THE INTERPRETATION OF SET THEORY IN PURE PREDICATION THEORY
preliminary report
by

Harvey M. Friedman
Department of Mathematics
Ohio State University
Columbus, Ohio 43210
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friedman@math.ohio-state.edu
www.math.ohio-state.edu/~friedman/

NOTE: This paper is in preparation, and the only proofs present are in Part II, A and B.

In fact, Godel gave an important model of pure predication, where he showed that restricted comprehension without parameters is valid, but where restricted comprehension with parameters is not (although this invalidity was not established until Cohen). This is the model based on ordinal definability in set theory.

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PART II. WITHOUT EXTENSIONALITY.

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INTRODUCTION

This paper was referred to in the Introductions to our papers [Fr97a], "The Axiomatization of Set Theory by Separation, Reducibility, and Comprehension," and [Fr97b], "The Interpretation of Set Theory in Mathematical Predication Theory."

In [Fr97a], all systems considered include the axiom of Extensionality and made unrestricted use of parameters. Extensionality and unrestricted use of parameters is appropriate in the context of mathematical predication.

In [Fr97b], all systems considered include unrestricted use of parameters, but not Extensionality. Unrestricted use of parameters is appropriate in the context of mathematical predication, and Extensionality is inappropriate.

Here we view the underlying concept as that of "pure predication." instead of set and instead of mathematical predication. This is the concept of predication that is used outside mathematics, where it is not appropriate to assume, e.g., that for every object x, there is a pure predicate that holds of exactly x. However, if x can be explictly defined, then such a pure predicate exists. First order definability without parameters approximates pure predication.

Godel proposed an important model of pure predication in the context of set theory which differs from that of mathematical predication. This is his concept of ordinal definable set. Parameterless separation is provably valid in the ordinal definable sets. However ordinary separation is not provably valid in the ordinal definable sets.

The most appropriate systems in this paper for pure predication are in Part II where we discuss systems without Extensionality and without parameters. For the sake of completeness and ease of exposition, we consider systems with Extensionality and without parameters in Part I.

As in [Fr97b] we will we will be exclusively concerned with interpretability. In each case, we opt for the simplest and cleanest set of axioms which allow for interpretability. The interpretations will at least preserve the arithmetic of natural numbers.

There is the legitimate concern of the strength of the set theories based on ZF with the higher of the large cardinal axioms that are interpreted here. Results of Hugh Woodin about forcing over universes with very large cardinals suggest that they are very strong - probably as strong as when the axiom of Choice is added. However at the time of this writing, this has not been resolved. In any case, it is known that these systems based on ZF are at least as strong as all large cardinals for which there exists a current inner model theory. One can of course merely add the axiom of Choice in its usual formulation to the axiomatizations considered here.

We close the paper with some conceptual interpretations of the axiomatizations proposed here. Although we regard this discussion as perhaps premature at this stage, it is intended to provide motivation for the axiomatizations, and the choice of terminology: reducibility and comprehension.

PART I. WITH EXTENSIONALITY

1. ONE SUBWORLD: ZF\P

In this section we treat the theory T2(W) in the language L(∈,W), as presented in the Introduction:

1. Extensionality. $(\forall x1)(x1 \in x2 \leftrightarrow x1 \in x3) \rightarrow (\forall x1)(x2 \in x1 \leftrightarrow x3 \in x1)$.
2. Comprehension. $(\exists x1)(\forall x2)(x2 \in x1 \leftrightarrow (x2 \in W \;\&\; \varphi))$, where φ is a formula in L(∈) with at most the free variable x2.
3. Reducibility. $(x1,...,xn \in W \;\&\; \varphi) \rightarrow (\exists xn+1 \in W)(\varphi)$, where $n >= 0$ and φ is a formula in L(∈) whose free variables are among x1,…,xn+1.

Here φ has at most the free variable y and does not mention W.

We define x = y if and only if x,y have the same elements. We let $x \in W$ abbreviate W(x).

LEMMA IA.1. The following are provable in T2(W).

i) $x1,\ldots,xn \in W \rightarrow (\varphi \leftrightarrow \varphi W)$, where $n \geq 0$ and φ is a formula in L(∈) whose free variables are among x1,…,xn;

ii) W is nonempty;

iii) $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$, where φ is a formula in L(∈,W) and x,y are not free in φ.

Proof: The first claim and third claims are by induction on φ. To see that W is nonempty, apply Reducibility to the sentence $\varphi = (\exists x1)(x1 \in x1 \rightarrow x1 \in x1)$. ∎

We write WFT(x) if and only if

i) x is transitive;

iii) for all y, if there is an element of x in y, then there is an ∈-minimal element of x in y.

Here WFT abbreviates "well founded and transitive."

We write $x \in= y$ for "$x \in y$ or $x = y$."

LEMMA IA.2. Let $\varphi(x,y1,\ldots,yn)$ be a formula with at most the free variables shown, and without W. The following is provable in T2(W). $(\mathrm{WFT}(x)\ \&\ x \in W\ \&\ \varphi W(x,y1,\ldots,yn)) \rightarrow (\exists z \in W)(z \in= x\ \&\ \varphi W(z,y1,\ldots,yn)\ \&\ (\forall w \in W)(w \in z \rightarrow \neg\varphi W(w,y1,\ldots,yn)))$.

Proof: Let $S = \{x \in W: (\exists y1\ldots yn)(\varphi W(y1,\ldots,yn)\ \&\ \neg(\exists z \in W)(z \in= x\ \&\ \varphi W(z,y1,\ldots,yn)\ \&\ (\forall w \in W)(w \in z \rightarrow \neg\varphi W(w,y1,\ldots,yn)))\}$. S exists by the third axiom.

It suffices to prove that S has no element x with WFT(x). We suppose that $x \in S$, WFT(x). Let x′ be an $\in$-minimal element of x in S.

Let $\varphi W(x',y1,\ldots,yn)$, $\neg(\exists z \in W)(z \in= x'\ \&\ \varphi W(z,y1,\ldots,yn)\ \&\ (\forall w \in W)(w \in z \rightarrow \neg\varphi(w,y1,\ldots,yn)))$. Then setting $z = x'$, we see that $\neg(\forall w \in W)(w \in x' \rightarrow \neg\varphi W(w,y1,\ldots,yn))$. Hence $(\exists w \in W)(w \in x'\ \&\ \varphi W(w,y1,\ldots,yn))$. Fix $u \in x'$ with $u \in W$, $\varphi W(u,y1,\ldots,yn)$. Note that $u \in x$ by the transitivity of x.

We claim that $u \in S$. To see this, suppose that $(\exists z \in W)(z \in= u\ \&\ \varphi W(z,y1,\ldots,yn)\ \&\ (\forall w \in W)(w \in z \rightarrow \neg\varphi W(w,y1,\ldots,yn)))$. Then $(\exists z \in W)(z \in= x\ \&\ \varphi W(z,y1,\ldots,yn)\ \&\ (\forall w \in W)(w \in z \rightarrow \neg\varphi W(w,y1,\ldots,yn)))$, which contradicts $x \in S$. Hence $\neg(\exists z \in W)(z \in= u\ \&\ \varphi W(z,y1,\ldots,yn)\ \&\ (\forall w \in W)(w \in z \rightarrow \neg\varphi W(w,y1,\ldots,yn)))$. Therefore $u \in S$.

We now have the desired contradiction since x′ is an $\in$-minimal element of x in S and $u \in x',S$.

LEMMA IA.3. Let $\varphi(x,y1,\ldots,yn)$ be a formula with at most the free variables shown, and without W. The following is provable in T2(W). $(\mathrm{WFT}(x)\ \&\ \varphi(x,y1,\ldots,yn)) \rightarrow (\exists z \in= x)(\varphi(z,y1,\ldots,yn)\ \&\ (\forall w \in z)(\neg\varphi(w,y1,\ldots,yn)))$.

Proof: By Lemma IA.1 ii, (∀x)(W(x) → (WFT(x) ↔ WFT^W(x))). By Lemma IA.2, (∀x,y1,…,yn ∈ W)(WFT^W(x) → φW(x,y1,…,yn) → (∃z ∈ W)(z ∈= x & φW(z,y1,…,yn) & (∀w ∈ W)(w ∈ z → ¬φW(w,y1,…,yn))). By Lemma IA.1 iii, (∀x,y1,…,yn)(WFT(x) → φ(x,y1,…,yn) → (∃z)(z ∈= x & φ(z,y1,…,yn) & (∀w)(w ∈ z → ¬φ(w,y1,…,yn))), as required.

LEMMA IA.4. Let φ(x,y1,…,yn) be a formula with at most the free variables shown. The following is provable in T2(W). (WFT(x) & φ(x,y1,…,yn)) → (∃z ∈= x)(φ(z,y1,…,yn) & (∀w ∈ z)(¬φ(w,y1,…,yn))).

Proof: Here we allow W to be mentioned in the property. But this is inconsequential since by Lemma IA.1 i, {x: W(x)} exists, and therefore can be used as a value for one of the free variables y1,…,yn.

LEMMA IA.5. The following is provable in T2(W). Let WTF(a). Then b ∈ c ∈ b ∈ a is impossible.

Proof: Suppose WFT(a) and b ∈ c ∈ b ∈ a. Apply Lemma IA.4 to the formula x = a or x = b or x = c. This formula holds of a and also an element of a (namely b). Hence there is an ∈-minimal element d of a for which it holds. This is a contradiction.

LEMMA IA.6. The following is provable in T2(W). (WFT(b) & c ∈ b & c is transitive) → WFT(c).

Proof: Let WFT(b) and c ∈ b. Let d be given, and assume that there is an element of c in d. Apply Lemma IA.4 to the formula x = b or (x ∈ c & x ∈ d). Clearly this formula holds for x = b. It also holds for some element of b. Hence there is an ∈-minimal element y of b for which it holds. Clearly y ≠ b. Hence y is an ∈-minimal element of c lying in d.

We define Ord(x) if and only if WTF(x) and x is ∈-connected. We say that x is an ordinal if and only if Ord(x).

LEMMA IA.7. The following is provable in T2(W). Every element of an ordinal is an ordinal. The ordinals are strictly linearly ordered under ∈. Transfinite induction can be applied to the ordinals with respect to any formula with any parameters. Every transitive set of ordinals is an ordinal.

Proof: Let x ∈ y where y is an ordinal. Obviously x is ∈-connected. To see that x is transitive, let a ∈ b ∈ x. Then a,b ∈ y, and so a ∈ x or x ∈ a or a = x. By Lemma IA.5, a ∈ x. And by Lemma IA.6, WFT(x). So x is an ordinal.

Assume the second claim is false. By Lemma IA.4, let x be an ∈-minimal ordinal which is not ∈-comparable with every ordinal. Thus every element of x is ∈-comparable with every ordinal. Let y be an ordinal. Then every element of x is ∈-comparable with y. Hence either every element of x is in y, or y ∈ x. So we can assume that x ⊆ y. If x = y then we are done. Otherwise, by Lemma IA.4, let z be an ∈-minimal element of y that is not in x. Then z ⊆ x. It suffices to prove that x ⊆ z. Let b ∈ x. Then b,z are comparable. So b ∈ z or z ∈ b or b = z. If z ∈= b then z ∈ x, which is a contradiction. Hence b ∈ z as required.

By Lemma IA.4, transfinite induction holds on the ordinals.

Let x be a transitive set of ordinals. Then x is ∈-connected. By transfinite induction on the ordinals, we see that WFT(x).

LEMMA IA.8. The following is provable in T2(W). There is a least ordinal not in W.

Proof: It suffices to prove that some ordinal is not in W. Suppose all ordinals lie in W. Since Ord(x) is equivalent to Ord(x)W for elements of W, we see that the set S of all ordinals exists. S is obviously a transitive set of ordinals. By Lemma IA.7, S is an ordinal. Hence S ∈ S, which contradicts Lemma IA.7.

We let OW be the least ordinal not in W.

LEMMA IA.9. The following are provable in T2(W). OW ⊆ W. OW is a limit ordinal; i.e., it is nonempty and has no greatest element.

Proof: The first claim is by the definition of OW. For the second claim, let x ∈ OW. Then x ∈ W. Now the assertion "x is an element of an ordinal" is true. Hence "x is an element of an ordinal"W is also true by Lemma IA.1. Hence let x ∈ y ∈ W and "y is an ordinal"W. By Lemma 1, x ∈ y ∈ W and y is an ordinal.

We still have to prove that OW is nonempty. Note that the empty set exists by the third axiom. It is obviously an ordinal. By applying Lemma IA.1 to "the empty set exists" we obtain "the empty set exists"W, and so the empty set lies in W.

LEMMA IA.10. The following is provable in T2(W). Let x be an ordinal. Then either x is empty, x is a limit ordinal, or there is a unique ordinal y such that x = {z: z ∈= y}. In the third case, x is called a successor ordinal, and y is called the predecessor of x.

Proof: Suppose x is an ordinal that is not empty and not a limit ordinal. Then x has a largest element, y. Let z ∈= y. Then obviously z ∈ x. On the other hand, let z ∈ x. Then z is comparable with y. So either z = y or z ∈ y. The case y ∈ z is impossIAle since y is the greatest element of x.

Also, if x = {z: z ∈= y'}, then y ∈= y' and y' ∈= y. By Lemma IA.7, y = y'.

A finite ordinal is an ordinal x which is not a limit ordinal, and where no element of x is a limit ordinal. The finite ordinals obviously form an ∈-connected transitive "class."

LEMMA IA.11. The following is provable in T2(W). Let x be an ordinal and y ∈ x. Then {z: z ∈= y} exists and is either x or a member of x. This is called the successor of y.

Proof: Let u be the least ordinal greater than y. This exists since y ∈ x. Now y ∈ u. Since u is transitive, y ⊆ u. Now suppose z ∈ u. Then z is not greater than y, and hence z ∈= y. We have thus shown that u = {z: z ∈= y}.

LEMMA IA.12. The following is provable in T2(W). Every finite ordinal lies in W. For every finite ordinal x, {y: y ∈= x} exists and is a finite ordinal. The set of all finite ordinals exists and is the first limit ordinal. It is a member and a subset of W.

Proof: Suppose not every finite ordinal lies in W. Let x be the least finite ordinal that does not lie in W. By Lemma IA.9, x is not empty. Hence x is a successor ordinal, and has a unique predecessor y. So y ∈ W. Now "{z: z ∈= y} exists" is a true statement. Hence "{z: z ∈= y} exists"W is a true statement. Let u in W be such that (∀z)(z ∈ u ↔ z ∈= y)W holds. Then (∀z ∈ W)(z ∈ u ↔ z ∈= y). But by the choice of x, (∀z)(z ∈= y → z ∈ W). Hence (∀z)(z ∈ u ↔ z ∈= y). Therefore u = x. Hence x ∈ W, which is the required contradiction.

Let x be a finite ordinal. By Lemma IA.9, x is not the largest ordinal in W. Hence by Lemma IA.11, {y: y ∈= x} exists. It is clearly transitive, ∈-connected, and well founded. It cannot be a limit ordinal since y is its greatest element. Furthermore, no elements can be limit ordinals since its elements are finite ordinals.

The set S of all finite ordinals exists since they are all in W. S is a transitive set of ordinals, and hence is itself an ordinal. By the previous paragraph, S has no greatest member, and so is a limit ordinal.

By the fourth claim of Lemma IA.1, S ∈ W. Clearly S ⊆ W since all finite ordinals lie in W.

We let ω be the set of all finite ordinals. For x ∈ ω we write x+ = {y: y ∈= x}.

The following two lemmas are painful enough that we are content to establish them in very special forms. This is all we need.

LEMMA IA.13. The following is provable in T2(W). For all x,y,z,w ∈ ω, {x,y,z,w} exists.

Proof: Suppose this is false. By four successive least element arguments, we obtain lexicographical least $x,y,z,w \in \omega$ such that $\{x,y,z,w\}$ does not exist. We have defined x,y,z,w without using W and without using any parameters. Since $x,y,z,w \in W$, we see that $\{x,y,z,w\}$ exists. This is the desired contradiction.

LEMMA IA.14. The following is provable in T2(W). For all distinct $x,y,z,w \in \omega$, $\{x,y,z,w\} \in W$.

Proof: By Lemma IA.13, $(\forall x,y,z,w \in \omega)(\exists u)(u = \{x,y,z,w\})$ holds. Hence by Lemma 1, $(\forall x,y,z,w \in \omega)(\exists u)(w = \{x,y,z,w\})W$ holds.

Let $x,y,z,w \in \omega$ be distinct. Then $x,y,z,w \in W$. Let $u \in W$ be such that $(u = \{x,y,z,w\})W$ holds. We can write $u = \{x,y,z,w\}$ as

$x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ u \subseteq \{x,y,z,w\}$,

where we don't care about the details of how $u \subseteq \{x,y,z,w\}$ is formalized.

Hence

$x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ (u \subseteq \{x,y,z,w\})W$.

Since x,y,z,w are distinct finite ordinals, we have $(x \neq y)W$ & $(y \neq z)W$ & $(z \neq w)W$. Hence

$(\exists x,y,z,w \in W)(x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ (x \neq y)W \ \& \ (y \neq z)W \ \& \ (z \neq w)W \ \& \ (u \subseteq \{x,y,z,w\})W)$.

By Lemma IA.1,

$(\exists x,y,z,w)(x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ x \neq y \ \& \ y \neq z \ \& \ z \neq w \ \& \ (u \subseteq \{x,y,z,w\}))$.

Hence z is a quadruple. But since $x \in u$ & $y \in u$ & $z \in u$ & $w \in u$, we have $u = \{x,y,z,w\}$.

Let $x,y,z \in \omega$. We now define the ordered triple $\langle x,y,z \rangle$ in the following special way. Enumerate the small number of possible order types of triples from ω.

For distinct x,y,z, define $\langle x,y,z \rangle = \{i,x+i+100,y+i+100,z+i+100\}$, where i is the index of the order type of x,y,z. If x,y,z are not distinct, then make the same definition, except make sure that the set has exactly four distinct terms by including $z+i+101$, etcetera.

LEMMA IA.15. The following is provable in T2(W). For all $x,y,z \in \omega$, $\langle x,y,z \rangle \in W$. For all x,y,z,a,b,c, $\langle x,y,z \rangle = \langle a,b,c \rangle \rightarrow (x = a \ \& \ y = b \ \& \ z = c)$.

Proof: To decode a triple, first look for the least element, i. This tells you the order type of the intended triple. In particular, it tells you how many of the next three terms to look at, and how to put them together in the right order.

LEMMA IA.16. Let $\varphi(x)$ be any formula. The following is provable in T2(W). $(\varphi(0) \ \& \ (\forall x \in \omega)(\varphi(x) \rightarrow \varphi(x+))) \rightarrow (\forall x \in \omega)(\varphi(x))$.

Proof: Suppose φ fails at some $x \in \omega$. Let x be the least such. Since x is a nonzero finite ordinal, write $x = y+$, where y is a finite ordinal. Then $\varphi(y)$. Hence $\varphi(x)$, which is the desired contradiction.

LEMMA IA.17. The following is provable in T2(W). There are functions S,+,x (successor, addition, and multiplication) on ω which obey the usual quantifier free axioms of Peano Arithmetic, and which are given by formulas without parameters and without W.

Proof: We handle + only. We construct + as a set of ordered triples from ω, defined without parameters and without W. Using the induction principle of Lemma IA.16, we see that for each $x \in \omega$, there can be at most one set of triples that defines + according to its recursion equations on exactly the pairs of arguments from x. Furthermore, the different functions (sets of triples) corresponding to the various $x \in \omega$ all agree on their common domains. We caution that it is not automatic, however, that if the appropriate set of triples exists for $x \in \omega$, then it exists for all $y \in x$.

Now if these appropriate sets of triples exist for every $x \in \omega$, then we can put them together into a single set of triples that defines the + we are looking for on all of ω. This is because we are taking the set of all triples from ω obeying a condition without parameters and without W, and that the triples from ω all lie in W.

So it remains to show that these sets of triples exist for every $x \in \omega$. Suppose this is false, and let $x \in \omega$ be the least counterexample. Then x is not zero, and we can let x = y+. Let S be the set of triples corresponding to y. Since S is unique, and since x is defined without parameters and without W, we see that S is defined without parameters and without W, as well as y. We now have to extend the set of triples to correspond to x = y+. This amounts to extending the set of triples so as to work with pairs of arguments, one of which is y and the other of which is $\in$= y. But this is no problem since we can refer to the previous set of triples corresponding to y at arguments involving the predecessor of y; if y is zero then the use of y in the triples is trivial.

This extension can be performed without parameters and without W, so that the set of triples exists. This is the desired contradiction.

LEMMA IA.18. The following is provable in EFA (exponential function arithmetic). There is an interpretation of the system pZ2 of parameterless second order arithmetic into T2(W).

Proof: We can interpret the natural numbers of pZ2 as elements of ω, and the S,+,x of pZ2 as the S,+,x on ω as given by Lemma IA.17. We can interpret the subsets of ω in pZ2 as the subsets of ω in T2(W). The interpretation of the induction axiom scheme in pZ2 is obtained from Lemma IA.7. The interpretation of the comprehension axiom scheme in pZ2 is obtained from the third axiom of T2(W).

Here pZ2 = parameterless second order arithmetic is as defined and discussed in my paper, On the necessary use of

abstract set theory, Advances in Mathematics, vol. 41, 1981, 209-280.

In that paper, we showed that pZ2 is equiconsistent with Z2 = ordinary second order arithmetic, which differs only in that the comprehension scheme is stated with all parameters. (Strictly speaking, the development there is for ω-models only, but it works for general models and equiconsistency).

From the proof in that paper, it follows that there is an interpretation of Z2 in pZ2. And it is well known that there is an interpretation of ZFC-P (ZFC without the powers set axiom) in Z2.

Let CST (countable set theory) be ZFC-P + "all sets are countable." It is also well known that CST is interpretable into ZFC-P.

THEOREM IA.19. The following is provable in EFA (exponential function arithmetic).

i) CST is interpretable into T2(W);

ii) T2(W) is consistent if and only if CST is consistent;

iii) every theorem of T2(W) without W is a theorem of ZF.

Proof: This is immediate from the preceding lemmas and discussion. For iii), let φ be a sentence without W that is provable in T2(W), and let it be provable in S, which is a finite fragment of T2(W). We can interpret S in ZF by setting W to be a transitive set which is an elementary substructure of the universe with respect to the formulas that appear in S. This construction is standard within ZF, but cannot be done if the power set is dropped.

We can identify a well known subsystem of ZF that is equiconsistent with ZFC-P which we can use in iii) above instead of ZF. We drop power set and replacement and use the reflection scheme

$$(\exists x)(x \text{ is nonempty and transitive } \& \ (\forall y1,\ldots,yk \in x)(\varphi^x \leftrightarrow \varphi)),$$

where φ is any formula whose free variables are among y1,…,yk. In the presence of the other axioms of ZF-P, replacement follows immediately from reflection.

THEOREM IA.20. Every theorem of T2(W) that does not mention W is a theorem of ZF-P with the reflection scheme.

B. TWO SUBWORLDS: INDESCRIBABLE AND SUBTLE CARDINALS

In this section IB, we treat the theory T2(W1,W2) in the language L(∈,W1,W2), as presented in the Introduction:

1. Extensionality. (∀x1)(x1 ∈ x2 ↔ x1 ∈ x3) → (∀x1)(x2 ∈ x1 ↔ x3 ∈ x1).
2. Comprehension. (∃x1)(W2(x1) & (∀x2)(x2 ∈ x1 ↔ (W1(x2) & φ))), where φ is a formula in L(∈,W1) with at most the free variable x2.
3. Resemblance. W(x1) → (φ ↔ φ[W1/W2]), where φ is a formula in L(∈,W1) with at most the free variable x1.

1. Transitivity

The first order of business is to interpret the more convenient system T3(W1,W2) in T2(W1,W2). The axioms of T3(W1,W2) extend those of T2(W1,W2) by the transitivity axioms

4. (W1(x1) & x2 ∈ x1) → W1(x2);
5. (W2(x1) & x2 ∈ x1) → W2(x2).

We define x = y if and only if x,y have the same elements. We let x ∈ W1 and x ∈ W2 abbreviate W1(x) and W2(x) respectively. In the next lemma, viii) guarantees that we can treat = just like ordinary equality.

LEMMA IB1.1. The following are provable in T2(W1,W2).

i) (∃x ∈ W2)(∀y)(y ∈ x ↔ y ∈ W1);

ii) (∃x)(∀y)(y ∈ x ↔ y ∈ W2);

iii) (∀x ∈ W1)(φ ↔ φ[W1/W2]), where φ has at most the free variable x, and does not mention W2;

iv) φ ↔ φ[W1/W2], where φ be a sentence not mentioning W2;

v) (∃y)(∀x)(x ∈ y ↔ (W2(x) & φ)), where φ has at most the free variable x, and does not mention W1;

vi) $(\exists x \in W2)(\varphi) \rightarrow (\exists x \in W1)(\varphi)$, where φ has at most the free variable x, and does not mention W1;

vii) $x \in W1 \rightarrow x \in W2$;

viii) $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$ where x,y are not free in φ.

Proof: The first claim is from the third axiom by setting φ to be x = x. The third claim is obvious from the third axiom. The fourth claim is a special case of the third claim. The second claim follows from the first claim and the fourth claim. The fifth claim follows from the fourth claim and the third axiom.

For the sixth claim, suppose $(\exists x \in W2)(\varphi)$. By the third claim, $(\exists x \in W1)(\varphi[W2/W1])$. By the third claim, $(\exists x \in W1)(\varphi[W2/W1][W1/W2])$, and hence $(\exists x \in W1)(\varphi)$. For the seventh claim, let $x \in W1$. By the third claim, $x \in W1 \leftrightarrow x \in W2$, and so $x \in W2$. For the eighth claim, note that this is standard for φ not mentioning W1,W2 (by induction on φ). But we have to handle the basis cases $y = z \rightarrow (y \in W1 \leftrightarrow z \in W1)$, and $y = z \rightarrow (y \in W2 \leftrightarrow z \in W2)$. First let x be as given in the first claim. Then $y = z \rightarrow (y \in x \leftrightarrow z \in x)$, and so $y = z \rightarrow (y \in W1 \leftrightarrow z \in W1)$. Finally let x be as given in the second claim. Then $y = z \rightarrow (y \in x \leftrightarrow z \in x)$, and so $y = z \rightarrow (y \in W2 \leftrightarrow z \in W2)$. ∎

We say that x is transitive if and only if $(\forall y,z)(y \in z \in x \rightarrow y \in x)$.

We are now ready to interpret the W1 and W2 of T3(W1,W2). We define W1′(x) if and only if $x \in W1$ & $(\exists y)$(y is transitive & $x \subseteq y \subseteq W1$). Similarly, we define W2′(x) if and only if $x \in W2$ & $(\exists y)$(y is transitive & $x \subseteq y \subseteq W2$).

Let φ be a formula in L(∈,W1,W2). Then the ′ interpretation of φ, which we write as φ', is the result of replacing W1 by W1′ and W2 by W2′, and expanding the resulting formula out with the definitions of W1′ and W2′.

LEMMA IB1.2. The following is provable in T2(W1,W2).
$(\exists x)(W1(x))'$.

Proof: This is (∃x)(W1′(x)). By axiom 3 the empty set ∅ ∈ W2. By Lemma IB1.1 vi, ∅ ∈ W1. Hence W1′(∅).

LEMMA IB1.3. The following is provable in T2(W1,W2). [(W1(x) & y ∈ x) → W1(y)]′. [(W2(x) & y ∈ x) → W2(y)]′.

Proof: The second claim follows from the first claim by Lemma IB1.1 iv. The first claim is (W1′(x) & y ∈ x) → W1′(y). Let x ∈ W1, x ⊆ z ⊆ W1, and z is transitive. Let y ∈ x. Then y ∈ z, and so y ∈ W1. Also every element of y lies in z by the transitivity of z. Hence y ⊆ z ⊆ W1. ∎

LEMMA IB1.4. Let φ be a formula in L(∈,W1,W2) with at most the free variable y which does not mention W2. The following is provable in T2(W1,W2). (∃x)(W2(x) & (∀y)(y ∈ x ↔ (W1(y) & φ) ↔ (W1(y) & φ[W1/W2])))′.

Proof: This is (∃x)(W2′(x) & (∀y)(y ∈ x ↔ (W1′(y) & φ′) ↔ (W1′(y) & φ′[W1/W2]))). The inside equivalence follows from axiom 3. It now suffices to prove that {y: W1′(y) & φ′} exists and has property W2′. It exists and lies in W2 by axiom 3. Now {y: W2′(y)} exists by Lemma IB1.1 v, and is transitive by Lemma IB1.3, and is obviously a subset of W2. Hence {y: W1′(y) & φ′} has property W2′. ∎

LEMMA IB1.5. Let φ be a theorem of T3(W1,W2). Then φ′ is a theorem of T2(W1,W2).

Proof: We have shown this for all axioms φ of T3(W1,W2). The result follows. ∎

In light of Lemma IB1.5, we work entirely within the system T3(W1,W2). We continue to use the abbreviations x = y, x ∈ W1, and x ∈ W2. We now know that W1,W2 are transitive sets, and W1 ∈ W2.

2. Ordinals and transfinite induction

We write WFT(x) if and only if

i) x is transitive;
iii) for all y, if there is an element of x in y, then there is an ∈-minimal element of x in y.

Here WFT abbreviates "well founded and transitive."

We write x ∈= y for "x ∈ y or x = y."

LEMMA IB2.1. Let φ(x,y1,…,yn) be a formula with at most the free variables shown, and without W1,W2. The following is provable in T3(W1,W2). (WFT(x) & x ∈ W1 & φ(x,y1,…,yn)) → (∃z ∈ W1)(z ∈= x & φ(z,y1,…,yn) & (∀w ∈ W1)(w ∈ z → ¬φ(w,y1,…,yn))).

Proof: Let S = {x ∈ W1: (∃y1…yn)(φ(y1,…,yn) & ¬(∃z ∈ W1)(z ∈= x & φ(z,y1,…,yn) & (∀w ∈ W1)(w ∈ z → ¬φ(w,y1,…,yn)))}. S exists by the third axiom.

It suffices to prove that S has no element x with WFT(x). We suppose that x ∈ S, WFT(x). Let x′ be an ∈-minimal element of x in S.

Let φ(x′,y1,…,yn), ¬(∃z ∈ W1)(z ∈= x′ & φ(z,y1,…,yn) & (∀w ∈ W1)(w ∈ z → ¬φ(w,y1,…,yn))). Then setting z = x′, we see that ¬(∀w ∈ W)(w ∈ x′ → ¬φ(w,y1,…,yn)). Hence (∃w ∈ W1)(w ∈ x′ & φ(w,y1,…,yn)). Fix u ∈ x′ with u ∈ W1, φ(u,y1,…,yn). Note that u ∈ x by the transitivity of x.

We claim that u ∈ S. To see this, suppose that (∃z ∈ W1)(z ∈= u & φ(z,y1,…,yn) & (∀w ∈ W1)(w ∈ z → ¬φ(w,y1,…,yn))). Then (∃z ∈ W1)(z ∈= x & φ(z,y1,…,yn) & (∀w ∈ W1)(w ∈ z → ¬φ(w,y1,…,yn))), which contradicts x ∈ S. Hence ¬(∃z ∈ W1)(z ∈= u & φ(z,y1,…,yn) & (∀w ∈ W)(w ∈ z → ¬φ(w,y1,…,yn))). Therefore u ∈ S.

We now have the desired contradiction since x′ is an ∈-minimal element of x in S and u ∈ x′,S. ∎

LEMMA IB2.2. Let φ(x,y1,…,yn) be a formula with at most the free variables shown, and without W1,W2. The following is provable in T3(W1,W2). (WFT(x) & x ∈ W2 & φ(x,y1,…,yn)) →

(∃z ∈ W2)(z ∈= x & φ(z,y1,…,yn) & (∀w ∈ W2)(w ∈ z → ¬φ(w,y1,…,yn))).

Proof: By Lemmas IB2.1 and IB1.1 iv. ∎

LEMMA IB2.3. Let φ(x,y1,…,yn) be a formula with at most the free variables shown. The following is provable in T3(W1,W2). (WFT(x) & x ∈ W2 & φ(x,y1,…,yn)) → (∃z ∈ W2)(z ∈= x & φ(z,y1,…,yn) & (∀w ∈ W2)(w ∈ z → ¬φ(w,y1,…,yn))).

Proof: By Lemmas IB2.2 and IB1.1 i,ii. W1,W2 can then be replaced by parameters. ∎

LEMMA IB2.4. The following is provable in T3(W1,W2). Let WTF(a), a ∈ W2. Then b ∈ c ∈ b ∈ a is impossible.

Proof: Suppose WFT(a) and b ∈ c ∈ b ∈ a. Apply Lemma IB2.3 to the formula x = a or x = b or x = c. This formula holds of a and also an element of a (namely b). Hence there is an ∈-minimal element d lying in W2 of a for which it holds. This is a contradiction. ∎

LEMMA IB2.5. The following is provable in T3(W1,W2). (WFT(b) & c ∈ b & c is transitive & b ∈ W2) → WFT(c).

Proof: Let WFT(b), c ∈ b, c is transitive, and b ∈ W2. Let d be given, and assume that there is an element of c in d. Apply Lemma IB2.3 to the formula x = b or (x ∈ c & x ∈ d). Clearly this formula holds for x = b. It also holds for some element of b lying in W2 (namely c). Hence there is an ∈-minimal element y of b for which it holds (every element of c lies in W2). Clearly y ≠ b. Hence y is an ∈-minimal element of c lying in d. Hence WFT(c). ∎

We define Ord(x) if and only if WTF(x) and x is ∈-connected. We say that x is an ordinal if and only if Ord(x).

LEMMA IB2.6. The following are provable in T3(W1,W2). Every element of an ordinal ∈ W2 is an ordinal ∈ W2. The ordinals ∈ W2 are strictly linearly ordered under ∈. Transfinite induction can be applied to the ordinals ∈ W2 with respect to any formula with any parameters. If x ⊆ W2 is a transitive set of ordinals, then x is an ordinal and x ∈= OW2.

Proof: Let x ∈ y where y is an ordinal ∈ W2. Obviously x is ∈-connected and in W2. To see that x is transitive, let a ∈ b ∈ x. Then a,b,y ∈ W2. Also a,b ∈ y, and so a ∈ x or x ∈ a or a = x. By Lemma IB2.4, a ∈ x. Since x is transitive, by Lemma IB2.5, WFT(x). So x is an ordinal ∈ W2.

For the second claim we have only to prove that for any two ordinals ∈ W2, the first is in the second, the second is in the first, or they are equal. Assume this is false. By Lemma IB2.3, let x be an ∈-minimal element of {x: x is an ordinal ∈ W2 which is not ∈-comparable with every ordinal ∈ W2}. Thus every element of x is ∈-comparable with every ordinal ∈ W2. Let y be an ordinal ∈ W2. Then every element of x is ∈-comparable with y. Hence either every element of x is in y, or y ∈ x. So we can assume that x ⊆ y. If x = y then we are done. Otherwise, by Lemma IB2.3, let z be an ∈-minimal element of y that is not in x. Then z ⊆ x. It suffices to prove that x ⊆ z. Let b ∈ x. Then b,z are comparable. So b ∈ z or z ∈ b or b = z. If z ∈= b then z ∈ x, which is a contradiction. Hence b ∈ z as required.

By Lemma IB2.3, transfinite induction holds on the ordinals ∈ W2, thus establishing the third claim.

For the fourth claim, let x be as given. Then x is ∈-connected. By transfinite induction on the ordinals, we see that WFT(x). Hence x is an ordinal. ∎

LEMMA IB2.7. Let φ(x1,…,xn,y) be a formula in L(∈,W1,W2) without W1 with at most the free variables shown. The following is provable in T3(W1,W2). (x1,…,xn ∈ OW2) → (∃z)(∀y)(y ∈ z ↔ (y ∈ W2 & φ)).

Proof: Let φ be as given and suppose the first claim is false. By Lemma IB2.6, we can successively choose least x1,…,xn ∈ OW2 such that the consequent is false. I.e., choose least x1 ∈ W2 such that for some x2,…,xn ∈ W2, the antecedent is false; then choose least x2 in W2 such that for some x3,…,xn ∈ W2, the antecedent is false; etcetera. This is n applications of Lemma IB2.6. Then x1,…,xn are defined without W1. Hence the antecedent is true after all by Lemma IB1.1 v. □

We now let OW1 be the set of all ordinals ∈ W1 and OW2 be the set of all ordinals ∈ W2.

LEMMA IB2.8. The following is provable in T3(W1,W2). OW1, OW2 are ordinals. OW1 ∈ W2. OW1 ∉ W1. OW2 ∉ W2. OW1 ∈ OW2.

Proof: By Lemma IB2.6, OW1 is a transitive sets of ordinals, each of which is ∈ W1. Hence by Lemma IB2.6, OW1 is an ordinal. Similarly, OW2 is an ordinal. OW1 ∈ W2 by axiom 3. If OW1 ∈ W1 then OW1 ∈ OW1, contradicting Lemma IB2.4. Similarly, OW2 ∉ W2. By axiom 3, OW1 ∈ OW2 since OW1 ⊆ W1 and is defined without W2. The forward direction of the last claim follows immediately from the axiom 3, and the backward direction is trivial. □

LEMMA IB2.9. The following are provable in T3(W1,W2). OW1 and OW2 are limit ordinals; i.e., they are nonempty and have no greatest element.

Proof: By axiom 3, ∅ exists, ∅ ∈ W2, and ∅ is an ordinal. By Lemma IB1.1 iv, ∅ ∈ W1. Hence ∅ ∈ OW1. Now let x ∈ OW1. Then "x is an element of an element of OW2." By Lemma 2.1, since x ∈ W1, we see that "x is an element of an element of OW1." Therefore OW1 is a limit ordinal. Hence by Lemma 2.1, OW2 is a limit ordinal. □

LEMMA IB2.10. The following is provable in T3(W1,W2). Let x be an ordinal in OW2. Then either x is empty, x is a limit ordinal, or there is a unique ordinal y such that x = {z: z ∈= y}. In the third case, x is called a successor ordinal, and y is called the predecessor of x.

Proof: Suppose x ∈ OW2 is not empty and not a limit ordinal. Then x has a largest element, y. Let z ∈= y. Then obviously z ∈ x. On the other hand, let z ∈ x. Then z is comparable with y. So either z = y or z ∈ y. The case y ∈ z is impossible since y is the greatest element of x.

Also, if x = {z: z ∈= y'}, then y ∈= y' and y' ∈= y. By Lemma IB2.6, y = y'. ∎

LEMMA IB2.11. The following is provable in T3(W1,W2). Let x ∈ OW2. Then {y: y ∈= x} exists, and lies in OW2. This is called the successor of y, written y+.

Proof: By Lemma IB2.6, let u be the least element of OW2 greater than x. Then every element of u is ∈= x. Let y ∈= x. Then by Lemma 2.7, y and u are comparable. Hence y ∈ u or u ∈ y or u = y. Since x ∈ u we see by Lemma IB2.5 that y ∈ u. ∎

3. Finite ordinals, strong ordinals, and arithmetic

A finite ordinal is an ordinal x which is not a limit ordinal, and where no element of x is a limit ordinal.

LEMMA IB3.1. The following is provable in T3(W1,W2). The set of all finite ordinals in OW1 is a limit ordinal. Every finite ordinal in OW2 lies in OW1. For every finite ordinal x ∈ OW1, {y: y ∈= x} exists and is a finite ordinal. The set of all finite ordinals in OW1 exists, and is the same as the set of all finite ordinals in OW2, and is the first limit ordinal in OW1.

Proof: By axiom 3, let A be the set of all finite ordinals in OW1. Then A is transitive, and hence by Lemma IB2.6, A is an ordinal. Obviously A ∈ W2. Hence A ∈ OW2. Now suppose A has a largest element, x. Then by Lemma IB2.11, x+ exists and x+ ∈ OW2. Also x+ is obviously a finite ordinal. Hence x+ ∉ OW1. Therefore x is the largest element of OW1, which contradicts Lemma IB1.9. Therefore A is a limit ordinal.

Let x be a finite ordinal in OW2. Then x is ∈-comparable with A. A ∈= x is impossible by the definition of finite ordinal. Hence x ∈ A.

Let x ∈ A. By Lemma IB2.11, x+ exists and lies in OW2. Since x is a finite ordinal, clearly x+ is a finite ordinal.

We have shown that A is the first limit ordinal in OW1, and is the set of all finite ordinals in OW2. Suppose there is an element of A that is a limit ordinal. Then there is an element of A that is higher than a limit ordinal. But this contradicts that every element of A is a finite ordinal. ∎

We let ω be the set of all finite ordinals in OW1 (or OW2). It is the first limit ordinal in OW1.

We say that x is a strong ordinal if and only if x is an ordinal and for all b, if there is an element of x not in b then there is an ∈-minimal element of x not in b.

LEMMA IB3.2. The following is provable in T3(W1,W2). Let x be a strong ordinal. Then x ∈ OW2 or OW2 ∈ x or x = OW2. OW2 and all of its elements are strong ordinals. ω is the set of all finite strong ordinals (i.e., strong ordinals that are finite ordinals).

Proof: First suppose that x ⊆ OW2. We can assume that x ≠ OW2. Let y be the least element of OW2 not in x. Then y ⊆ x. Now y is ∈-comparable with every element of x. Hence x ⊆ y or y ∈ x. Hence x ⊆ y, and so x = y, in which case x ∈ OW2.

Now suppose that there is an element of x not in OW2. Let y be an ∈-minimal element of x not in OW2. By the transitivity of x, y ⊆ OW2. To see that OW2 ⊆ y, let b ∈ OW2. Then b ∈ x, and so b,y are ∈-comparable. Hence b ∈ y. So we have shown that y = OW2. Hence OW2 ∈ x.

The second claim follows from Lemma IB2.6.

For the third claim, it suffices to show that every finite strong ordinal x lies in OW2. But x is comparable with OW2. Since OW1 is a limit ordinal in x, we see that x ∈ OW2. ∎

We are not claiming the ∈-comparability of strong ordinals. Just their comparability with OW2.

We now define {x1,…,xn} = {y: y = x1 or … y = xn}. This definition is meant to be made for each standard integer n. Of course, we cannot prove in T2(W1,W2) that, e.g., for all x, {x} exists; or even for all x ∈ W1, {x} exists. So we must tread very carefully.

The {}-terms are inductively defined in the metatheory as follows:

i) each variable x1,x2,… is a {}-term;

ii) if t1,…,tm are {}-terms, n ≥ 0, then {t1,…,tm} is a {}-term.

We do not claim that t(x1,…,xn) exists for all x1,…,xn. We have to be very careful about existence.

LEMMA IB3.3. Let t(x1,…,xn) be a {}-term whose variables are among those shown. The following are provable in T3(W1,W2). (∀x1,…,xn ∈ OW2)(t(x1,…,xn) ∈ W2). (∀x1,…,xn ∈ OW1)(t(x1,…,xn) ∈ W1).

Proof: The second claim follows from the first claim by Lemma IB1.1 iv. We prove the first claim by induction on t. Assume (∀x1,…,xn ∈ OW2)(ti(x1,…,xn) ∈ W2), for 1 ≤ i ≤ m. We must show that (∀x1,…,xn ∈ OW2)({t1(x1,…,xn),…,tm(x1,…,xn) ∈ W2).

Suppose this is false. By n successive applications of transfinite induction, let x1,…,xn be lexicographically least elements of OW2 such that ¬({t1(x1,…,xn),…,tm(x1,…,xn)} ∈ W2). Then x1,…,xn are each definable by formulas not mentioning W1. By Lemma 2.1 vi we see that x1,…,xn ∈ W1. Now by induction hypothesis, each ti(x1,…,xn) ∈ W2. Hence again by Lemma IB1.1 vi, each ti(x1,…,xn) ∈ W1. Hence by axiom 3, {t1(x1,…,xn),…,tm(x1,…,xn)} ∈ W2, which is the required contradiction. □

We now define <x,y> = {{x},{x,y}}. Of course <x,y> may not exist. Define <x> = x. For n ≥ 2, inductively define <x1,…,xn> = <x1,<x2,…,xn>>. Again, we have to be careful about existence.

LEMMA IB3.4. Let n ≥ 1. The following is provable in T3(W1,W2). (∀x1,…,xn ∈ OW2)(<x1,…,xn> ∈ W2). (∀x1,…,xn ∈ OW1)(<x1,…,xn> ∈ W1).

Proof: This is a special case of Lemma IB3.3. ∎

For sets x and n ≥ 2 we write xn for {<y1,…,yn>: y1,…,yn ∈ x}. This may not necessarily exist. We take x1 = x.

We also write x≤n for the union of the xm, 1 ≤ m ≤ n. Again, this may not necessarily exist.

LEMMA IB3.5. Let n ≥ 2. The following is provable in T3(W1,W2). (∀x ∈ OW2)(xn,x≤n ∈ W2). (∀x ∈ OW1)(xn,x≤n ∈ W1). $OW2^n$ and OW≤n exist.

Proof: Suppose that the first claim is false and let x be the least counterexample. By Lemma IB1.1 vi we see that x ∈ OW1. By Lemma IB3.3, we can define xn as a subset of W1 without W1. Hence by axiom 3, xn ∈ W2, which is the required contradiction. The second claim follows. And OW2n exists by Lemma IB3.3 and Lemma IB1.1 v. The ≤n exponent versions are proved analogously. ∎

We say that f is a function if and only if f is a univalent set of ordered pairs. We write f:x → y if f is a function whose domain is x and all of whose values lie in y. When we write f:x → y we are committed to the existence of x,y as objects.

We now want to give some variants of Lemma IB2.7.

LEMMA IB3.6. Let φ(x1,…,xn,y) be a formula in L(∈,W1,W2) without W1 with at most the free variables shown, and m ≥ 1. The following is provable in T3(W1,W2). (x1,…,xn,z ∈ OW2) → (∃w ∈ W2)(∀y)(y ∈ w ↔ (y ∈ z≤m & φ)).

Proof: As in the proof of Lemma IB2.7, successively minimize x1,…,xn,z ∈ OW2 so that the conclusion is false. Then x1,…,xn,z ∈ W1, and so x1,…,xn,z ∈ OW1. Hence z≤m ∈ W1, and so we can define w = {y ∈ z≤m: φ} as a subset of W1 without mentioning W1. Hence w ∈ W2, which is the required contradiction. ∎

LEMMA IB3.7. Let φ(x1,…,xn,y) be a formula in L(∈,W1,W2) without W1 with at most the free variables shown. The following are provable in T3(W1,W2).

i) (x1,…,xn,z ∈ OW1) → (∃w ∈ W1)(∀y)(y ∈ w ↔ (y ∈ z≤m & φ[W2/W1]));

ii) (x1,…,xn,y ∈ OW1) → (φ ↔ φ[W2/W1]);

iii) (x1,…,xn,z ∈ OW1) → (∃w ∈ W1)(∀y)(y ∈ w ↔ (y ∈ z≤m & φ));

Proof: The first claim is immediate from Lemma IB3.6 and Lemma IB1.1 iv. For the second claim, write ψ(x) = (∃x1…xnz)(x = {x1,…,xn,z} & φ). By Lemma IB1.1 iii, x ∈ W1 → (ψ ↔ ψ[W2/W1). The claim follows from x1,…,xn,z ∈ OW1 → {x1,…,xn,z} ∈ W1, by Lemma IB3.5. The third claim follows immediately from the first two claims. ∎

We now develop some arithmetic of ω.

LEMMA IB3.8. The following is provable in T3(W1,W2). The function S:ω → ω given by S(x) = x+ exists. S is definable without W1,W2. S ∈ W1.

Proof: S = {<x,y>: x,y ∈ ω & y = x+} exists and lies in W2 by axiom 3, because it is defined without W1,W2 as a subset of W1. By Lemma IB1.1 vi, S ∈ W1. ∎

LEMMA IB3.9. The following is provable in T3(W1,W2). There is a unique function f:ω2 → ω such that f(<x,0>) = x and f(<x,y+>) = f(<x,y>)+. This f is definable without W1,W2, and f ∈ W1.

Proof: By transfinite induction, we see that there is at most one such f. We now have to prove existence. We claim that for every x ∈ ω, there exists a unique g:ω → ω such that for all y ∈ ω, g(0) = x and g(y+) = g(y)+. Uniqueness follows from existence. Suppose this is false, and least x be the least counterexample. Then x ≠ 0, because we can explicitly define the desired function without W1,W2 as a subset of W1 in case x = 0. So let x = y+ and g be for y. By induction, g is the unique such function for y. Hence g is defined without W1,W2 since y is. Hence we can define the function h:ω → ω without

W1,W2 as a subset of W1 by h(z) = g(z)+. Then h(0) = y+ = x and h(z+) = g(z+)+ = g(z)++ = h(z)+. Then h corresponds to x. This is the required contradiction.

Finally, we define f(<x,y>) = g(y) where g corresponds to x.

For the second claim, note that f it is defined as a subset of W1, and hence f ∈ W2. By Lemma IB1.1 vi, f ∈ W1. □

We write +:ω2 → ω for the unique function provided by Lemma IB3.9. We often write x+y, which is defined only for x,y ∈ ω.

LEMMA IB3.10. The following is provable in T3(W1,W2). There is a unique function f:ω2 → ω such that f(<x,0>) = 0 and f(<x,y+>) = f(<x,y>)+x. This function is definable without W1,W2, and lies in W1.

Proof: Similar to the proof of Lemma IB3.9. We prove that for each x ∈ ω there is a unique function g:ω → ω such that for all y ∈ ω, g(0) = 0 and g(y+) = g(y)+x. Again we have the least counterexample x which is defined without W1,W2. But this time we don't consider the predecessor of x. Instead, we build up g for x by finite approximations. The details are left to the reader. □

We write x:ω2 → ω for the unique function provided by Lemma IB3.10.

LEMMA IB3.11. The following is provable in T3(W1,W2). ω under S,+,x forms a model of the Peano arithmetic axioms in which induction may be applied with any formula of T3(W1,W2).

Proof: Immediate by transfinite induction. □

4. Finite sequence codes

Let f:x → y. We write f|z for f restricted to z; i.e., for {<u,v> ∈ f: u ∈ z}. When we write f|z we are committed to the existence of z as an object. Of course, we have to worry about the existence of f|z.

We define the relation <* between pairs of objects <a1,a2> and <b1,b2>. We say <a1,a2> <* <b1,b2> if and only if

i) both a's are elements of both b's; or

ii) some a equals some b, and a1 ∈ b1; or

iii) a1 = b1 and a2 ∈ b2.

We define <*(<a,b>) = {<c,d>: <c,d> <* <a,b>}. We have to worry about the existence of this set.

We say that f is an <a,b> pairing function if and only if there exists a strong ordinal y such that

i) <*(<a,b>) exists;

ii) f:<*(<a,b>) → y;

iii) the range of f is y;

iv) for all x ∈ <*(<a,b>), f(x) = the ∈-least element z of y which is not equaled to any f(z), z <* x.

LEMMA IB4.1. The following is provable in T3(W1,W2). Let a,b ∈ OW2. There is a unique <a,b> pairing function. Furthermore, this function is an element of W2, and its range is an element of OW2.

Proof: Suppose that there exists a,b ∈ OW2 such that the two claims are false. By transfinite induction, let <a,b> be <*-least such that the two claims are false. By Lemma IB1.1 vi and Lemma IB2.8, a,b ∈ OW1 and <a,b> ∈ W1 and <*(<a,b>) ∈ W1. For each y ∈ <*(<a,b>), let fy be the unique <a,b> pairing function, and let y* be the range of fy. Each fy ∈ W2 and each y* ∈ OW2. Hence each fy ⊆ W2.

Now each fy and y* is defined from y without mentioning W1 and without mentioning W2. So we can apply axiom 3 to the true statement "y*,fy ∈⊆ W2." Hence "y*,fy ∈⊆ W1." Using transfinite induction, we see that the fy's are comparable under ⊆.

Now suppose that <a,b> has no <* predecessor. Then we can define the union of the fy, y ∈ <*(<a,b>), as a subset of W1 without mentioning W1. By the previous paragraph, this union will be a function with domain <*(<a,b>), which must exist by Lemma IB2.7. The range will be the union of the y*, y ∈ x, which is defined as a subset of W1 without mentioning W1. Hence it exists and lies in W2. Also the range is a transitive subset of OW1. Hence by transfinite induction, it is an ordinal in W2. So it lies in OW2. So the union function is an <a,b> pairing function lying in W2 whose range is an element of OW2. Also by transfinite induction, the union function is the unique <a,b> pairing function. (Here we also use Lemma IB3.3). But this contradicts the choice of a,b.

Clearly <a,b> ≠ <0,0>. Finally suppose that <a,b> is a <* successor. Let <c,d> be the greatest element of <*(<a,b>). Let f be the unique <c,d> pairing function. Also let f:<*(<c,d>) → y* be onto, where y* ∈ W2 and f ∈ W1. Since y* is an element of W2 defined without W1, we have y* ∈ OW1.

We can obviously extend f to f′ by taking f′(<c,d>) = y*, which is defined as a subset of W1 without mentioning W1. Hence it lies in W2. Clearly f′ is a <a,b> pairing function with range y*+. Clearly y*+ ∈ W2 since OW2 is a limit ordinal. By transfinite induction, there is a unique <a,b> pairing function. Again this contradicts the choice of a,b, and completes the proof. □

We say that f is a pairing function on x if and only if there exists y such that

i) x,y are strong ordinals;
ii) x2 exists;
iii) f:x2 → y;
iv) the range of f is y;
v) for all z ∈ x2, f(z) = the ∈-least element z of y which is not equaled to any f(w), w <* x.

LEMMA IB4.2. The following is provable in T3(W1,W2). Let x ∈ OW2. There is a unique pairing function on x. Furthermore, this function is an element of W2, and its range is an element of OW2. Let x ∈ OW1. There is a unique pairing function on x. Furthermore, this function is an element of W1, and its range is an element of OW1.

Proof: Let x ∈ OW2. Apply Lemma IB4.1 with a = 0 and b = x. Note that <*(<0,x>) = x2.

The part about OW1 follows by Lemma 2.1 iv. □

LEMMA IB4.3. The following is provable in T3(W1,W2). There is a unique pairing function f on OW2. The range of f is OW2. For all x,y ∈ OW1, f(<x,y>) ∈ OW1. There is a unique pairing function f on OW1. It lies in W2, and its range is OW1.

Proof: For the first claim, let fx be the unique pairing function on x ∈ OW2. By transfinite induction, the fx are comparable under ⊆. Also by transfinite induction, we see that for all x ∈ OW2, x ⊆ rng(fx). We can define the union of the fx, x ∈ OW2, as a subset of W2 without mentioning W1. Hence the union, f, exists. It is clear that f is a pairing function on OW2 whose range is OW2. Let x,y ∈ OW1. Let z = max(x,y)+. Then z ∈ OW1, and so by Lemma IB3.7, fz ∈ W1 has its range in OW1. Since fz ⊆ f, we see that f(<x,y>) ∈ OW1.

There is a unique pairing function f on OW1 because OW1 ∈ OW2. By Lemma IB4.2, it also lies in W2. Rng(f) is an element of OW2 that is included in W1. Since OW1 is a limit ordinal, we see that OW1 ⊆ rng(f). Hence rng(f) = OW1. □

We now fix P to be the unique pairing function P:OW22 → OW2. P has range OW2.

We say that x is P-closed if and only if x ∈= OW2 and for all y,z ∈ x, P(<y,z>) ∈ x.

LEMMA IB4.4. The following is provable in T3(W1,W2). (∀x,y ∈ OW2)(x,y ⊆ P(<x,y>)). Every P-closed x is 0,1, or a limit ordinal. 0,1,ω are the first three P-closed ordinals. (∀x ∈= OW2)(x is P-closed if and only if there is a pairing function on x whose range is x).

Proof: The first claim is proved by transfinite induction. For the second claim, let 1 ∈ x = y+ ∈ OW2. Then x ⊆ P(<0,y>). Hence x ∈ P(<1,y>), and so x is not P-closed.

For the third claim, clearly 0,1 are P-closed. To see that ω is P-closed, recall that ω is the first limit ordinal in OW2, lies in OW1, and also the set of all finite ordinals. By transfinite induction, choose x,y to be $<^*$-least elements of OW2 such that $P(<x,y>) = \omega$. If $x,y \in \omega$ then $<x,y>$ has an immediate predecessor in $<^*$, which contradicts $P(<x,y>) = \omega$. Hence either x or y is an infinite ordinal. Therefore for all finite ordinals u,v, we see that $<u,v> <^* <x,y$, and hence $P(<u,v>) \in \omega$. So ω is P-closed.

For the fourth claim, let $x \in= OW2$ be P-closed. Let f be the unique pairing function on x, and let y be its range. Then $f \subseteq P$. By the first claim, $x \subseteq y$. By P-closure, $y \subseteq x$, and so $x = y$ as required. On the other hand, let $x \in= OW2$ be such that the unique pairing function f on x has range x. Since $f \subseteq P$, we see that x is P-closed. □

LEMMA IB4.5. The following is provable in T3(W1,W2). OW1,OW2 are P-closed. $(\forall x \in OW2)(\exists y \in OW2)(x \in y$ & y is P-closed). $(\forall x \in OW1)(\exists y \in OW1)(x \in y$ & y is P-closed).

Proof: OW1 is P-closed by Lemma IB4.3. OW2 is obviously P-closed. Suppose the second claim is false, and let x be the least counterexample in OW2. Then x is an element of W2 that is definable without mentioning W1. Hence $x \in OW1$. But OW1 is P-closed. This is the required contradiction.

By Lemma IB4.4, we can rewrite $(\forall x \in OW2)(\exists y \in OW2)(x \in y$ & y is P-closed) in the form $(\forall x \in OW2)(\exists y \in OW2)(\exists f)(x \in y$ & f is a pairing function on y with range y). Then $(\forall x \in OW1)(\exists y \in OW1)(\exists f)(x \in y$ & f is a pairing function on y with range y). The third claim follows by Lemma IB4.3. □

Let $g:y \to y$ be a pairing function on y with range y. We say that $z \in y$ is a strong point of g if and only if $(\forall u,v \in z)(g(<u,v>) \in z)$.

We say that f,g is a pairclosure system on x if and only if there exists y such that

i) $g:y \to y$ is a pairing function on y;
ii) x is a strong ordinal;
iii) $f:x \to y$;

iv) for all z ∈ x, f(z) = the ∈-least strong point of g which is not equaled to any f(z), z ∈ x;

v) every strong point of g is a value of f.

LEMMA IB4.6. The following are provable in T3(W1,W2).

i) let x ∈ OW2. There is a unique pairclosure system f,g on x. Furthermore, f,g,dom(g),rng(f) ∈ W2;

ii) let x ∈ OW1. There is a unique pairclosure system f,g on x. Furthermore, f,g,dom(g),rng(f) ∈ W1;

Proof: For the first claim, note that uniqueness is clear by transfinite induction. Suppose that there exists x ∈ OW2 such that ¬(there is a pairclosure system f,g on x with f,g,dom(g),rng(f) ∈ W2). By transfinite induction, let x be least with this property. By Lemma 2.1 vi, x ∈ OW1. For each y ∈ x, let fy,gy be the unique pairclosure system on y, and let y* be the domain of gy. Each fy,gy,rng(f) ∈ W2 and each y* ∈ OW2. Then each gy is simply the unique pairing function on y*; it has range y*.

Now each fy,rng(f),y* is an element of W2 defined from y without mentioning W1 and without mentioning W2. So we can apply axiom 3 to the true statement "fy,rng(f),y* ∈ W2." Hence "fy,rng(f),y* ∈ W1." Using transfinite induction, we see that the fy's are comparable under ⊆.

Now suppose that x is a limit ordinal. Then we can define the union of the fy, y ∈ x, as a subset of W1 without mentioning W1. We can also define the union of the y* as a subset of W1 without mentioning W1. These result in a function f* with domain x and an element x* of OW2. Let g* be the unique pairing function on x*. Then g* ∈ W2. We also see that f*,g*,dom(f*),rng(f*) ∈ W2. Note that f*,g* is a pairclosure system on x. This is a contradiction.

Clearly x ≠ 0. Finally suppose that x is a successor ordinal, x = y+. Let f,g be the unique pairclosure system on y. We have f,g,dom(g),rng(f) ∈ W2. Since these four sets are elements of W2 defined without W1, we have f,g,dom(g),rng(f) ∈ W1. Since dom(g) is an ordinal in OW2, we see that dom(g) ∈ OW1.

We want to extend f. By Lemma IB4.5, let z be the least P-closed ordinal with dom(g) ∈ z. Let g* be the unique pairing function on z. Then g*:z → z and g* ∈ W2. Extend f to f*:x → z by setting f*(y) = dom(g). f* exists because it is defined as a subset of W1 without mentioning W1. rng(f*) exists for the same reason. And f*,rng(f*) ∈ W2. This is the desired contradiction.

The second claim follows from the first claim by Lemma 2A iv. □

LEMMA IB4.7. The following is provable in T3(W1,W2). There is a unique pairclosure system f,g on OW2. Rng(f) exists and is the set of all P-closed elements of OW2, and g = P. For all x ∈ OW1, f(x) ∈ OW1. There is a unique pairclosure system f,g on OW1. Rng(f) is an unbounded subset of OW1, and g is the restriction of P to OW1. f,g,rng(f) ∈ W2.

Proof: Using Lemma IB4.6, let f,g be the union of the pairclosure systems fx,gx on x ∈ OW2. f,g exist because they are defined as subsets of W2 without mentioning W1. Then f,g is a pairclosure system on OW2. Uniqueness follows by transfinite induction. Since g is a pairing function on OW2, g = P. Using transfinite induction, we see that every P-closed element of OW2 is a value of f.

Now let x ∈ OW1. Then x+ ∈ OW1, fx+ ⊆ f, and rng(fx+) ⊆ W1. Hence f(x) ∈ W1 and f(x) ∈ OW2. Therefore f(x) ∈ OW1.

There is a unique pairclosure system f,g on OW1 by Lemma IB4.6. The unique pairing function on OW1 is P restricted to OW1. And the P-closed elements of OW1 are unbounded in OW1 by Lemma IB4.5. Also f,g,rng(f) ∈ W2 by Lemma IB4.2. □

We let P*:OW2 → OW2 be such that for the unique pairclosure system f,g on OW2, f = P*.

LEMMA IB4.8. The following is provable in T3(W1,W2). There exist unique functions I,J:OW2 → OW2 such that for all x ∈ OW2, P(<I(x),J(x)>) = x. For all x ∈ OW2, I(x),J(x) ∈= x.

Proof: Since P:OW22 → OW2 is one-one onto, the I,J are unique if they exist. But I,J can be defined as subsets of W2 without W1. Therefore I,J exist. ∎

LEMMA IB4.9. The following is provable in T3(W1,W2). Let x ∈ OW2. There exists unique fx,gx:ω → x+ such that

i) fx(0) = I(x);

ii) gx(0) = J(x);

iii) (∀y ∈ ω)(fx(y+) = I(gx(y)));

iv) (∀y ∈ ω)(gx(y+) = J(gx(y))).

Furthermore, f,g ∈ W2.

Proof: Let x be a least counterexample in OW2. Then x ∈ OW1. We first claim that for all z ∈ ω there exists fz,gz:z → x+ obeying the four clauses, lying in W2. It is clear that if they exist then they are unique. Suppose this is false, and let z ∈ ω be the least counterexample. Clearly z ≠ 0. Let z = w+, and consider fw,gw ∈ W2. Clearly fw,gw ⊆ W1. We can appropriately extend them at w as subsets of W1 defined without W1. Hence these extensions lie in W2. Thus the claim is established. ∎

Think of fx as the infinite sequence coded by x. Lemma IB4.10 says that we can find a code for the result of appending any y in front of the infinite sequence coded by x.

LEMMA IB4.10. The following is provable in T3(W1,W2). Let x,y ∈ u ∈ OW2, where u is a P-closed limit. There exists z ∈ u such that

i) gz(0) = x;

ii) fz(0) = y.

It follows that for all m ∈ ω, gz(m+) = gx(m) and fz(m+) = fx(m).

Proof: Let x,y,u be as given. Set z = P(<y,x>). The final part is by induction on m. For m = 0 note that gz(1) = J(gz(0)) = J(x) = gx(0). And fz(1) = I(gz(0)) = I(x) = fx(0). Assume true for m. Note that gz(m++) = J(gz(m+)) = J(gx(m)) = gx(m+). And fz(m++) = I(gz(m+)) = I(gx(m) = fx(m+). ∎

We now need to know that we can code the result of chopping off the front of any infinite sequence.

LEMMA IB4.11. The following is provable in T3(W1,W2). Let x ∈ u ∈ OW2, where u is a P-closed limit. There exists y ∈ u such that gy(0) = gx(1). It follows that for all m ∈ ω, gy(m) = gx(m+) and fy(m) = fx(m+).

Proof: Let x,u be as given. Set y = <fx(1),gx(1)>. Then gy(0) = J(y) = gx(1) and fy(0) = I(y) = fx(1). Assume gy(m) = gx(m+) and fy(m) = fx(m+). Then gy(m+) = J(gy(m)) = J(gx(m+)) = gx(m++), and fy(m+) = I(fy(m)) = I(gy(m)) = I(gx(m+)) = fx(m++). □

With Lemmas IB4.10 and IB4.11, we have the information needed to give an appropriate coding of finite sequences. It is conceptually clearer, although not absolutely necessary, to arrange that certain elements of OW2 code a unique finite sequence from OW2, and that every finite sequence from OW2 have a unique code.

Accordingly, we define the set FSC of finite sequence codes as follows. FSC is the set of all x ∈ OW2 such that

i) I(x) ∈ ω;
ii) (∀y ∈ x)(J(y) = J(x) → (∃i ∈ J(x))(fI(x)(i) ≠ fI(y)(i))).

FSC is defined as a subset of OW2 without mentioning W1, and therefore exists by Lemma 2A v.

Let x ∈ FSC. We write lth(x) = I(x). J(x) does the real coding, whereas I(x) just indicates the (finite) length. Here ii) is the minimality property that guarantees that every finite sequence has a unique code.

Let x ∈ FSC and i ∈ lth(x). We write x[i] for fx(i). This is the i-th term of the finite sequence coded by x, where we count from 0.

The following summarizes the essential properties of our finite sequence coding.

LEMMA IB4.12. Let n ≥ 1. The following is provable in T3(W1,W2). Let u ∈ OW2 be a P-closed limit.

i) (∀x,y ∈ FSC)(lth(x) = lth(y) & (∀i ∈ lth(x))(x[i] = y[i]) → x = y);

ii) 0 is the unique element of FSC such that lth(0) = 0;

iii) (∀x ∈ u,FSC)(∃y ∈ u,FSC)(lth(x) ≠ 0 → (lth(y) = lth(x)-1 & (∀m ∈ lth(y)(y[m] = x[m+])));

iv) (∀x ∈ u,FSC)(∀y ∈ u)(∃z ∈ u,FSC)(lth(z) = lth(x)+ & z[0] = y & (∀m ∈ lth(z))(m ≠ 0 → z[m] = x[m-1]));

v) (∀x ∈ FSC)(∀m ∈ lth(x))(x[m] ∈ u) → x ∈ u.

Proof: All claims except v) follow from Lemmas IB4.9 - IB4.11. In order to guarantee that we stay in FSC, we have to perform a minimization. The existence before minimization comes from Lemmas IB4.9 - IB4.11.

We establish v) by induction on lth(x). The case lth(x) = 0 is trivial since then x = 0. Suppose this is true for length n. Now let lth(x) = n+. By iii), we can chop off the front term of x to get x′ ∈ FSC of length n (of course we may not have x ∈ u, but this doesn't cause a problem). We can apply the induction hypothesis to x′ getting x′ ∈ u. But then we can put the front term back by iv), this time getting an equivalent element of FSC that is in u. Hence we get back exactly x, and so x ∈ u. □

We can prove all of the usual facts about finite sequences from OW2 using Lemma IB4.12 and induction (Lemma IB3.11). Two particularly useful facts are the existence of replacements of a term in a sequence by another term, and the existence of a code for a specified list of elements.

LEMMA IB4.13. The following is provable in T3(W1,W2). Let u ∈ OW2 be a P-closed limit. (∀x ∈ u,FSC)(∀y ∈ u)(∀n ∈ ω)(∃z ∈ u,FSC)(lth(x) = lth(z) & x[n] = y & (∀m ∈ lth(x))(m ≠ n → x[m] = z[m])).

Proof: Fix u,y as given. Prove by induction on n that (∀x ∈ u,FSC)(∀y ∈ u)(n ∈ lth(x) → (∃z ∈ u,FSC)(lth(x) = lth(z) & x[n] = y & (∀m ∈ lth(x))(m ≠ n → x[m] = z[m]))).

For the case n = 0, use Lemma IB4.12 to chop off the front term of x, put y in front, and minimize. Assume true for n and let n+ ∈ lth(x). Apply the induction hypothesis to the result of chopping off the front term of x. Then put x[0] in front and minimize. □

For x ∈ FSC, y ∈ OW2, and n ∈ lth(x), we let x[n/y] be the z given by Lemma IB4.13. Thus x[n/y] is the code for the finite sequence obtained by replacing the term of the finite sequence coded by x at position n with y.

LEMMA IB4.14. Let n ≥ 1. The following is provable in T3(W1,W2). Let u ∈ OW2 be a P-closed limit, and x1,…,xn ∈ u. Then there exists y ∈ u such that lth(y) = n & y[0] = x1 & … & y[n-1] = xn. Here 1,…,n are given by the closed terms S(0),…,S…S(0), where there are 1,…,n S's.

Proof: Obvious from Lemma IB4.12 by external induction on n.□

5. Arithmetization, structures, and satisfaction relations

Armed with Lemma IB3.11, we can arithmetize formulas of set theory. For this purpose, we will use L(∈) = first order predicate calculus without equality and with only the binary relation symbol ∈, variables x0,x1,…, ∀, &, and ¬. In the usual way, we explicitly define certain syntactic operations which provably obey the appropriate properties. This is standard material.

In particular, one develops Fmla ⊆ ω, Var:ω → ω, Neg:Fmla → Fmla, And:Fmla2 → Fmla, Or:Fmla2 → Fmla, Imp:Fmla2 → Fmla, Iff:Fmla2 → Fmla, Eps:ω2 → Fmla, All:ωxFmla → Fmla, Ex:ωxFmla → Fmla, #:Fmla → ω, and ρ:Fmla → ω. Informally, Var(n) is the index of the n-th variable. Fmla is the set of all indices of formulas in L(∈), Neg(n) is the index of the negation of the formula with index n. And(n,m) is the index of the conjunction of the formulas with indices n,m. Or(n,m) is the index of the disjunction of the formulas with indices n,m. Imp(n,m) is the index of the implication of the formula with index n to the formula with index m. Iff(n,m) is the index of the if and only if of the formula with index n to the formula with index m. Eps(n,m) is the index of the formula Var(n) ∈ Var(m). All(n,m) is the result of universally quantifying the formula with index m by Var(n).

Ex(n,m) is the result of existentially quantifying the formula with index m by Var(n). Also #(n) is the greatest m such that the m-th variable appears in the formula with index n. Finally ρ(n) be the number of occurrences of variables in the formula with index n.

LEMMA IB5.1. The following is provable in T3(W1,W2). No element of OW2 is an ordered pair.

Proof: Suppose <x,y> = {{x},{x,y}} ∈ OW2. Then <x,y> is transitive and also x,y ∈ OW2. Hence x = {x} or x = {x,y}. In either case, x ∈ x, violating transfinite induction. □

An OW2-structure is a set R consisting of elements of OW2 and OW22, where (∀x,y)(<x,y> ∈ R → (x,y ∈ R & x,y ∈ OW2)). Note that by Lemma IB5.1, R is really an arbitrary subset of OW2 together with a subset of its Cartesian square - if we have enough comprehension, which we don't.

It turns out to be important at one point in this proof to stay close to OW2. Ordered pairs from OW2 is quite close to OW2, and sets of ordered pairs are close to OW2. However, ordered pairs of subsets of OW2 are too far away.

For OW2-structures R, we write |R| for the least P-closed limit u ∈= OW2 such that every element of R is either in u or in u2.

Also it will be convenient to write x ∈' R to indicate that x ∈ R & x ∈ OW2. We also speak of the elements' of R.

Let R be an OW2-structure. We say that x is adequate for b in R if and only if

i) b ∈ Fmla;
ii) x ∈ FSC;
iii) #(b) ∈ lth(x);
iv) (∀i ∈ lth(x))(x[i] ∈' R).

We say that y is a satisfaction relation for R if and only if y is the set of all <b,x> such that x is adequate for b, and

i) b = Eps(n,m), n,m ∈ ω, and <x[n],x[m]> ∈ R; or

ii) b = Neg(n), n ∈ Fmla, and <n,x> ∉ y; or

iii) b = And(n,m), n,m ∈ Fmla, and <n,x>,<m,x> ∈ y; or

iv) b = Or(n,m), n,m ∈ Fmla, and <n,x> ∈ y or <m,x> ∈ y; or

v) b = Imp(n,m), n,m ∈ Fmla, and either <n,x> ∉ y or <m,y> ∈ y; or

vi) b = Iff(n,m), n,m ∈ Fmla, and either (<n,x>,<m,x> ∈ y) or (<n,x>,<m,x> ∉ y); or

vii) b = All(n,m), n ∈ ω, m ∈ Fmla, and for all u ∈ x, <m,x[n/u]> ∈ y; or

viii) b = Ex(n,m), n ∈ ω, m ∈ Fmla, and there exists u ∈ x such that <m,x[n/u]> ∈ y.

The idea here is to view R as the relational structure <R ∩ OW2,R ∩ OW22> and do the Tarski definition of truth.

We define SAT(R,n,x) if and only if R is an OW2 structure and there exists a satisfaction relation y for R such that <n,x> ∈ y.

LEMMA IB5.2. The following is provable in T3(W1,W2). Let R be an OW2-structure. There is at most one satisfaction relation y for R. Furthermore, y ⊆ |R|2. If there is a satisfaction relation for R then SAT(R,n,x) obeys the usual inductive clauses for satisfaction (provided x is adequate for n in R).

Proof: By induction on the number of occurrences of variables. I.e., one shows that for all n ∈ ω, any two satisfaction relations on R have the same elements whose first component m has ρ(m) ∈= n. For the second claim, note that condition i) implies that x lies in every P-closed strict upper bound on the elements' of R, by Lemma IB4.12 v. For the third claim, use the uniqueness in Lemma IB5.1 □

It will be convenient to have the following version of Russell's Paradox in this context.

LEMMA IB5.3. The following is provable in T3(W1,W2). Let R be an OW2 structure which has a satisfaction relation. Then there exists n ∈ Fmla such that the following fails for all x ∈' R and y ∈ FSC: for all a ∈' R, <a,x> ∈ R ↔ SAT(R,n,y[0/a]). In particular, this is provable in T3(W1,W2) when n is taken to be the numeral (closed S-term) which is the index of the formula ¬x0 ∈ x0.

Proof: Left to the reader.

We now give a sufficient condition for the existence of the satisfaction relation for an OW2-structure R.

LEMMA IB5.4. Let φ be a formula in L(∈,W1,W2) with at most the free variable x, not mentioning W1. The following is provable in T3(W1,W2). Suppose φ has a unique solution, and it is an OW2-structure R.

i) R has a unique satisfaction relation;

ii) the satisfaction relation is ⊆ |R|2;

iii) if |R| ∈ OW2 then R and its satisfaction relation lie in W1, and |R| ∈ OW1.

Proof: Let φ,R be as given. First assume |R| ∈ OW2. Then |R| ∈ OW2. Now |R| is definable without mentioning W1. By Lemma IB1.1 vi, |R| ∈ W1, and hence in OW1. Hence R is a subset of W1 definable without mentioning W1. By axiom 3, R ∈ W2. Therefore by Lemma IB1.1 vi, R ∈ W1.

Still assuming |R| ∈ OW2, the satisfaction relation of R, if it exists, is a subset of |R|2 by inspection. So if it exists, it is a definable subset of W1 without mentioning W1. So the satisfaction relation is also ∈ W1, if it exists.

We also see by inspection that the satisfaction relation of R, if it exists, is a subset of |R|2 even in the case |R| = OW2.

It remains to show that R has a satisfaction relation. Uniqueness is from Lemma IB5.1. We no longer assume |R| ∈ OW2.

For n ∈ ω, we can define the n-satisfaction relations for R; we use the same 6 clauses as in the definition of satisfaction relation for R, except that we strengthen clause ii) by insisting that the number of occurrences of variables in the formula of index b is at most n; i.e., ρ(b) ≤ n. Again we have that there is at most one n-satisfaction relation for R.

Now there is a direct construction of a 2-satisfaction relation for R since the only formulas with at most 2 occurrences of symbols are atomic formulas.

We then prove that for each n ∈ ω, there exists an n-satisfaction relation for R. Assume this is false, and let n be least. Then 2 ∈ n. Let y be the (n-1)-satisfaction relation for R. Now we can extend y to a (the) n-satisfaction relation for R rather explicitly, using Lemma IB1.1 v.

We also argue by induction that the various n-satisfaction relations for R cohere in the obvious way. Hence we can take the union of the various n-satisfaction relations and see that this union is a (the) satisfaction relation for R. Here we again use Lemma IB1.1 v. □

Let R be an OW2-structure with a satisfaction relation y. We say that v is a preferred name over R if and only if

i) v is of the form <b,x> where x is adequate for b in R;

ii) there is no <b′,x′> ∈ <b,x> such that for all a ∈ A, SAT(R,b′,x′[0/a]) ↔ SAT(R,b,x[0/a]);

iii) there is no c ∈′ R such that for all a ∈′ R SAT(R,b,x[0/a]) ∈ y ↔ <a,c> ∈ R.

The idea here is that the preferred names 'name' the subsets of R ∩ OW2 that are first order definable over (the relational structure associated with) R by means of the satisfaction relation for R, although we are careful not to commit to the existence of these subsets of R as objects. (The subset of R ∩ OW2 'named' by <b,x> is the set of all a ∈′ R that SAT(R,b,x[0/a])). We don't even commit to the existence of R ∩ OW2. The preferred names are those that are minimal among those names that 'name' the same subset, and we also require of a preferred name that it not 'name' any set of the form {a: <a,c> ∈ R}, c ∈′ R, where we again are careful not to commit to the existence of this {a: <a,c> ∈ R}. It will eventually turn out that we can prove the existence of all these relevant subsets of R in the contexts we are interested in, and we need this existence only considerably later.

We introduce a second absolute value on OW2-structures R. We write ||R|| for the least d ∈= OW2 such that for all x ∈’ R, I(x) ∈ d. Note that ||A|| ∈= |A|.

Let R be an OW2-structure, |R| ∈ OW2, ||R|| = d, and y be the satisfaction relation for R. We define the OW2- structure R* = S as follows. The elements of S consist of

i) the elements’ of R together with the P(<d,v>) such that v is a preferred name over R;
ii) the elements of R which are pairs together with the pairs <a,P(<d,v>)> such that v = <b,x> is a preferred name over R, a ∈’ R, and SAT(R,b,x[0/a]).

Note that the P(<d,v>) in i) are new (i.e., are not in R) because ||R|| = d.

We say that an OW2-structure R is extensional if and only if for all x,y ∈’ R, if (∀z ∈’ R)(<z,x> ∈ R ↔ <z,y> ∈ R) then x = y.

LEMMA IB5.5. Let φ be a formula in L(∈,W1,W2) with at most the free variable x, not mentioning W1. The following is provable in T3(W1,W2). Suppose φ has a unique solution, and this solution is R, where R is an OW2-structure such that |R| ∈ OW2. Then R,R* ∈ W1. If R is extensional then R* is extensional. And ||R*|| = ||R||+.

Proof: By Lemma IB5.4, R and its satisfaction relation are both in W1. Then R* can be defined as a subset of W1 without mentioning W1, and hence exists by axiom 3. By Lemma IB1.1 vi, R* ∈ W1. The extensionality claim is obvious by construction. For the final claim, note that by Lemma IB5.3, a new element P(<||R||,v>) in clause i) survives clause ii).
□

LEMMA IB5.6. Let R be an OW2-structure with a satisfaction relation. Let n ∈ Fmla and x ∈ FSC, where x is adequate for n in R. There exists x ∈’ R* such that for all a ∈’ R, <a,x> ∈ R* if and only if SAT(R,n,x[0/a]).

Proof: By construction of R*. Left to the reader.

6. Constructible universe structure

We are now ready to complete the construction of the constructible universe on OW2. Recall the function P*:OW2 → OW2 which enumerates the P-closed elements of OW2.

The most obvious definition of an L-function is a function f with domain some x ∈= OW2 such that the following holds.

i) every value of f is a structure;
ii) if 0 ∈ x then F(0) = ∅;
iii) if y+ ∈ x then f(y+) = f(y)*;
iv) if y ∈ x is a limit then f(y) is the union of the f(z), z ∈ y.

However, these are set valued functions and some technical problems crop up when we get too far from OW2. So we instead make the following definitions. Let T,x be arbitrary. We write Tx = {y: <x,y> ∈ T}. Of course, we have to worry about whether this set exists.

We say that T is an L-system on x if and only if

i) x ∈= OW2;
ii) T ⊆ x2;
iii) for all y ∈ x, Ty exists and is an OW2-structure;
iv) if 0 ∈ x then T0 = ∅;
v) if y+ ∈ x then Ty+ = Ty*;
vi) if y ∈ x is a limit then f(y) is the union of the Tz, z ∈ y.

Note that T is a set of ordered pairs and triples from OW2, and every Ty is a set of elements and ordered pairs from OW2.

Let R and S be OW2-structures. We write R ⊆* S if and only if R ⊆ S, and (∀x,y ∈′ R)(<x,y> ∈ R ↔ <x,y> ∈ S).

LEMMA IB6.1. The following is provable in $T3(W1,W2)$. Let $x \in y \in= OW2$, T be an L-system on x, and T' be an L-system on y. Then for all $z \in x$, $Tz = T'z$.

Proof: By transfinite induction, using the fact that clauses iii) - vi) in the definition of L-systems are entirely deterministic. □

LEMMA IB6.2. The following are provable in $T3(W1,W2)$. Let $x \in= OW2$ and T be an L-system on x.

i) for all $y \in x$, $||Ty|| = y$ and $|Ty| \in= \max(\omega,P^*(y))$;
ii) for all $z \in y \in x$, $Tz \subseteq^* Ty$;
iii) $T \subseteq \max(\omega,P^*(x))\leq 3$.

Proof: Let $x \in= OW2$ and T be an L-system on x. Now if $0 \in x$ then $|T0)| = |\emptyset| = \omega$, and $||T0|| = ||\emptyset|| = 0$. So by transfinite induction, $y \in x \rightarrow ||Ty|| = y$. Also $|f(1)| = |\emptyset^*| = \omega$ and $|T2(W1,W2)| = |\emptyset^{**}| = \omega$. And by starting the transfinite induction at 2, we obtain $1 \in y \in x \rightarrow |Ty| \in= P^*(y)$.

For the second claim, show that the following holds for all $y \in x$: for all $z \in y$, $Tz \subseteq^* Ty$. The basis case $y = 0$ is trivial. Suppose $y = w+$. Clearly $Tw \subseteq^* Ty$. If $z \in w$ then $Tz \subseteq^* Tw$, and so $Tz \subseteq^* Ty$. Finally suppose y is a limit. Then the Tz, $z \in y$, are ordered under $\subseteq^*$. Let $z \in y$. Then clearly $Tz \subseteq Ty$. Also let $u,v \in' Tz$. If $\langle u,v\rangle \in Tz$ then $\langle u,v\rangle \in Ty$. If $\langle u,v\rangle \notin Tz$ then $\langle u,v\rangle$ is outside every Tv, $v \in y$. Hence $\langle u,v\rangle \notin Ty$.

The third claim is immediate from the first claim. □

LEMMA IB6.3. The following is provable in $T3(W1,W2)$. Let $x \in= OW2$ and T be the L-system on x. If $v \in x$ then $Tv \in W2$. If $v \in x$ and $v \in OW1$ then $Tv \in W1$. If $x \in OW2$ then $T \in W2$. If $x \in OW1$ then $T \in W1$.

Proof: For the first claim, first note that by Lemma IB6.1, Tv does not depend on T. So let $v \in OW2$ be least such that there exists an L-system on some x, $v \in x \in= OW2$, such that

Tv ∉ W2. Then Tv is an OW2-structure with |Tv| ∈ OW2 (Lemma IB6.2) which is definable without mentioning W1, and so according to Lemma IB5.3 iii, Tv ∈ W1. This is a contradiction.

For the second claim, we see from Lemma IB6.2 that Tv ⊆ max(ω,P*(v))≤2. Hence Tv is defined as a subset of max(ω,P*(v))≤2 without mentioning W1 but with parameters for elements of W1. And by Lemma IB4.7, P*(v) ∈ OW1. Hence Lemma IB3.7 iii applies. So f(v) ∈ W1.

For the third claim, let x ∈ OW2 be least such that there exists an L-system T on x such that T ∉ W2. Then as usual, x ∈ OW1. By Lemma IB6.2, T ⊆ max(ω,P*(x))≤3. By Lemma IB4.7, P*(x) ∈ OW1. Hence T is defined as a subset of W1 without mentioning W1, and so exists as an element of W2 by axiom 3. This is a contradiction.

For the fourth claim, let x ∈ OW1. As above, T ⊆ max(ω,P*(x))≤3 and P*(x) ∈ OW1. Hence t is defined as a subset of max(ω,P*(x))≤3 without mentioning W1, but with parameters from W1 (namely x). Then T ∈ W1 by Lemma IB3.7 iii.

LEMMA IB6.4. The following is provable in T3(W1,W2). For all x ∈= OW2 there exists a unique L-system T on x. For all y ∈ x, Ty is an extensional structure.

Proof: Note that uniqueness follows from Lemma IB6.2. So it suffices to prove that for all x ∈= OW2, there exists an L-system on x all of whose cross sections are extensional. Let x ∈ OW2 be least such that there is no L-system T on x, all of whose values are extensional. Clearly x ≠ 0 since the empty function is the L-system on 0. First suppose x is a limit. Now for each y ∈ x let T^y be the unique L-system with domain y. By Lemma IB6.1, these T^y are all comparable in the strong sense that they have the same appropriate cross sections. Now the union of the T^y, y ∈ x, if it exists, must have the same appropriate cross sections as the T^y, y ∈ x. So this union, if it exists, must be the L-system on x. So in the case of x being a limit, it suffices to prove the existence of this union. By Lemma IB6.2, for all y ∈ x, T^y ⊆ max(ω,P*(x))≤3. Hence this union can be defined as a subset of W2 without mentioning W1. Therefore the union exists by Lemma IB1.1 v, and we have the required contradiction.

Finally suppose x = y+, and let T be the L-system on y all of whose values are extensional. We wish to extend T to T′ so that T′ is the L-system on x, all of whose cross sections are extensional, thereby obtaining the required contradiction. If y = 0 then define T′ = ∅. It is easy to see that T′ is the L-system on 1 (and also the L-system on 0). Assume first that y = z+. Then y,z ∈ OW1 by definability considerations. Also the extensional OW2-structure Tz is definable without mentioning W1 and |Tz| ∈ OW2, by Lemma IB6.2. Hence by Lemma IB5.5, Tz* ∈ W1 and Tz* is extensional. And by Lemma IB6.3, T ∈ W1. Now take T′ = T ∪ {<y,u>: u ∈ Tz*}. This union exists by axiom 3 since it defines a subset of W1 without mentioning W1. This is the required contradiction.

Finally, assume that x = y+ where y is a limit. We again have T ∈ W1 and x,y ∈ OW1. By Lemma IB6.3, for all z ∈ y, Tz ∈ W1 and Tz is extensional. By Lemma IB6.2 ii, these Tz are comparable under ⊆*. Therefore the union R of the Tz, z ∈ y, if it exists, is an extensional OW2-structure. Now by Lemma IB6.3, each Tz, z ∈ y, lies in W1. Therefore this union, R, is defined as a subset of W1 without mentioning W1, and so exists in W2 by axiom 3. By Lemma IB1.1 vi, R ∈ W1. Now take T′ = T ∪ {<y,u>: u ∈ R}. This union exists by axiom 3 since it defines a subset of W1 without mentioning W1. It is easy to check that T′ is the L-system on x all of whose cross sections are extensional. This is the required contradiction.

∎

We now introduce some notation. For A ⊆ OW2 write |A| for the least P-closed limit x ∈ OW2 such that A ⊆ x, and ||A|| for the least x ∈ OW2 such that A ⊆ x. |A| exists if and only if ||A|| ∈ OW2.

For x ∈ OW2 we define A[x] = {u: u ∈′ f(x)} and R[x] = {<u,v>: <u,v> ∈ f(x)}, where f is the L-system for x+. We define A[OW2] = {u: (∃x ∈ OW2)(u ∈ A[u]} and R[OW2] = {v: (∃x

∈ OW2)(v ∈ R[x])}. But we need to know that these sets exist. The definition is well formed because it is independent of the choice of f by Lemma IB6.1.

LEMMA IB6.5. The following are provable in T3(W1,W2).

i) for all x ∈= OW2, A[x],R[x],A[x]2,A[x] ∪ R[x] exist and are ∈= W2;

ii) for all x ∈ OW1, A[x],R[x],A[x]2,A[x] ∪ R[x] ∈⊆ W1;

iii) for all x ∈= OW2, A[x] ⊆ OW2 and R[x] ⊆ A[x]2;

iv) for all x ∈ OW2, |A[x]| ∈= max(ω,P*(x));

v) for all x ∈= OW2, ||A[x]|| = x;

vi) A[0] = R[0] = A[1] = R[1] = ∅;

vii) for all x ∈ OW2, A[x+] ∪ Rx+ = (A[x] ∪ R[x])*;

viii) for all limits x ∈= OW2, A[x] = {y: (∃z ∈ x)(y ∈ A[z])}, R[x] = {y: (∃z ∈ x)(y ∈ R[z])};

ix) for all x ∈= OW2, there is a unique satisfaction relation on A[x] ∪ R[x], and is lies in W2;

x) for all x ∈ OW1, the satisfaction relation on A[x] ∪ R[x] lies in W1;

xi) for all x ∈= OW2 and w, {a: <a,w> ∈ R[x]} exists and is ∈⊆ W2;

xii) for all x ∈ OW1 and w, {a: <a,w> ∈ R[x]} ∈⊆ W1;

xiii) for all x ∈ OW2 and y ∈= OW2, A[x] ⊆ A[y] and R[x] ⊆ R[y];

xiv) for all x ∈ OW2 and a,b, <a,b> ∈ R[x] ↔ <a,b> ∈ R[OW2];

xv) for all x ∈ OW2 and a ∈ b ∈ OW2, if b ∈ A[x] and a ∈ A[OW2] then a ∈ A[x];

xvi) for all x ∈= OW2 and y,z ∈ A[x], if for all a ∈ A[x], <a,y> ∈ R[x] if and only if <a,z> ∈ R[x], then y = z.

Proof: We first prove that for all x ∈ OW2 and L-systems T on x+, the two components of Tx exist; i.e., the elements of Tx in OW2 and the elements of Tx in OW22, separately exist. This is proved by taking the least counterexample x ∈ OW2 and then constructing these two sets as subsets of W1, using Lemma IB6.3. Thus for all x ∈ OW2, A[x],R[x] exist and are just the components of Tx, where T is the L-system on x+. Also by Lemma IB1.1 v, A[OW2],R[OW2] exist. The rest of this is straightforward, using Lemmas IB5.4,IB5.5,IB3.6 and IB3.7. ∎

7. Axioms of ZFC

We now aim to prove in T3(W1,W2) that (A[OW2],R[OW2]) satisfies the axioms of ZFC, formulated in L(∈). This fact can easily be stated in T3(W1,W2) using Lemma IB3.11.

We let SAT(A,R,n) be (∀x ∈ FSC)(if x is adequate for n in A,R then SAT(A,R,n,x).

Externally, let φ be a formula in L(∈) and let t be the closed S-term representing the index of φ; i.e., S…S(0), where the number of S's is the index of φ. Also let x ∈= OW2. We define the formula $\varphi^{(x)}$ in L(∈,W1,W2) to be the result of relativizing the quantifiers to A[x] and replacing each x ∈ y with <x,y> ∈ R[x] and then expanding into primitive notation. In general, W1 will never appear in $\varphi^{(x)}$, but W2 will.

We say that w is x-adequate for n if and only if

- i) n ∈ Fmla;
- ii) w ∈ FSC;
- iii) #n ∈ lth(w);
- iv) (∀m ∈ lth(w))(w[m] ∈ A[x]).

This is a focusing of the previous definition of adequacy made after Lemma IB5.1 to fit the present context. Reacll the previous definition of SAT within T3(W1,W2).

LEMMA IB7.1. Let φ(x0,…,xk) be a formula in L(∈) and t be the closed S-term representing the index of φ. The following is provable in T3(W1,W2). Let x ∈= OW2 and w be adequate for t in A[x]. Then SAT(A[x],R[x],t,w) ↔ $\varphi^{(x)}$(w[0],…,w[k]).

Proof: By induction on the complexity of φ.

LEMMA IB7.2. Let φ be a sentence in L(∈) and t be the closed S-term representing the index of φ. The following is provable in T3(W1,W2). x ∈= OW2 → SAT(A[x],R[x],t) ↔ $φ^{(x)}$.

Proof: From Lemma IB7.1.

Lemma IB7.2 supports the following way of proving that a specific sentence holds in A[x],R[x], formulated in terms of SAT. One simply informally argues for the relativization of the sentence within T3(W1,W2).

LEMMA IB7.3. Let φ be a formula in L(∈) and t be the closed S-term representing the index of φ. The following is provable in T3(W1,W2). Let x ∈= OW2 and w be adequate for t in A[x]. Then SAT(A[x],R[x],t,w) ↔ φ(x).

Proof: By induction on the number of occurrences of variables in φ. ▯

LEMMA IB7.4. The following is provable in T3(W1,W2). x ∈ A[OW2] → (x ∈ A[I(x)+] & x ∉ A[I(x)]). <x,y> ∈ R[OW2] → I(x) ∈ I(y).

Proof: The first claim is proved by transfinite induction. Suppose this is true for all x ∈ A[y], y ∈ OW2. Let x ∈ A[y+]. By the construction of A[y+], either x ∈ A[y], it which case we are done, or I(x) = y. In the latter case, x ∈ A[I(x)+]. It remains to see that x ∉ A[I(x)]. Suppose x ∈ A[I(x)] = A[y]. Then by induction hypothesis, x ∉ A[I(x)], which is the required contradiction. The second claim is clear by transfinite induction. ▯

1. Extensionality. Let Ext be the closed S-term representing the index of this sentence.

2. Pairing. Let Pair be the closed S-term representing the index of this sentence.

LEMMA IB7.5. The following is provable in T3(W1,W2). SAT(A[OW2],R[OW2],Ext). SAT(A[OW2],R[OW2],Pair).

Proof: The first claim follows from Lemma IB6.5 xvi. For the second claim, let x,y ∈ A[z]. Then the pair in (A[OW2],R[OW2]) lies in A[z+].

3. Union. Let Union be the closed S-term representing the index of this formula.

4. Foundation. Let Found be the closed S-term representing the index of this formula.

LEMMA IB7.6. The following is provable in T3(W1,W2). SAT(A[OW2],R[OW2],Union). SAT(A[OW2],R[OW2],Found).

Proof: Let x ∈ A{OW2]. By Lemma IB7.4, in ([A[OW2],R[OW2]), the elements of the elements of x all lie in A[I(x)], and so the union lies in A[I(x)+]. The second claim follows from IB7.4. ∎

5. Infinity. Let Inf be the closed S-term representing the index of this formula.

LEMMA IB7.7. The following is provable in T3(W1,W2). SAT(A[OW2],R[OW2],Inf).

Proof: Show by induction on n ∈ ω that there is a unique map f:n → OW2 such that

i) if 0 ∈ x then f(0) is the empty set of (A[OW2],R[OW2]);
ii) for all m+ ∈ n, f(m+) is the f(m) ∪{f(m)} of (A[OW2],R[OW2]).
iii) for all m ∈ n, f(m) ∈ A[m+].

Then argue by induction that for all n, f(n) is ∈-connected and transitive in A[ω],R[ω]). Next show, using Lemma IB7.6, that for all n ∈ ω, the ∈-connected transitive sets of ([A[ω],R[ω]) lying in A[n] are exactly the f(m), m ∈ n. Hence the ∈-connected transitive sets of (A[ω],R[ω]) are exactly the range of f. Therefore the ω of (A[OW2],R[OW2]) lies in A[ω+1]. ∎

6. Bounded separation. We make the following definition in T3(W1,W2). Let m be the index of a formula φ of L(∈) and let n be the smallest index of a variable other than x0,x1 that does not appear in φ. Let B(m) be the index of the formula φ' resulting from relativizing all quantifiers to membership in xn. Let Bsep(m) be the index of the formula (∀x1)…(∀xn)(∃xn+1)(∀x0)(x0 ∈ xn+1 ↔ (x0 ∈ xn & φ')).

LEMMA IB7.8. The following is provable in T3(W1,W2). m ∈ Fmla → SAT(A[OW2],R[OW2],Bsep(m)).

Proof: We argue in T3(W1,W2). Let m ∈ Fmla. Let w be OW2-adequate for Bsep(m). We must show SAT(A[OW2],R[OW2],Bsep(m),w). It suffices to show SAT(A[OW2],R[OW2],Bsep'(m),w), where Bsep'(m) is the result of deleting the block of universal quantifiers in front of Bsep(m). Let n be the length of this block of universal quantifiers. Without loss of generality we can assume lth(w) = n+. It suffices to show that there exists y ∈ A[OW2] such that for all z ∈ A[OW2], <z,y> ∈ R[OW2] if and only if <z,w[n]> & SAT(A[OW2],R[OW2],B(m),w[0/z]). We can then put this in a form so that y can be chosen in A[u+] where for all m ∈ n+, w[m] ∈ A[u]. ∎

Since we have used bounded separation instead of separation, we need to verify the following strong form of replacement.

7. Collection. Let φ be a formula of L(∈) and let n be the smallest index of a variable other than x0,x1 that does not appear in φ. (∀x1)…(∀xn-1)(∃xn)(∀x0)((x0 ∈ xn & (∃x1)(φ)) → (∃x1)(x1 ∈ xn & φ)). Define Coll:Fmla → Fmla by Coll(m) = the index of the above formula where φ is the formula with index m.

LEMMA IB7.9. Let φ be a formula in L(∈,W1,W2) without W1, in which x0 is not free. The following is provable in T3(W1,W2). x2,…,xn ∈ OW2 → (∃x0 ∈ OW2)(∀x1 ∈ x2)((∃x3 ∈ OW2)(φ) → (∃x3 ∈ x0)(φ)).

Proof: Argue in T3(W1,W2). Suppose this is false for some choice of x2,…,xn ∈ OW2, and let x2,…,xn ∈ OW2 be minimized.

Then x2,…,xn are defined without mentioning W1, and so x2,…,xn ∈ OW1. According to Lemma IB3.7 ii, (x1 ∈ x2 & (∃x3 ∈ OW2)(φ)) ↔ (x1 ∈ x2 & (∃x3 ∈ OW1)(φ[W2/W1])) ↔ (x1 ∈ x2 & (∃x3 ∈ OW1)(φ)). The second equivalence is more easily seen by reversing the order. Thus we can take x0 = OW1 and obtain the required contradiction. ∎

LEMMA IB7.10. The following is provable in T3(W1,W2). m ∈ Fmla → SAT(A[OW2],R[OW2],Coll(m)).

Proof: Use Lemma IB7.10. We can argue semiformally as in Lemma IB7.8, starting with an OW2-adequate w for Coll(m). we omit the details. ∎

8. Power set. Let Pow be the closed term representing the index of this sentence.

This is the most interesting case, and we draw on a new idea. We need to develop a strong kind of absoluteness of the L-systems. In particular, we need to generalize L-systems to strong ordinals without mentioning W1 or W2. Recall the important comparability result, Lemma IB3.2, which says that strong ordinals are ∈-comparable with OW2. We use this to establish that general L-systems on strong ordinals are comparable with A[OW2],R[OW2].

We say that x is a very strong ordinal if and only if

i) x is a strong ordinal;
ii) x2 exists;
iii) the ordered pair of any two elements of x exists;
iv) there is a unique pairing function f on x; and f:x → x;
v) there is a unique pairclosure system f,g on x; and g:x → x;
vi) no element of x is an ordered pair;
vii) finite sequence coding relative to the unique pairclosure system exists and has all of the relevant properties;
viii) there is no ∈-greatest element of x.

Let x be a very strong ordinal. An x-structure is a set R consisting of elements of x and x2, where (∀a,b)(<a,b> ∈ R →

(a,b ∈ R & a,b ∈ x)). We define x* in the appropriate way as in the case of OW2-structures, using the finite sequence coding given by vii) above.

We say that T is a generalized L-system on x if and only if

i) x is a very strong ordinal;

ii) T ⊆ x2;

iii) for all y ∈ x, Ty exists and is an x-structure;

iv) if 0 ∈ x then T0 = ∅;

v) if y ∈ x is a limit then f(y) is the union of the Tz, z ∈ y;

vi) if y+ ∈ x then Ty+ = Ty*;

vii) the sets T/1 = {y ∈ x: (∃z ∈ x)(y ∈ Tz)} and T/2 = {y ∈ x2: (∃z ∈ x)(y ∈ Tz)} exist, (T/1)2 exists, and T/2 ⊆ (T/1)2.

LEMMA IB7.11. The following is provable in T3(W1,W2).

i) let x ∈= OW2 be 0 or a limit. The L-system on x is the unique generalized L-system T on x. T/1 = A[x] and T/2 = R[x];

ii) let x be a very strong ordinal such that OW2 ∈ x, and let T be a generalized L-system on x. Then for all y ∈= OW2, Ty = A[y] ∪ R[y].

Proof: Obviously every x ∈= OW2 that is 0 or a limit is a very strong ordinal. The rest of the first claim is by transfinite induction. The second claim is also proved by transfinite induction. The unique pairclosure system on x is used to control the L-systems. □

LEMMA IB7.12. The following is provable in T3(W1,W2). SAT(A[OW2],R[OW2],Pow).

Proof: Let x ∈ OW2 be least such that x has no power set in A[OW2] in the sense of A[OW2],R[OW2]. Then as usual, x ∈ OW1, and so rk(x) ∈ OW1. We write y ⊆′ x to indicate that y ⊆ x in the sense of A[OW2],R[OW2], which is the same as saying that for all a, if <a,y> ∈ R[OW2] then <a,x> ∈ R[OW2]. This is also the same as saying that for all a ∈ OW1, if <a,y> ∈ R[OW2] then <a,x> ∈ R[OW2].

Now we claim that there exists w ∈ OW2 such that w ⊆′ x and w ∉ OW1. For otherwise, we would get the power set of x in the sense of A[OW2],R[OW2] in A[OW1+], contradicting the choice of x.

Now fix w to be the least such w ∈ OW2. Note that w can be alternatively defined as the unique w such that

there exists a very strong ordinal u and a generalized L-system T on u such that OW1 ∈ u and w is the least element of u such that w ⊆ x in the sense of (T/1,T/2), and w ∉ OW1.

That this uniquely defines the same w follows from Lemmas IB7.13 and IB3.2.

Note that this definition of w does not mention W2. Instead, it uses x as a parameter and mentions W1. Therefore by Lemma IB1.1 iv there is also a unique w′ such that

there exists a very strong ordinal u and a generalized L-system T on u such that OW2 ∈ u and w is the least element of u such that w ⊆ x in the sense of (T/1,T/2), and w′ ∉ OW2.

Recall that x ∈ OW1 and so <a,x> ∈ R[OW2] → a ∈ OW1. Hence by Lemma IB3.7 ii, we see that for all a, if <a,x> ∈ R[OW2] then a ∈ w ↔ a ∈ w′. Hence w = w′. But w ∈ OW2 and w′ ∉ OW2. This is the required contradiction. ∎

We make the following definition in T3(W1,W2). Let m be the index of a formula φ of L(∈) and let n be the smallest index of a variable other than x0,x1 that does not appear in φ. Let B(m) be the index of the formula φ′ resulting from relativizing all quantifiers to membership in xn. Let RF(m) be the index of the formula (∃xn)(∀x0)…(∀xn-1)(φ ↔ φ′). (RF for "reflection.")

LEMMA IB7.13. The following is provable in T3(W1,W2). #(m) → SAT(A[OW2],R[OW2],RF(m)).

Proof: This is not the strongest result of this kind, but it suffices for our purposes. Argue by induction on the number of occurrences of variables in m. Lemma IB7.9 is used ω times to obtain a suitable ω-sequence. Then Lemma IB7.9 is used again to get an upper bound. ∎

9. Separation. Let φ be a formula of L(∈) and let n be the smallest index of a variable other than x0,x1 that does not appear in φ. (∀x1)…(∀xn)(∃xn+1)(∀x0)(x0 ∈ xn+1 ↔ (x0 ∈ xn & φ)). Define Sep:Fmla → Fmla by Sep(m) = the index of the above formula where φ is the formula with index m.

LEMMA IB7.14. The following is provable in T3(W1,W2). m ∈ Fmla → SAT(A[OW2],R[OW2],Sep(m)).

Proof: From Lemmas IB7.13 and IB7.8. ∎

LEMMA IB7.15. The following is provable in T3(W1,W2). (A[OW2],R[OW2]) satisfies ZF.

Proof: By Lemmas IB7.5, IB7.6, IB7.7, IB7.8, IB7.10, IB7.12, and IB7.14. Strictly speaking, Replacement is used instead of Collection in ZF. But the triviality that Replacement follows from Collection in the presence of Separation can be lifted to this context and formalized within T3(W1,W2).

We now come to the axiom of choice, which is the most complicated of the verifications of the axioms of ZFC. But now that we have ZF in (A[OW2,R[OW2]), we can rely heavily on previous developments of the constructible universe within ZF, due to Gödel.

This is a good point to talk about the soundness theorem, which we have avoided doing up to now.

We have been using the notions of satisfaction relation and SAT for OW2-structures. Here the domain lives within OW2. In this context, we have no trouble stating and proving the following soundness theorem:

LEMMA IB7.16. The following is provable in T3(W1,W2). Let R be an OW2-structure with a satisfaction relation. Then R satisfies every valid sentence of the usual Hilbert style axioms and rules of inference for L(∈).

Proof: By suitably formalizing the standard proof of soundness of predicate calculus. □

Lemma IB7.16 will allow us to argue that if a sentence is provable in ZF then it is satisfied by (A[OW2],R[OW2]), as well as related facts.

LEMMA IB7.17. The following is provable in T3(W1,W2). For each x ∈ OW2 there is a unique o(x) ∈ A[OW2] such that I(o(x)) = x and (A[OW2],R[OW2]) satisfies that o(x) is an ordinal. Furthermore, o(x) ∈ A[x+], o(x) ∉ A[x].

Proof: Uniqueness follows from the fact that (A[OW2],R[OW2]) satisfies ZF and therefore (A[OW2],R[OW2]) satisfies that the ordinals are ∈-connected. Suppose existence is true for all x ∈ y, where y ∈ OW2 is a limit. Now use "being an ordinal" over A[y] in order to obtain o(y) = z such that I(z) = y and (A[OW2],R[OW2]) satisfies z is an ordinal. □

We will use the notation o(x) below.

We now make the following standard definition within ZF. We define L(0) = ∅, L(α+1) = the set of all subsets of L(α) that are first order definable over L(α) with parameters allowed for elements of L(α), L(λ) = the union of the L(β), $\beta < \lambda$.

LEMMA IB7.18. The following is provable in T3(W1,W2). Let x ∈ OW2, y ∈ A[OW2], and (A[OW2],R[OW2]) satisfies "y = L[o(x)]," Then (∀v)(<v,y> ∈ R[OW2] ↔ v ∈ A[y]). (A[OW2],R[OW2]) satisfies ZF + V=L.

Proof: Argue by transfinite induction on x ∈ OW2 within T3(W1,W2). The crucial case is the successor case, x = u+. By induction hypothesis, the L(u) in (A[OW2],R[OW2]) is the element representing the set A[u]. Let L(o(u)+) be the L(o(u)+) of (A[OW2],R[OW2]). Then the elements of L(o(u)+) in the sense of (A[OW2],R[OW2]) are exactly the elements of A[OW2] whose extension is first order definable over

(A[u],R[u]). This is because the formalization of "first order definable over" in ZF is accurate. But these are exactly the elements of A[u+]. The last claim is now obvious.

10. Choice. Let Choice be the index of the axiom of choice.

LEMMA IB7.19. The following is provable in T3(W1,W2). SAT(A[OW2],R[OW2],Choice).

Proof: By Lemma IB7.18. ∎

LEMMA IB7.20. The following is provable in T3(W1,W2). (A[OW2],R[OW2]) satisfies ZFC + V=L.

Proof: By Lemmas IB7.15, IB7.18, and IB7.19. ∎

8. Indescribable cardinals, subtle cardinals, and ∈-models

We will first show that o(OW1) is a strongly inaccessible cardinal in (A[OW2],R[OW2]).

LEMMA IB8.1. The following is provable in T3(W1,W2). o(OW1) is an uncountable regular cardinal in (A[OW2],R[OW2]).

Proof: If o(OW1) is a regular cardinal in (A[OW2],R[OW2]) then surely it is an uncountable regular cardinal in (A[OW2],R[OW2]), because $\omega \in$ OW2 and o(OW1) serves as the first infinite ordinal in (A[OW2],R[OW2]).

Now suppose o(OW1) is not a regular cardinal in (A[OW2],R[OW2]). Since (A[OW2],R[OW2]) satisfies ZFC, there exists w ∈ A[OW2] such that in (A[OW2],R[OW2]), w is satisfied to be an unbounded map from an element of o(OW1) into o(OW1). I.e., (∃b ∈ OW1)(∃w ∈ A[OW2])((A[OW2,R[OW2]) satisfies that w is an unbounded map from o(b) into 0(OW1). Choose α to be least, and then choose w to be least.

Then w can be defined as the unique w such that

(∃u)(∃T)(u is a very strong ordinal,
T is a generalized L-system on u,

OW1 ∈ u, and w ∈ u is least such that
(T/1,T/2) satisfies that w is an unbounded
map from o(b) into o(OW1)).

Note that this definition of x does not mention W2 but uses b as a parameter and mentions W1. Therefore by Lemma IB1.1 iv there is also a unique w′ such that

(∃u)(∃T)(u is a very strong ordinal,
T is a generalized L-system on u,
OW2 ∈ u, and w′ ∈ u is least such that
(T/1,T/2) satisfies that w′ is an unbounded
map from o(b) into o(OW2).

We can view w as defining an unbounded map from b into OW1 and w′ as defining an unbounded map from b into OW2. According to Lemma IB3.7 ii, we see that for all v ∈ α and v′ ∈ OW1, w(b) = v′ ↔ w′(b) = v′. Hence for all v ∈ b, w(b) = w′(b). This is the desired contradiction. ∎

LEMMA IB8.2. The following is provable in T3(W1,W2). o(OW1) is a strongly inaccessible cardinal in (A[OW2],R[OW2]).

Proof: By Lemma IB8.1, it remains to show that o(OW1) is a strong limit. Fix x ∈ OW1 and consider the cardinal of the power set of x according to (A[OW2],R[OW2]). This makes appropriate sense since the latter is a model of ZFC. Now this cardinal is some y ∈ OW2 which is defined without mentioning W1 using the parameter x. Hence by Lemma IB3.7 ii, y ∈ OW1. ∎

Using this technique, we can go somewhat farther - say into higher order inaccessibles. However, to get into indescribable territory we must use V=L in (A[OW2],R[OW2]).

We say that T is a distinguished generalized L-system on x if and only if

i) T is a generalized L-system on x;
ii) (T/1,T/2) has a satisfaction relation;
iii) (T/1,T/2) satisfies ZFC + V=L;
iv) if T′ is a generalized L′-system on x′ then T and T′ are comparable in the sense that:
v) x ∈ x′ or x′ ∈ x or x = x′; and

vi) for all y in both x and x', T[y] = T'[y].

LEMMA IB8.3. The following is provable in T3(W1,W2). The L-system on OW2 is the unique distinguished generalized L-system T on OW2.

Proof: From Lemmas IB7.20 and IB3.2. ∎

In ZFC + V=L, there is the standard construction of a well ordering of the universe, <c. This ordering is defined by transfinite recursion on each L(α), where all elements of L(α) come before all other sets, and where <c is defined on L(α+1) is defined in terms of the least finite sequence of parameters need to define the elements over L(α); the finite sequences of parameters are in turn ordered according to the lexicographic ordering over <c defined on L(α). Thus in order to compare two sets under <c, we need only look at the first place in the constructible hierarchy where these sets appear.

LEMMA IB8.4. The following is provable in T3(W1,W2). Let T be a distinguished generalized L-system for u and T' be a distinguished generalized L-system for u'. Then the <c of (T1,T2) and the <c of (T'1,T'2) are comparable.

Proof: From the above remarks, the comparability of (T1,T2) and (T1',T2') given by the definition of distinguished generalized L-systems carries over to the comparability in question. ∎

LEMMA IB8.5. The following is provable in T3(W1,W2). Let T be a distinguished generalized L-system on u. Furthermore let x ∈ OW2, x ∈ u, y ∈ T/1, and y ⊆ o(x) hold in (T/1,T/2). Then y ∈ A[OW2].

Proof: Suppose this is false and fix x ∈ OW2 be least such that this is false for some u,T. Then x is definable without mentioning W1, and so x ∈ OW1. We claim that there is an ∈-least y such that there exists a distinguished generalized L-system T on some u such that

*) x ∈ u, y ∈ T/1, y ⊆ o(x) holds in (T/1,T/2), and y ∉ A[OW2].

This is by the comparability property of distinguished generalized L-systems. We fix this necessarily unique y. Also fix a strong ordinal u and a distinguished generalized L-system on u such that *) holds. Now since (T/1,T/2) satisfies ZFC + V=L, we see that (T/1,T/2) satisfies "there is a one-one map f from o(x) onto o(OW2)." (This is according to the main facts Gödel established in order to prove the generalized continuum hypothesis in ZFC + V=L). Now within (T/1,T/2) we can minimize this f with respect to <c. According to Lemma IB8.4, the minimized internal f that we obtain in this process is independent of the choice of u and distinguished generalized L-system satisfying *). From this it follows that f is defined without mentioning W1. Now f can also be viewed as a map f′ from x onto OW2. Therefore f′ is defined without mentioning W1. But this contradicts Lemma IB7.9. ∎

LEMMA IB8.6. The following is provable in T3(W1,W2). Let T be a distinguished generalized L-system on u, where OW2 ∈= u. Let x ∈ OW2. Then the V(o(x)) of (T/1,T/2) is the same as the V(o(x)) of (A[OW2],R[OW2).

Proof: By transfinite recursion on x ∈ OW2. The nontrivial case is the successor case y+. Here we take any internal map from V(o(y)) one-one onto an internal ordinal o(x) in order to apply Lemma IB8.5 for subsets of o(x). ∎

Let n ≥ 1 and κ be a cardinal. We say that κ is n-th order indescribable if and only if for all R ⊆ V(κ) and first order sentence φ, if (V(κ+n),∈,R) satisfies φ, then there is an α < κ such that (V(α+n),∈,R ∩ V(α)) satisfies φ.

We say that κ is totally indescribable if and only if κ is n-th order indescribable for all n ≥ 1. This definition agrees with the one given in, say,

A. Kanamori, The Higher Infinite, Springer-Verlag, 1994, p. 59.

It is natural to go a bit further, and so we say that κ is extremely indescribable if and only if for all R ⊆ V(κ) and first order sentence φ, if (V(κ+κ),∈,R) satisfies φ, then there is an α < κ such that (V(α+α),∈,R ∩ V(α)) satisfies φ.

LEMMA IB8.7. The following is provable in T3(W1,W2). o(OW1) is an extremely indescribable cardinal in (A[OW2],R[OW2]).

Proof: Suppose this is false. Then there exists x, and sentence φ such that the following holds in (A[OW2],R[OW2]):

x ⊆ V(o(OW1)) and (V(o(OW1)+o(OW1)),∈,x) satisfies φ,

but for all z ∈ o(OW1),

(V(o(z)+o(z)),∈,x ∩ V(o(z))) satisfies ¬φ.

Choose φ to have least index, and then x ∈ A[OW2] to be <c-least with this property.

Now x can be defined as the unique x such that

(∀u)(∀T)(if T is a distinguished generalized L-system on u then (∃u')(∃T')(u ∈= u' & T' is a distinguished generalized L-system on u', OW1 ∈ u', and x ∈ u' is <c-least such that: (T'/1,T'/2) satisfies "x ⊆ V(o(OW1)) and (V(o(OW1)+o(OW1)),∈,x) satisfies φ," but for all z ∈ OW1, T'/1,T'/2) satisfies "(V(o(z)+o(z)),∈,x ∩ V(o(z))) satisfies ¬φ")).

To see that this definition is correct, we first check that this statement holds of the actual x. Let T be a distinguished generalized L-system on u. If OW2 ∈= u then we set u' = u and T' = T. Otherwise, we set u = OW2 and T = the L-system on OW2. By the suitable preservation of <c and the V's (Lemma IB8.6) as well as ordinal addition (left to the reader), we see that the statement holds of the actual x.

Now suppose the above statement holds of an x. We set u = OW2 and T = the L-system on OW2. Then argue by preservation as above in order to obtain that this x is the actual x.

Note that this definition of x does not mention W2 but uses φ as a parameter and mentions W1. Therefore by Lemma IB1.1 iv there is also a unique x′ such that

(∀u)(∀T)(if T is a distinguished generalized L-system on u then (∃u′)(∃T′)(u ∈= u′ & T′ is a distinguished generalized L-system on u′, OW1 ∈ u′, and x′ ∈ u′ is <c-least such that: (T′/1,T′/2) satisfies "x′ ⊆ V(o(OW1)) and (V(o(OW1)+o(OW1)),∈,x′) satisfies φ," but for all z ∈ OW1, T′/1,T′/2) satisfies "(V(o(z)+o(z)),∈,x′ ∩ V(o(z))) satisfies ¬φ")).

From the original definition of x, we have (A[OW2],R[OW2]) satisfies "x ⊆ V(o(OW1)) & (V(o(OW1)+o(OW1)),∈,x) satisfies φ."

From the above definition of x′, we can set z = OW1 and obtain (T′/1,T′/2) satisfies "(V(o(OW1)+o(OW1)),∈,x′ ∩ V(o(OW1)) satisfies ¬φ." By choosing u = OW2, we have OW2 ∈ u′, and so by Lemma IB8.6, we see that (A[OW2],R[OW2]) satisfies "(V(o(OW1)+o(OW1)),∈,x′ ∩ V(o(OW1)) satisfies ¬φ."

We have our required contradiction provided we can show that in (A[OW2],R[OW2]), the equation x′ ∩ V(o(OW1)) = x holds. Now x′ is defined in exactly the same way as x except with W1 replaced by W2 (and there is no mention of W2 in the definition of x). Now by the strong inaccessibility of o(OW1) in (A[OW2],R[OW2]) given by Lemma IB8.2, we see that in (A[OW2],R[OW2]), the elements of V(o(OW1)) are externally elements of OW1. The equation now follows from Lemma IB3.7 ii. □

THEOREM IB8.8. The following is provable in T3(W1,W2). (A[OW2],R[OW2]) satisfies ZFC + V=L + "there exists an extremely indescribable cardinal." The following is provable in T2. ZFC + V= L + "there exists an extremely indescribable cardinal" is consistent. The following is provable in EFA (exponential function arithmetic). If T2 is consistent then ZFC + V=L + "there exists an extremely indescribably cardinal" is consistent.

Proof: By Lemmas IB7.20, IB8.7, and IB7.16. We also need the interpretation of T3(W1,W2) in T2(W1,W2) given in section A. T2 also has an adequate way of discussing arithmetic. ∎

We would like to replace (A[OW2],R[OW2]) by a set model. We can either go the route of relativizations, avoiding a discussion of satisfaction relations. Or we can consider satisfaction relations. Here we have to be careful in that it is not assumed that the underlying set is made up of ordinals. So the finite sequence apparatus we have developed cannot be used. For this reason, we prefer to go the relativization route.

LEMMA IB8.9. The following is provable in T3(W1,W2). There is a unique function f such that

i) dom(f) = A[OW2];
ii) for all x,y ∈ A[OW2], <x,y> ∈ R[OW2] ↔ x ∈ y;
iii) rng(f) exists and is a transitive set.

Proof: Uniqueness is clear by transfinite induction and the fact that (A[OW2],R[OW2]) satisfies extensionality. We now prove that for all x ∈ OW2 there is a function f such that

i) dom(f) = A[x];
ii) for all y,z ∈ A[x], <y,z> ∈ R[OW2] ↔ f(y) ∈ f(z);
iii) r,rng(f) ∈ W2, and rng(f) is transitive.

Suppose this is false and let x ∈ OW2 be least such that it is false. Then x is defined without mentioning W1, and so x ∈ OW1. By Lemma IB6.5 ii, for all y ∈= x, A[y] ∈ W1. Now by induction hypothesis, for all y ∈ x, the unique f satisfying I)-iii) lies in W2, as well as its range. By axiom 3, for all y ∈ x, the unique f satisfying I)-iii) lies in W1. Hence for all y ∈ x, the unique f satisfying i)-iii) lies in W1, as well as its range. Clearly x ≠ 0. Suppose x is a limit. Then the union of the f's associated with the y ∈ x defines a subset of W1 without mentioning W1. Hence it exists, and satisfies i)-iii) for x. This is the required contradiction.

Now suppose x = y+ and let f satisfy i)-iii) with y. Then f ∈ W1. We want to extend f to the domain A[x+]. Now each new element w of A[x+] is to be sent to a unique corresponding subset of rng(f). This unique corresponding subset Bw of rng(f) is defined in the obvious way and lies in W1 according to Lemma IB3.7 iii. But we also need to know that each <w,Bw> ∈ W1. To see this, suppose w is the ∈-least element of A[x+] such that <w,Bw> ∉ W1. Then w is defined without mentioning W2. Hence by axiom 3, w is defined without mentioning W1. Then we can form {w,Bw} as a subset of W1 defined without mentioning W1, and hence it exists. Now consider the statement about w that {w,Bw} ∈ W2. This statement does not mention W1. Hence {w,Bw} ∈ W1. The same argument works to show that {w} ∈ W1 (something we already knew). Now we can repeat the argument to obtain the existence of {{w},{w,Bw}} = <w,Bw> and see that it lies in W1. Hence we can define the set of all such pairs, together with the pairs in f, as a subset of W1 without mentioning W1. This is the extension of f that satifies I)-iii) for x = y+ and it lies in W2. This is the required contradiction.

Finally, we can define the union of all the f's for the various x ∈ OW2 as a subset of W2 without mentioning W1. This is the required function, and it and its range exists by Lemma IB1.1 ii.

THEOREM IB8.10. i) let φ be any conjunction of theorems of ZFC + V=L + "there exists an extremely indescribable cardinal." The following is provable in T2. There is a set x such that the relativization of φ to x holds. The set can be taken to be a nonempty transitive set.

ii) there is a single explicit definition {x: ψ}, where ψ is in the language of T2, which provably defines a set in T2, such that the following holds: let φ be any conjunction of theorems of ZFC + V=L + "there exists an extremely indescribable cardinal." Then the relativization of φ to {x: ψ} is provable in T2. Furthermore, we can insist that {x: ψ} is provably transitive in T2.

Proof: Lemma IB8.9 provides an isomorphism between (A[OW2],R[OW2]) and a transitive set. Apply Theorem IB8.8. We also use the interpretation of T3(W1,W2) in T2 as given in section A. □

We make the following definitions in ZFC. Let κ be a cardinal. We write S(κ) for the set of all subsets of κ. We say that f:κ → S(κ) is regressive if and only if for all α < κ, f(α) ⊆ α. We say that A ⊆ κ is closed unbounded if and only if the union of any bounded subset of A with no greatest element lies in A, and A is unbounded in κ. We say that κ is a subtle cardinal if and only if for all stationary A ⊆ κ and regressive f:κ → S(κ), there exists α < β ∈ A such that f(α) = f(β). This definition is due to Kunen (unpublished).

Subtle cardinals are inaccessible. The first subtle cardinal is known to be greater than the first weakly compact cardinal, and even greater than the first extremely indescribable cardinal. It is smaller than the first cardinal which arrows ω, where we partition all of the finite subsets into finitely many pieces.

THEOREM IB8.11. The following is provable in ZFC + "there exists a subtle cardinal." There are three transitive sets x ∈ y ∈ z such that every theorem of T2 holds in (z,∈) with W1 interpreted as x and W2 interpreted as y. T2 is consistent. The following is provable in EFA. If ZFC + "there exists a subtle cardinal" is consistent then T2 is consistent.

Proof: Let κ be a subtle cardinal. Define f:κ → S(κ) by f(α) = {<x,n>: x ∈ V(α) & n is the index of a sentence true in (V(κ),∈) about x,V(α)} provided α is a limit ordinal; 0 otherwise. Let A ⊆ κ be the set of all limit ordinals < κ. By the subtlety of κ, fix α < β < κ to be limit ordinals such that f(α) = f(β). Then set x = V(α), y = V(β), and z = V(κ). □

PART II. WITHOUT EXTENSIONALITY