending -1,0,1,n,m has the following form:
-1
0
1
n
m
nm
n+1
n+m

...

Proof: Use the equation (v[5],v[1])•(v[4],v[6]) = 0 to force v[6] = nm. Use the equation (v[4],v[3])•(v[1],v[7]) = 1 to force v[7] = n+1. We force v[8]-v[5] = v[4] using the very definition of d-b = e given by Lemma 3.4, which involves adding a lot of additional variables for the displayed ... . Specifically, the extra 14 variables used in Lemma 3.4, and some additional variables to implement Lemma 3.3, are used here, with copies of the setup displayed above to force multiplication and addition. Successor is used here in combination with multiplication in order to force addition. QED

Now that we have controlled addition and multiplication via Lemma 3.6, we essentially repeat the proofs of Theorems 2.12 and 2.16, now for NBIPS(Z,{-1,0,1}), starting with Lemma 2.7, which provides the link with H10(Z). We don't have to go through squaring as we did in Lemma 2.8 because we already have addition and multiplication forced. Thus we have established the following.

THEOREM 3.7. NBIPS(Z,{-1,0,1}) is complete r.e. There exists n such that NBIPS(Z,14,000) is complete r.e.

We also can give an upper bound for n. The calculation will proceed exactly as before, except at one spot. Following the proof of Theorem 2.18, we again start with the 100 from [Jo82], over N. We don't need to reduce to squaring, with still have to pass from N to Z, bringing us up to 700. The last factor to be applied corresponds to the blowup involved in Lemma 3.6. We estimate this factor to be about 20, arriving at 14,000.

THEOREM 3.8. NBIPS(Z,14,000) is complete r.e.

4. ADDITIONAL VARIANTS AND OPEN QUESTIONS

We first show solvability dimension k = 1 as promised.

THEOREM 4.1. PIPS(Z) is solvable.

Proof: Let C be an n-condition. If we require the Z realizations to have no 0 terms, then this is immediate, since then all terms v[i], $1 \leq i \leq n$, would have to be bounded by maximum of the absolute values of the constants

used in the condition, and we can use exhaustive search. For the general case, define the n-profile/0 P of a realization α as the true set of statements v[i] =≠ 0, 1 ≤ i ≤ n, about α. An n-condition C has a Z realization if and only if there is an n-profile/0 P such that C has a realization α with the n-profile/0 P. Note that there are 2^n n-profiles/0. This is solvable for each particular n-profile/0 by first plugging the zeros in P into C, reducing to a Z realization problem where the Z realizations are required to have no 0 terms, solved by exhaustive search as indicated above. QED

The strongest solvability results use the widest conditions, which, in our case, is PIPS(Zk), NBIPS(Z). The strong unsolvability results use the narrowest conditions, which, for us, is PIPS(Zk,≠,{-1,0,1}), PIPS(Zk,≠,n), NBIPS(Z,{-1,0,1}), NBIPS(Z,n).

It is natural to put some basic conditions on Z^k realizations, and revisit our problems. We will consider only three such conditions.

1. No term is the zero vector.
2. No repeated vector terms.
3. No zero vector terms and no repeated vector terms.

THEOREM 4.2. For all k ≥ 3, PIPS(Z^k,≠,{-1,0,1}) with restrictions 1,2, or 3 above, is unsolvable. For all k ≥ 3 there exists n such that PIPS(Z^k,≠,n) with restrictions 1,2, or 3 above, is unsolvable.

Proof: We first use the restriction 1, no term is the zero vector. We use the obvious notion of profile of Z^k realizations, which just indicates which vector terms are zero vectors and which are not. To solve the original unrestricted problem (which in fact is unsolvable) we need only solve the original problem for Z^k realizations under a given profile. However, requiring a given profile is tantamount to the requirement that no term is the zero vector, for a closely related problem - namely one where variables set to the zero vector are removed replaced by 0 in the atoms. If in that atom, we use c ≠ 0 is used, then we already know to return false. If in that atom, 0 is used, then that atom is removed. This shows how to solve the original problem if we can solve the restricted problem with restriction 1. This argument establishes the first

claim for restriction 1, no term is the zero vector. The same argument works instead for restriction 2, no repeated vector terms, except that instead we retain only one variable in each equivalence class defined by the given profile, replacing the others in the atoms with the ones retained. This shows how to solve the original problem if we can solve the restricted problem with restriction 2. This procedure works if we combine restrictions 1,2, which is restriction 3.

The same method works for establishing the second claim, and in fact here, the same n can be used since the above process of simplification does not introduce any new variables. QED

We can easily revisit Theorem 4.1 using these restrictions 1-3.

THEOREM 4.3. PIPS(Z) with restrictions 1,2, or 3 above, is unsolvable.

Proof: By Theorem 4.2, it suffices to solve PIPS(Z) where the realizations are required to avoid 0 and have no repeated terms. The v[i]v[j] = c present must have $c \neq 0$, for otherwise we trivially have no Z realization. But then the c's provide upper bounds, and so we can use exhaustive search. QED

We also direct this development to the NBIPS of section 3.

THEOREM 4.4. NBIPS(Z,{-1,0,1}) is unsolvable if we use restrictions 1,2, or 3. There exists n such that NBIPS(Z,n) is unsolvable if we use restrictions 1,2 or 3.

Proof: This method of profiles works the same way for restriction 2. For restriction 1, we need to be more careful. In the process, some (v[i],v[j])•(v[p],v[q]) = c gets replaced by, say, (0,v[j])•(v[p],v[q]) = c, which is equivalent to v[j]v[q] = c. There can be more than one 0 substituted, in which case we either arrive at the same shape v[j]v[q] = c, or 0 = c, the latter which is trivial to handle. Now since we are operating under no 0's, c = 0 tells us to retain false, and $c \neq 0$ tells us to look only at finitely many v[j],v[q], and thus the profiles get further refined, with elimination of variables and atoms. There is no difficulty in combining both procedures, to handle restriction 3. Since no event are any new variables

created, the second claim is also established in this way. QED

Note the word "partial" used throughout the paper.

DEFINITION 4.1. In FIPS(Z^k), FIPS(Z^k,{-1,0,1}), FIPS(Z^k,≠), FIPS(Z^k,≠,{-1,0,1}), FIPS(Z^k,n), FIPS(Z^k,≠,n), we use F for "full" rather than P for "partial". In the ones without ≠, we require of the condition that the set of variables used is some v[1],...,v[n] (with the same n in case "n" is mentioned), with all dot products v[i]•v[j], $1 \leq i,j \leq n$, used. In the ones with ≠, we also require of the condition that the set of variables used is some v[1],...,v[n] (again with the same n in case "n" is mentioned), with all dot products v[i]•v[j], $1 \leq i \neq j \geq n$, used.

THEOREM 4.5. FIPS(Z^k) is solvable.

Proof: This is because every relevant v[i]•v[i] is present, and so the relevant Z^k realizations are all placed within the finite set $(-\max(|c_1|,...,|c_n|),...,\max(|c_1|,...,|c_n|)$, which allows us to simply conduct exhaustive search. QED

We leave the status of FIPS(Z^k,≠), FIPS(Z^k,≠,{-1,0,1}), FIPS(Z^k,≠,n) open, although the case k = 1 is solvable using Theorem 4.3.

We can also consider the full forms of NBIPS(Z), NBIPS(Z,{-1,0,1}). This is solvable, as any use of nonzero c throws the associated variables into a finite set, and the uses of c = 0 are easily handled.

We now list some open questions.

1. Is PIPS(Z^2) and its variants solvable or unsolvable? We claim no result of this kind.
2. Is PIPS(Z^3,n) solvable for tiny n? We have not investigated this question, but suspect that this becomes difficult for tiny n.
3. We ask 2 also in dimension 2 with Z^2.
4. We ask 2 also for the various variants of PIPS considered here.
5. Is NBIPS(Z,n) solvable for tiny n?
6. For which k is FIPS(Z^k,≠), FIPS(Z^k,≠,{-1,0,1}) solvable?

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