SUM BASE TOWERS, INVARIANT MAXIMAL CONTINUATIONS, AND CONCRETE INCOMPLETENESS EXTENDED ABSTRACT

Harvey M. Friedman*
August 2, 2013

*This research was partially supported by an Ohio State University Presidential Research Grant and by the John Templeton Foundation grant ID #36297. The opinions expressed here are those of the author and do not necessarily reflect the views of the John Templeton Foundation.

Abstract. In this extended abstract, we present an account of new examples of concrete mathematical statements that can be proved from large cardinal hypotheses but not in the usual ZFC axioms for mathematics (assuming ZFC is consistent). Many of the statements are provably equivalent to Π^0_1 sentences (purely universal statements, logically like FLT) - in particular the consistency (not the 1-consistency) of strong set theories. Some are in explicitly Π^0_1 form. The examples are thematic, and fall into two major groups. The first is sum base towers, which are of finite length, made up of sets of positive integers (both finite and infinite sets are treated). These basic objects are motivated through an analogy with elements in physical science. They are related to towers in Boolean Relation Theory, but are more concrete and combinatorial, and involve only the ordering of Z^+ and addition on Z^+. We present a substantial theory of sum base towers for both arbitrary $R \subseteq Z^{+k}$ and tame $R \subseteq Z^{+k}$. Tameness here is taken to mean integral piecewise linear relations or the more extensive Presburger relations. These substantial theories cannot be carried out in ZFC, but can be carried out using certain large cardinal hypotheses, represented by the SRP axiom system. If unrestricted $R \subseteq Z^{+k}$ are used then the results are mostly equivalent to 1-Con(SRP), and often have simple Π^0_2 or Π^0_3 forms. If tame $R \subseteq Z^{+k}$ are used then the results are mostly equivalent to Con(SRP), and often have simple Π^0_1 forms. The second major group of examples lives in the rationals, with only order. The basic shape of the results assert that any finite set of negative rational vectors has a "maximal continuation" which is invariant under certain shift operators (they add 1 to only some coordinates, leaving others fixed). These and many other statements are shown to be provably equivalent to the widely believed Con(SRP), and hence unprovable in ZFC (assuming ZFC is consistent). Modifications are made to "maximal continuations" which allow for stronger invariance properties. One version is particularly strong, and a simple cross sectional condition is added which propels the statement beyond the huge cardinal hierarchy to be equivalent to Con(HUGE). Another kind of weakening of maximality which we call source maximality, supports a series of finite source maximal continuations of a negative set of rational vectors. These are explicitly Π^0_2 and become explicitly Π^0_1 because a relevant and easy exponential type bound. This also applies to statements corresponding to Con(HUGE). We also present some nondeterministic constructions of infinite and finite length with the same metamathematical properties. These lead to practical computer investigations designed to provide arguable confirmation of Con(ZFC) and more.

1. Introduction.

This extended abstract reports on the state of the art in Concrete Mathematical Incompleteness. For a detailed discussion of Concrete Mathematical Incompleteness, see the Introduction in [Fr14].

The previous report on the state of the art is [Fr13], which was based on invariant maximal cliques in order invariant graphs. Much of that approach is present here in section 5.

There are two major new initiatives introduced here. These are sum base towers, and invariant maximal continuations (and related concepts).

The sum base towers are certain towers of subsets of Z+ associated with a given relation R ⊆ Z+k. The key definition is rather simple - so simple that for many readers, it does not require motivational discussion. However, we have presented a significant amount of motivating material in section 2.1 (also see the introductory material for section 2). We give the definition of sum base towers here for the reader's convenience.

DEFINITION 2.2. Let $R \subseteq Z^{+k}$ and $S \subseteq Z^+$. n is R related to S if and only if $(\exists m_1,\ldots,m_{k-1} \in S)(R(n,m_1,\ldots,m_{k-1}))$.

DEFINITION 2.1.4. Let $R \subseteq Z^{+k}$ and $1 \leq t \leq \infty$. A sum base tower of length t for R consists of a sequence of sets $S_1 \subseteq S_2 \subseteq \ldots \subseteq Z^+$ of length t, where
i. $1 \in S_1$.
ii. For all $i < t$ and $n,m \in S_i$, n+m is either in S_{i+1} or is R related to $S \cap [1,n+m)$, but not both.

In section 2.1, we first motivate the yet simpler notion of base for R with an analogy with atomic elements in physical science. Every $R \subseteq Z^{+k}$ has a unique base. We calculate the unique base for some particularly simple R. We suggest the project of calculating the unique base for other simple R. We present an algorithmic unsolvability result that limits the scope of calculations of bases.

Then the weaker notion of sum base for R is introduced, also motivated with a further physical science analogy. The sum base pairs are then introduced as a weakening, or staggering, of sum bases. In the setup, if (A,B) is a sum base pair then A ⊆ B. Also (A,A) is a sum base pair if and only if A is a sum base.

In section 2.1, a substantial list of theorems are presented to the effect that every R ⊆ Z^{+k} has a sum base pair of a certain kind. The move to sum base pairs from sum bases (and bases) is essential here, since not every R ⊆ Z^{+k} has a sum base of these kinds. In fact, there does not appear to be any interesting theorems to the effect that every R ⊆ Z^{+k} has a sum base of a certain kind - except the basic result that every R ⊆ Z^{+k} has an infinite sum base.

The sum base towers are introduced in the most obvious way by iterating the sum base pair concept.

Section 2.1 concludes with the obvious question: can the substantial lists of theorems about sum base pairs be extended to sum base towers of longer finite length?

In section 2.2, we see that the answer is yes, but only if we go well beyond ZFC. In the list, only the most basic result - the existence of a sum base pair starting with an infinite set - can be extended to sum base towers of finite length if we are working in ZFC. None of the other results about sum base pairs lift within ZFC - yet they all lift if we extend ZFC by certain large cardinal hypotheses.

We suspect that there is a deep underlying principle here - that all "simple" statements about the existence of sum base pairs whose first term is infinite lift to sum base towers. But of course we have seen that such a transfer theorem can only be proved by going far beyond the usual ZFC axioms for mathematics.

To summarize, it is clear that a proper investigation of sum base towers cannot be conducted within ZFC, but can be conducted quite far using certain large cardinal hypotheses - or at least their 1-consistency.

Here are some more details about section 2. We only consider sum base towers of finite length, where the sets are all finite, or the sets are all infinite. The former are called finitary, and the latter are called infinitary.

Sections 2.2 - 2.7 consists of presentations of statements of the following forms:

1. Every $R \subseteq Z^{+k}$ has an infinitary sum base tower of finite length with some simple properties.
2. Every tame $R \subseteq Z^{+k}$ has an infinitary sum base tower of finite length with some simple properties.
3. Every $R \subseteq Z^{+k}$ has a finitary sum base tower of finite length with some simple properties.
4. Every tame $R \subseteq Z^{+k}$ has a finitary sum base tower of finite length with some simple properties.

Five of the statements are highlighted with names, which are, respectively, of forms 1,2,3,3,4.

INFINITE EVEN INTEGER TOWER "THEOREM" (IEIT). For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower of length t whose starting set contains an even integer.

INFINITE GEOMETRIC PRGRESSION TOWER "THEOREM" (IGPT). For all k,t, every integral piecewise linear $R \subseteq Z^{+k}$ has a sum base tower of length t starting with some $\{1,r,r^2,...\}$, $r > 1$.

FINITE GEOMETRIC GROWTH TOWER "THEOREM" (FGGT). Let $R \subseteq Z^{+k}$ and $r,p > (8kt)!!$. R has a finitary sum base tower of length t, with base r geometric growth, starting with a set of p odd integers.

FINITE SIMILARITY TOWER "THEOREM" (FST). For all k,t, every $R \subseteq Z^{+k}$ has a finitary sum base tower $\{1 < n < m < r\} = S_1 \subseteq ... \subseteq S_t$, where for all $i < n$ from St, $n+m+i \in S_t \leftrightarrow m+r+i \in S_t$.

FINITE GEOMETRIC PRGRESSION TOWER "THEOREM" (FGPT). Let $R \subseteq Z^{+k}$ be integral piecewise linear and $r,p > (8ktIPL(R))!!$. R has a sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$, which lives in $[1,r^{p+1}]$.

Note that FGPT is in explicitly Π^0_1 form. The double factorial expressions in FGGT and FGPT are merely convenient and safe.

In section 2.6, we explore the effect of restricting the statements in sections 2.4 and 2.5 to tame $R \subseteq Z^{+k}$. Basically, the statements go from being equivalent to 1-Con, to being equivalent to Con. Also for the tame case, estimates can be placed to create explicitly Π^0_1 forms.

In section 2.7 we give explicitly Π^0_2 forms of most of the results in sections 2.4 and 2.5, including FGGT and FST.

In section 2.8 we consider the effect of allowing higher length sums in the definition of sum base towers. This does not change any results. However, it does allow us to fix the lengths of sum base towers to $t = 3$, without changing any results. So the step from sum base pairs to sum base triples is unexpectedly profound.

We view bases, sum bases, sum base pairs, and sum base towers are fundamental and inevitable. We propose to back up this belief formally by proving that they are the unique "simple" concepts that have certain "simple" properties. But this is for the future.

The second major new initiative is invariant maximal continuations (and related concepts).

DEFINITION 3.1.1. S is a continuation of $E \subseteq J^k$ if and only if $E \subseteq S \subseteq J^k$, and every element of $S^2 \subseteq Q^{2k}$ is order equivalent to some element of $E^2 \subseteq Q^{2k}$. S is a maximal continuation of $E \subseteq J^k$ if and only if S is a continuation of $E \subseteq J^k$ such that for all $x \in J^k \setminus S$, $S \cup \{x\}$ is not a continuation of $E \subseteq J^k$.

See Definitions 3.1 and 3.4.

Note that the maximality here is simply the usual maximality with respect to inclusion, or inclusion maximality.

INVARIANT MAXIMAL CONTINUATION "THEOREM" (IMC). For all k,n, every finite negative $E \subseteq Q_{\leq n}{}^k$ has a ≥ 0,N shift invariant maximal continuation.

Here "negative" means "all coordinates of elements are < 0". We need to explain the invariance condition.

There are many ways to formulate invariance, and we have picked a convenient one that supports a considerable amount of uniformity over the ambient spaces $Q\leq^{nk}$.

DEFINITION 3.6. (Abbreviated). Let f be any function. $S \subseteq X$ is f invariant if and only if for all $x,f(x) \in X$, $x \in S \leftrightarrow f(x) \in S$.

This allows us to use functions $f:Q^* \to Q^*$ for the invariance, where Q^* is the set of all finite sequences from Q. The most natural such map is the shift which simply adds 1 to all coordinates. However, invariance under the shift turns out to be inappropriate for our purposes.

In IMC, we use a variant of the shift operator, where we add 1 to some coordinates and leave the remaining coordinates fixed.

DEFINITION 3.1.4. (Abbreviated). The ≥p,N shift of $x \in Q^*$ results from adding 1 to all coordinates of x that are ≥p, provided they are all integers.

THEOREM 3.1.4. (Abbreviated). IMC is provably equivalent to Con(SRP) over WKL_0. IMC is provably equivalent to an explicitly Π^0_1 sentence via the Gödel Completeness Theorem.

So in particular, IMC is provable in SRP+, but not in ZFC (assuming ZFC is consistent).

There is nothing special about pairs in the definition of continuation. We would have the same results if we used m-tuples, for any fixed m. I.e., we can use a multiple form of continuation. This remark applies throughout the entire abstract.

We have not settled the status of IMC with ambient space Q^k instead of ambient space $Q_{\geq n}{}^k$. We know that it is provable in WKL_0 + Con(SRP), but it may be provable in ZF, or far less. This open question compelled us to use ambient spaces J^k, J a rational interval, instead of just Qk. However, in sections 3.2 - 3.4, we are able to work in the single ambient space $Q_{\geq -1}{}^k$ and achieve equivalence with Con(SRP).

In section 3.2, we use the same notion of continuation for subsets of Q^k. However, the notion of maximality is strengthened from inclusion maximality.

DEFINITION 3.2.1. S is a ceiling maximal continuation of $E \subseteq J^k$ if and only if S is a continuation of $E \subseteq J^k$ such that for all $x \in Jk \backslash S$, $S{\leq}\lceil x\rceil \cup E \cup \{x\}$ is not a continuation of $E \subseteq Jk$.

See Definitions 3.2 and 3.3.

INVARIANT CEILING MAXIMAL CONTINUATION THEOREM (ICMC). For all k, every finite negative $E \subseteq Q_{\geq -1}k$ has a ≥0,N shift invariant ceiling maximal continuation.

THEOREM 3.2.3. (Abbreviated). ICMC is provably equivalent to Con(SRP) over WKL_0. ICMC is provably equivalent to an explicitly Π^0_1 sentence via the Gödel Completeness Theorem.

What are we to make of the fact that we are keeping the same notion of continuation but strengthening inclusion maximality? At first glance, this would appear to be undesirable. However, the whole point of maximality is that something bad happens if we add any excluded point from the ambient space. The weakest bad thing that can happen is that we no longer have a continuation. But it is interesting to demand that if we add any excluded point from the ambient space, even worse things happen.

In section 3.3, we maintain the same notion of continuation for subsets of Q^k, but modify the maximality notion in a way that is incomparable with inclusion maximality and ceiling maximality. We begin with the very natural naïve maximality (naïve inductive), which is stronger than ceiling maximality.

DEFINITION 3.3.1. S is a naive inductive continuation of $E \subseteq Q^k$ if and only if S is a continuation of $E \subseteq Q^k$ such that for all $x \in Q^k \setminus S$, $S_{<x} \cup E \cup \{x\}$ is not a continuation of $E \subseteq Q^k$.

Theorem 3.3.1 tells us that naïve inductive continuations are far too strong. We then weaken naïve inductive continuations to inductive continuations, using the # construction. Let $S \subseteq Q^k$. S# consists of the vectors that built out of S. Officially, $S\# = (\text{fld}(S) \cup \{0\})^k$, where fld(S) is the set of all coordinates of elements of S.

DEFINITION 3.3.2. S is an inductive continuation of $E \subseteq Q^k$ if and only if S is a continuation of $E \subseteq Q^k$ such that for all $x \in S\#\setminus S$, $S_{<x} \cup E \cup \{x\}$ is not a continuation of $E \subseteq Q^k$.

Thus if S is an inductive continuation, and a new vector is added that is built out of S, then something bad happens involving x and a vector lower down. (Vectors lower down if we use a multiple form of continuation).

INVARIANT INDUCTIVE CONTINUATION THEOREM (IIC). Every finite negative $E \subseteq Q^k$ has an inductive continuation containing its ≥ 0 shift.

Here ≥ 0 shift adds 1 to all nonnegative coordinates.

THEOREM 3.3.4. IIC is provably equivalent to Con(SRP) over WKL_0. IIC is provably equivalent to an explicitly Π^0_1 sentence via the Gödel Completeness Theorem.

Let's reconsider the use of continuations of subsets of Q^k, but with modified notions of maximality. Inclusion maximality takes the form: S is a continuation of $E \subseteq Q^k$ such that

$$(\forall x \in Qk\backslash S)(S \cup \{x\} \text{ is not a continuation of } E)$$
$$(\forall x \in Qk\backslash S)(\alpha(E,S,x) \text{ is not a continuation of } E)$$

where α is a Boolean combination of E,S, and sets simply associated with x. We propose to completely determine the status of IMC for all such α.

The choice of sets simply associated with x is crucial here. If we use only S,{x}, then we have covered inductive maximality. We can be a little more ambitious and widen the class to E,S,{x} and {y: u < v}, where u,v are among max(x),max(y). We propose to completely determine the status of ICMC for all such α.

For ceiling maximality, we use E,S,{x}, and {y: u < v}, where u,v are among max(x),max(y),$\lfloor x\rfloor$,$\lfloor y\rfloor$,$\lceil x\rceil$,$\lceil y\rceil$. We only need max(y),$\lceil x\rceil$. We propose to completely determine the status of ICMC for all such α.

Inductive maximality takes the form:

$$(\forall x \in S\#\backslash S)(S_{<x} \cup E \cup \{x\} \text{ is not a continuation of } E)$$
$$(\forall x \in S\#\backslash S)(\alpha(E,S,x) \text{ is not a continuation of } E)$$

where α is a Boolean combination of E,S, and sets simply associated with x. We can use E,S,{x}, and {y: u < v}, where u,v are among max(x),max(y),$\lfloor x\rfloor$,$\lfloor y\rfloor$,$\lceil x\rceil$,$\lceil y\rceil$. We only need max(x),max(y). We propose to completely determine the status of IIC for all such α.

We now come to section 3.4. It seems premature to seriously try to give a corresponding general treatment of inductive ≤=continuations at this time.

In section 3.4, we modify continuations of subsets of Q^k in order to give special status to $x \in Q^{k\neq} = \{x \in Q^k: x_1,\ldots,x_k$ are distinct}. They are treated very differently than the $x \in Q^k\backslash Q^{k\neq}$.

The Invariant Inductive ≠-Continuation "Theorem", II≤C, imposes a convenient invariance condition weaker than that used in IIC, but imposes a cross sectional condition involving fixing the first k-1 coordinates to be $(3/2)^n$. This cross sectional condition has the effect of making the elementary embedding to be internal rather than external. The elementary embedding to be is +1 on the nonnegative elements of the field of S. External elementary embeddings of models of set theory correspond to SRP, whereas Internal elementary embeddings of models of set theory correspond to HUGE.

In section 3.5, we use one dimensional embeddings for the invariance conditions instead of f:Q* → Q*. We begin with the definition of the order theoretic subsets of Q^k, and give a characterization of the order theoretic partial f:Q → Q. We then define "f is an embedding of S ⊆ J^k" and introduce the following order theoretic f, assuming A is finite.

DEFINITION 3.5.2. Let A ⊆ Q. sh[J,A] is the partial function from J into J defined by sh[J,A](p) = p if p ∈ $J_{<\min(A)}$; p+1 if p ∈ J ∩ A and p+1 ∈ J; undefined otherwise. By convention, min(∅) = -∞. If A is finite then we sometimes replace A with an enumeration of A.

We observe that S ⊆ Jk is ≥0,N invariant if and only if S ⊆ J^k is sh(J,N) invariant. We restate IMC, ICMC, IIC as follows.

EMBEDDED MAXIMAL CONTINUATION "THEOREM" (EMC). For all k,n, every finite negative E ⊆ $Q_{\leq n}$k has a sh[$Q_{\leq n}$,0,...,n] embedded maximal continuation.

EMBEDDED CEILING MAXIMAL CONTINUATION THEOREM (ECMC). For all k, every finite negative E ⊆ $Q_{\geq -1}$k has a sh[$Q_{\leq n}$,0,...,n] embedded ceiling maximal continuation.

EMBEDDED INDUCTIVE CONTINUATION THEOREM (EIC). Every finite negative E ⊆ Q^k has a sh[$Q_{\leq n}$,0,...,n] embedded inductive continuation.

These are provably equivalent to Con(SRP) over WKL_0.

In section 3.6, we give some necessary and some sufficient conditions on an order theoretic partial f:J → J so that it can be used for embedded maximal continuations in J^k. Strictly increasing is necessary, no matter what rational interval J is. Strictly increasing and continuous is sufficient if J is not closed. If J is a nontrivial closed interval then strictly increasing, continuous, and no iterate of m maps one endpoint to a different endpoint, is sufficient. In all cases, the set E ⊆ Jk must be finite and have all of its elements fixed by f.

We do not have a complete characterization of the order theoretic partial f:J → J that can be used here. We created the following Template.

TEMPLATE 3.6.6. Given order theoretic partial f:J → J. For all k, every finite E ⊆ J^k, where all coordinates of elements are fixed by f, has an f embedded maximal continuation.

CONJECTURE 3.6.7. Every instance of Template 3.6.6 is provable or refutable in SRP.

In section 4, we work with finite series of successive finite continuations of negative E. The difficulty of course is that a finite negative E ⊆ Q^k may have no finite maximal continuation. However, finite inductive continuations of E do exist, but there may be none that are ≤0,N invariant, or contain either ≥0 shift.

We use source maximal, source ceiling maximal, source inductive, and source ≠-continuations. These are weaker than maximal, ceiling maximal, inductive, and ≠-continuations. The idea here is that Q^k\S is replaced by E#\S, and S#\S is replaced by E#\S. Here are the main statements.

FINITE INVARIANT MAXIMAL CONTINUATION "THEOREM" (FIMC). For all k,n,t, every finite negative E ⊆ $Q_{\geq n}$k has t successive finite source maximal continuations that are ≥0,N shift invariant.

FINITE INVARIANT CEILING MAXIMAL CONTINUATION "THEOREM" (FICMC). For all k,t, every finite negative E ⊆ $Q_{\geq -1}{}^k$ has t successive finite source ceiling maximal continuations that are ≥0,N shift invariant.

FINITE INVARIANT INDUCTIVE CONTINUATION "THEOREM" (FIIC). For all k,t, every finite negative E ⊆ Q^k has t successive finite source inductive continuations, where the ≥0 shift of each continuation is a subset of all successive continuations.

FINITE INVARIANT INDUCTIVE ≠-CONTINUATION "THEOREM" (FII≠C). Every finite negative $E \subseteq Q^k$ has t successive finite source inductive ≠-continuations, $S_1,\ldots,S_t$, where $S_t+1 \cap Q(0,t)^k$ is a subset of St with field $St<(3/2)^t>$.

The first three are provably equivalent to Con(SRP) over EFA. The fourth is provably equivalent to Con(HUGE) over EFA.

In section 5, we take up E followers, E sets, and graphs. Followers, maximal followers, ceiling maximal followers, inductive followers, and inductive ≠-followers are like continuations, maximal continuations, ceiling maximal continuations, inductive continuations, and inductive ≠-continuations except that the requirement $E \subseteq S$ is dropped. We still demand that every $x \in S^2$ is order equivalent to some $y \in E^2$. Dropping $E \subseteq S$ allows us to use arbitrary subsets E of J^k, $Q_{\geq -1}{}^k$, Q^k, Q^k, respectively.

E sets, maximal E sets, ceiling maximal E sets, inductive E sets, and ≠-inductive E sets are the same, except that $E \subseteq Q^{2k}$, and we demand that every $x \in S^2$ is order equivalent to some $y \in E$. We can use arbitrary subsets E of J^{2k}, $Q_{\geq -1}{}^{2k}$, Q^{2k}, Q^{2k}, respectively.

The graph formulations are essentially the same as the E set formulations. The key observation is that the cliques in an order invariant graph on Jk are the same as the E sets, where E is the edge set of the graph together with the diagonal. The graph theoretic formulations are explicitly presented in section 5.

In section 6, we take two of our statements, and present a corresponding infinite length nondeterministic algorithm for each, with an eye toward the section 7 computer investigations. These are Proposition 3.3.3 (E sets) and II≠/Eset. These correspond to SRP and HUGE, respectively.

For the purposes of the computer investigations in section 7, it is important to have some control on the search space. This involves restricting the possible choices that need to be considered in the computer searches.

In section 7, we discuss the implementation of the nondeterministic algorithms presented in section 6. Resource issues are addressed.

SRP and HUGE, respectively, are used in an apparently essential way to prove that the two nondeterministic algorithms can be navigated for a definite small finite number of steps. At this point, we rely on heuristics for this claim. If an appropriately exhaustive search for such a limited navigation comes up negative, then this would establish the inconsistency of SRP and HUGE, respectively. Hence if a search comes up positive, then some sort of confirmation appears to be taken place that at least substantial fragments of SRP and substantial fragments of HUGE are consistent.

But what is the nature of this arguable confirmation? We refer the reader to the last three paragraph of section 7 for some relevant thoughts.

Throughout the abstract, we have highlighted those statements that are equivalent in a weak system to an explicitly Π^0_1 sentence, and each case the explicitly Π^0_1 sentence takes the form of the consistency of a standard formal system. These include almost all of the statements presented in section 2 involving tame R, none of the ones involving all R, and all statements in sections 3-7.

There is a vivid way of restating this property without referring to Π^0_1 sentences.

DEFINITION 1.1. A statement φ has the refutation property if and only if ACA_0 proves that: if φ is false then RCA_0 proves that φ is false.

Here we have chosen the reasonably weak system ACA_0 and the yet weaker system RCA_0 for this purpose.

THEOREM 1.2. All of the statements presented here that were claimed to be provably equivalent to consistency (not 1-consistency) statements, have the refutation property.

2. Sum base towers.

DEFINITION 2.1. Z,Z^+,N are the set of all integers, positive integers, and nonnegative integers, respectively. We use k,n,m,i,j,r,s,t for positive integers, unless indicated otherwise.

DEFINITION 2.2. Let $R \subseteq Z^{+k}$ and $S \subseteq Z^+$. n is R related to S if and only if $(\exists m_1,\ldots,m_{k-1} \in S)(R(n,m_1,\ldots,m_{k-1}))$.

We will draw stronger conclusions when we hypothesize that $R \subseteq Z^{+k}$ is tame in either of the following two senses.

DEWFINITION 2.3. $R \subseteq Z^k$ is integral piecewise linear if and only if R is a Boolean combination of integral half planes

$$\{(n_1,\ldots,n_k) \in Z^k: c_1n_1 + \ldots + c_kn_k > d\}, \text{ where } c_1,\ldots,c_k,d \in Z.$$

DEFINITION 2.4. $R \subseteq Z^k$ is Presburger if and only if R is a Boolean combination of integral half planes and congruence sets

$$\{(n_1,\ldots,n_k) \in Z^k: c|n_i\}, \text{ where } 1 \le i \le k \text{ and } c \in Z.$$

DEFINITION 2.5. Let $R \subseteq Z^{+k}$ be integral piecewise linear. IPL(R) is the least s such that R can be represented as in Definition 2.3 above with coefficients from [-s,s]. Let $R \subseteq Z^s$ be Presburger. PRES(R) is the least s such that R can be represented as in Definition 2.4 with coefficients from [-s,s].

Presburger sets arise in the following important application of quantifier elimination in model theory.

THEOREM 2.1. $R \subseteq Z^k$ is Presburger if and only if R is first order definable over (Z,<,+). $R \subseteq Z^{+k}$ is Presburger if and only if R is first order definable over (N,+).

We highlight five statements by giving them titles, which are also the titles of sections 2.2 - 2.6. These five statements are all provable in ZFC extended by certain large cardinal hypotheses, and not provable in ZFC alone (assuming ZFC is consistent).

The first two of these highlighted statements use infinite sets in the hypothesis and in the conclusion. The first is provably equivalent to 1-Con(SMAH) over ACA', and the second is provably equivalent to Con(SMAH) over ACA'.

The third and fourth of these highlighted statements use infinite sets in the hypothesis but only finite sets in the conclusion. The third is provably equivalent to 1-Con(SMAH) over ACA', and the fourth is trapped between two 1-Con statements above 1-Con(ZFC).

We eliminate all uses of infinite sets in the third and fourth highlighted statements in section 2.8, using a standard method for giving explicitly $\Pi^0{}_2$ forms via compactness, or an infinite finitely branching tree argument.

The fifth highlighted statement is already explicitly $\Pi^0{}_1$, and is provably equivalent to Con(SMAH) over ACA'.

2.1. Bases, sum bases, sum base pairs, sum base towers.

DEFINITION 2.1.1. S is a base for $R \subseteq Z^{+k}$ if and only if
i. $S \subseteq Z^+$.
ii. For all n, $n \in S$ or n is R related to $S \cap [1,n)$, but not both.

The idea is that every n is either accepted as a base element, or is replaced by lesser R related base elements.

As a motivation, think of atomic elements in physical science. Every substance is either accepted as an atomic element, or is broken down into atomic elements, but not both. I.e., if a substance is broken down into atomic elements, then it is not regarded as an atomic element.

THEOREM 2.1.1. Every $R \subseteq Z^{+k}$ has a unique base. For $k \geq 1$, the unique bases for the various $R \subseteq Z^{+k}$ are the subsets of Z^{+k} containing 1.

We have calculated the unique base for some very simple R's.

THEOREM 2.1.2. The unique base for the relation Z^{+k} is {1|. The unique base for the relation $R(n_1,...,n_{k+1}) \leftrightarrow n_1,...,n_{k+1} \in Z^+ \wedge n_1 = n_2+...+n_{k+1}$, is {$n \in Z^+$: the residue of n modulo k^2-k is among 1,...,k-1}. The unique base for the relation $R(n,m,r) \leftrightarrow n,m,r \in Z^+ \wedge n = m \bullet r$, is the set of all primes together with 1.

The calculation of the unique base, even for simple $E \subseteq Z+^k$, seems rather difficult. Here is an impossibility theorem.

THEOREM 2.1.3. There exists a "small" k such that the following holds. The question of whether the base of a given tame subset of Z^{+k} is finite is algorithmically unsolvable. The question of whether two tame subsets of Z^{+k} have the same base is algorithmically unsolvable.

Here we can use either integral piecewise linear or Presburger R for tame (see Definitions 2.3 and 2.4).

However, if k is "tiny" then algorithmic solvability is expected. We will not go into this matter further here.

We now give a natural restriction of the base concept in order to obtain greater flexibility.

DEFINITION 2.1.2. S is a sum base for $R \subseteq Z^{+k}$ if and only if
i. $1 \in S$.
ii. For all $n,m \in S$, n+m is either in S or is R related to $S \cap [1,n+m)$, but not both.

We can return to the motivation from atomic elements in physical science. In physical science, we are concerned not with arbitrary imagined substances, but rather with substances that actually exist, physically. So the restriction to substances built up from atomic elements is natural, even before one has identified what the atomic elements are.

Of course, this motivation would suggest that we go beyond just binary sums in Definition 2.1.2, but the theory that we present is already surprisingly deep with just binary sums. In section 2.9, we consider longer sums, with further unexpected results.

THEOREM 2.1.4. Every $R \subseteq Z^{+k}$ has an infinite sum base. For all k, $\emptyset \subseteq Z^{+k}$ has continuumly many sum bases, and Z^{+k} has only the sum base {1}. For all $k \geq 2$, there are $R,R' \subseteq Z^{+k}$ whose only sum bases are {1,2,4,6,...} and {1,3,5,7,...}, respectively. In particular, there is no $n > 1$ such that every $R \subseteq Z^{+k}$ has a sum base containing n.

By Theorem 2.1.4, finite sum bases may not exist. This suggests that we investigate finite approximations to sum bases. Here is a length 2 approximation.

DEFINITION 2.1.3. Let $R \subseteq Z^{+k}$. A sum base pair for R consists of two sets $S_1 \subseteq S_2 \subseteq Z^+$, where
i. $1 \in S_1$.
ii. For all $n,m \in S_1$, n+m is in S_2 or n+m is R related to $S_2 \cap [1,n+m)$.

The idea of considering finite approximates to sum bases is also natural from the point of view of atomic elements in physical science. Although real numbers are used extensively in physical science, finite aspects of reality arguably dominate our thinking.

Sum base pairs support considerably more flexibility than we have either with bases or sum bases.

THEOREM 2.1.5. Every $R \subseteq Z^{+k}$ has a sum base pair starting with an infinite set.

The study of sum base pairs for $R \subseteq Z^{+k}$ is substantial. We give a sample of further results.

THEOREM 2.1.6. Every $R \subseteq Z^{+k}$ has a sum base pair starting with an infinite set containing an even integer.

There is nothing special here about even integers as we now see with this much stronger statement.

THEOREM 2.1.7. Let $A \subseteq Z^+$ be infinite. Every $R \subseteq Z^{+k}$ has a sum base pair starting with an infinite subset of A.

THEOREM 2.1.8. For every $R \subseteq Z^{+k}$, every sufficiently large integer lies in the infinite starting set of some sum base pair.

There is an unexpected uniformity here.

THEOREM 2.1.9. For all $n \gg k$, every $R \subseteq Z^{+k}$ has a sum base pair starting with an infinite set containing n.

We can arrange for the elements of the first set to be in various senses "inaccessible" to the second set. Here is a very weak and a much stronger result of this kind.

THEOREM 2.1.10. Every $R \subseteq Z^{+k}$ has a sum base pair S,S', $|S| = \infty$, where the least element of $S\backslash\{1\}$ is not the successor of any integer in S'.

THEOREM 2.1.11. Every $R \subseteq Z^{+k}$ has a sum base pair S,S', $|S| = \infty$, where no $[\log(n),n]$, $n \in S$, contains a sum of pairs of integers from S'.

We now present a common strengthening of Theorems 2.1.5 - 2.1.11.

THEOREM 2.1.12. Let $A \subseteq Z^+$ be infinite and $n \gg k$. Every $R \subseteq Z^{+k}$ has a sum base pair S,S', where $|S| = \infty$, $n \in S \subseteq A \cup$

{n}, and no [log(n),n], n ∈ S, contains a sum of pairs of integers from S'.

THEOREM 2.1.13. Theorems 2.1.5 - 2.1.12 are provable in ACA'.

We now straightforwardly iterate the sum base pair concept.

DEFINITION 2.1.4. Let R ⊆ Z+k and 1 ≤ t ≤ ∞. A sum base tower of length t for R consists of a sequence of sets $S_1 \subseteq S_2 \subseteq \ldots \subseteq Z^+$ of length t, where
i. $1 \in S_1$.
ii. For all i < t and n,m ∈ S_i, n+m is either in S_{i+1} or is R related to S ∩ [1,n+m), but not both.

Here is the sense in which sum base towers approximate sum bases.

THEOREM 2.1.14. The sum bases for $R \subseteq Z^{+k}$ are exactly the unions of the infinite length sum base towers for $R \subseteq Z^{+k}$, and these are exactly the unions of the infinite length sum base towers for $R \subseteq Z^{+k}$ whose terms are finite.

THEOREM 2.1.15. Theorem 2.1.5 holds for infinite length sum base towers. However, Theorems 2.1.6 - 2.1.12 fail for infinite length sum base towers.

We will now only work with sum base towers of finite length.

DEFINITION 2.1.5. A finitary sum base tower for $R \subseteq Z^{+k}$ is a sum base tower of finite length whose terms are finite. An infinitary sum base tower for $R \subseteq Z^{+k}$ is a sum base tower of finite length whose terms are infinite.

An obvious question is: do Theorems 2.1.6 - 2.1.12 extend from sum base pairs to sum base towers of finite length?

We shall see that finite length sum base towers - even finitary sum base towers - cannot be properly investigated within the usual ZFC axioms for mathematics.

2.2. Infinite even integer tower "theorem".

We now restate Theorems 2.1.5 - 2.1.12 for infinitary sum base towers of finite length.

THEOREM 2.1.5 (infinitary towers). For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower of length t.

PROPOSITION 2.1.6 (infinitary towers). For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower of length t whose starting set contains an even integer.

PROPOSITION 2.1.7 (infinitary towers). Let $A \subseteq Z^+$ be infinite. For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower of length t starting with a subset of A.

PROPOSITION 2.1.8 (infinitary towers). For all k,t and $R \subseteq Z^{+k}$, every sufficiently large integer lies in the starting set of some infinitary sum base tower of length t.

PROPOSITION 2.1.9 (infinitary towers). For all n >> k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower of length t whose starting set contains n.

PROPOSITION 2.1.10 (infinitary towers). For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower $S_1,...,S_t$, where the least element of $S_1\backslash\{1\}$ is not the successor of any integer in S_t.

PROPOSITION 2.1.11 (infinitary towers). For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower $S_1,...,S_t$, where no [log(n),n], $n \in S$, contains a sum of pairs of integers from S_t.

PROPOSITION 2.1.12 (infinitary towers). Let $A \subseteq Z^+$ be infinite and n >> k,t. Every $R \subseteq Z^{+k}$ has an infinitary sum base tower $S_1,...,S_t$, where $n \in S_1 \subseteq A \cup \{n\}$, and no [log(n),n], $n \in S_1$, contains a sum of pairs of integers from S_t.

THEOREM 2.2.1. Theorem 2.1.5 is provable in ACA'. Propositions 2.1.6 - 2.1.12 are provably equivalent to 1-Con(SMAH) over ACA'. This holds even if we weaken Proposition 2.1.6 by replacing "starting" with "last". In Propositions 2.1.9 and 2.1.12, the optimal function for >> eventually dominates all provably recursive function of SMAH, but is provably recursive in SMAH+.

We give a name to Proposition 2.1.6.

INFINITE EVEN INTEGER TOWER "THEOREM" (IEIT). For all k,t, every $R \subseteq Z^{+k}$ has an infinitary sum base tower of length t whose starting set contains an even integer.

2.3. Infinite geometric progression tower "theorem".

We now work with tame $R \subseteq Z^{+k}$.

INFINITE GEOMETRIC PRGRESSION TOWER "THEOREM" (IGPT). For all k,t, every integral piecewise linear $R \subseteq Z^{+k}$ has a sum base tower of length t starting with some $\{1,r,r^2,...\}$, $r > 1$.

PROPOSITION 2.3.1. For all k,t, every integral piecewise linear $R \subseteq Z^{+k}$ has a sum base tower of length t starting with $\{1,r,r^2,...\}$, r = (8ktIPL(R))!!.

PROPOSITION 2.3.2. For all Presburger $R \subseteq Z^{+k}$ and r > (8ktPRES(R))!!, R has a sum base tower of length t starting with $\{1,r,r^2,...\}$. Furthermore, the sets in the tower can be taken to be definable in the tame structure $(N,<,+,2^x)$.

THEOREM 2.3.3. IGPT and Propositions 2.3.1, 2.3.2 are provably equivalent to Con(SMAH) over RCA_0. IGPT is refutable in RCA_0 for general $R \subseteq Z^{+k}$, even if we just use $\{1,r,r^2,...,r^r\}$.

2.4. Finite geometric growth tower "theorem".

We now work with finitary sum base sequences of finite length. Here we can start with finite geometric progressions without assuming that $R \subseteq Z^{+k}$ is tame. However, this does not take us out of ZFC. But see section 2.6, where we use finite geometric progressions with strong uniformity to go beyond ZFC.

THEOREM 2.4.1. For all k,p,t, every $R \subseteq Z^{+k}$ has a finitary sum base tower of length t starting with some $\{1,r,r^2,...,r^p\}$, $r > 1$, (starting with some p element set of odd integers).

THEOREM 2.4.2. For all k,p,t there exists $r > 1$ such that the following holds. Every $R \subseteq Z^{+k}$ has a finitary sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$ (starting with some p element set containing r}).

THEOREM 2.4.3. Let r >> k,p,t. Every $R \subseteq Z^{+k}$ has a finitary sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$ (starting with some p element set containing r).

THEOREM 2.4.4. Theorems 2.4.1 - 2.4.3 are provably equivalent to 1-Con(ATR($<\omega^\omega$)) over ACA'. The optimal functions for ∃ and >> eventually dominates all provably recursive functions of ATR($<\omega^\omega$), but is a provably recursive function of ATR(ω^ω).

Moving the quantification over p to the inside has a major effect in strength.

THEOREM 2.4.5. For all k,t there exists r > 1 such that the following holds. For all p, every R ⊆ Z^{+k} has a finitary sum base tower of length t starting with some p element set containing r.

THEOREM 2.4.6. Let r,p >> k,t. Every R ⊆ Z^{+k} has a finitary sum base tower of length t starting with some p element set containing r.

THEOREM 2.4.7. Theorems 2.4.5 and 2.4.6 are provably equivalent to 1-Con(WZC) over ACA'. The optimal functions for ∃ and >> eventually dominates all provably recursive functions of WZC, but is a provably recursive function of Z. Theorems 2.4.5 and 2.4.6 with "some p element set containing r" replaced by $\{1,r,r^2,\ldots,r^p\}$ results in statements refutable in RCA_0.

We now take a different approach and place a growth condition on the finitary sum base towers.

DEFINITION 2.4.1. Let r,t ≥ 1. We say that $S_1 \subseteq \ldots \subseteq S_t \subseteq$ Z+ has base r geometric growth if and only if for all $n \in S_1$, $|S_t \cap [1,n]| = r^{|S\_1 \cap [1,n)|}$.

FINITE GEOMETRIC GROWTH TOWER "THEOREM" (FGGT). Let R ⊆ Z+k and r,p > (8kt)!!. R has a finitary sum base tower of length t, with base r geometric growth, starting with a set of p odd integers.

THEOREM 2.4.8. FGGT is provably equivalent to 1-Con(SMAH) over ACA'.

The double exponential expression here is merely chosen to be convenient and safe.

2.5. Finite similarity tower "theorem".

Let n < m be from $S \subseteq Z^+$. There is a very simple way of talking about n,m being "similar" in S. For all i < n from S, n+i ∈ S ↔ m+i ∈ S.

THEOREM 2.5.1. For all k,t, every R ⊆ Z+k has a finitary sum base tower $\{1 < n < m\} = S_1 \subseteq \ldots \subseteq S_t$, where for all i < n from S_t, $n+i \in S_t \leftrightarrow m+i \in S_t$.

We now want to talk about (n,m) and (m,r) being "similar" in S, where n < m < r are from S. We use the condition for all i < n from S, n+m+i ∈ S ↔ m+r+i ∈ S.

FINITE SIMILARITY TOWER "THEOREM" (FST). For all k,t, every R ⊆ Z+k has a finitary sum base tower $\{1 < n < m < r\} = S_1 \subseteq \ldots \subseteq S_t$, where for all i < n from S_t, $n+m+i \in S_t \leftrightarrow m+r+i \in S_t$.

We can go further with similarity.

PROPOSITION 2.5.2. For all k,p,t, every $R \subseteq Z^{+k}$ has a finitary sum base tower $\{1 < n_1 < \ldots < n_p\} = S_1 \subseteq \ldots \subseteq S_t$, where for all $i < n_1$ from S_t, $n_1+\ldots+n_{p-1}+i \in S_t \leftrightarrow n_2+\ldots+n_p+i \in S_t$.

PROPOSITION 2.5.3. For all k,p,t, every R ⊆ Z+k has a finitary sum base tower $\{1 < n_1 < \ldots < n_p\} = S_1 \subseteq \ldots \subseteq S_t$, where for all $i < n_{i1} < \ldots < n_{iq}$ and $1 < n_{j1} < \ldots < n_{jq}$, $n_{i\_1}+\ldots+n_{i\_q}+i \in S_t \leftrightarrow n_{j\_1}+\ldots+n_{j\_q}+i \in S_t$.

THEOREM 2.5.4. Theorem 2.5.1 implies 1-Con(ZFC\P), and follows from 1-Con(Z3), over ACA'. FST implies 1-Con(ZFC + ∃λ)(λ is a subtle cardinal)) and follows from 1-Con(ZFC + (∃λ)(λ has the 2-SRP)), over ACA'. Propositions 2.5.2 and 2.5.3 are provably equivalent to 1-Con(SRP) over ACA'.

2.6. Finite geometric progression tower "theorem".

We now work with tame $R \subseteq Z+^k$, as in section 2.3. We first examine the effect of tameness on the results in sections 2.4 and 2.5.

THEOREM 2.6.1. In Theorems 2.4.1 - 2.4.3, 2.4.5, 2.4.6, FGGT, Theorem 2.5.1, FST, and Propositions 2.5.2, 2.5.3, replace $R \subseteq Z^{+k}$ by "integral piecewise linear $R \subseteq Z^{+k}$" or "Presburger $R \subseteq Z^{+k}$". These modified statements have the same provable equivalence except with 1-Con replaced by Con. Using the decision procedure for Presburger arithmetic, and obvious estimates on the reduction of size of sets in finitary sum base towers, these modified statements become explicitly Π^0_3. In fact, many become explicitly Π^0_2, including FGGT and FST. In each case, an exponential type expression can be used as a bound for the existential quantifiers, resulting in explicitly Π^0_1 sentences that have the same provable equivalences over ACA'. In some cases, including FGGT and FST, these bounds for the existential quantifiers are immediate from the decision procedure for Presburger arithmetic. In other cases, special features of finitary sum base towers need to be exploited. The former includes Theorem 2.4.1, FGGT, Theorem 2.5.1, FST, and Propositions 2.5.2, 2.5.3. The latter includes the remaining Theorems and Propositions.

We now exploit tameness of $R \subseteq Z^{+k}$ in a direct manner with finite geometric progressions where the ratio is independent of the length.

FINITE GEOMETRIC PRGRESSION TOWER "THEOREM" (FGPT). Let $R \subseteq Z^{+k}$ be integral piecewise linear and r,p > (8ktIPL(R))!!. R has a sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$, which lives in $[1,r^{p+1}]$.

PROPOSITION 2.6.1. Let $R \subseteq Z^{+k}$ be Presburger and r,p > (8ktPRES(R))!!. R has a sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$, which lives in $[1,r^{p+1}]$.

Note that FGPT and Proposition 2.6.1 are explicitly Π^0_1.

THEOREM 2.6.2. FGPT and Proposition 2.6.1 are provably equivalent to Con(SMAH) over ACA'.

As in section 4, the double exponential expressions here are merely chosen to be convenient and safe.

2.7. $R \subseteq [1,s]^k$.

Many of the statements we have presented in sections 2.4 - 2.6 make finite existential conclusions about arbitrary $R \subseteq Z^{+k}$. Such statements are generally equivalent to a corresponding statement about $R \subseteq [1,s]^k$, where s is sufficiently large relative to the data. This is a standard way of generating finite forms. Usually, this equivalence is by a compactness, or finitely branching tree argument.

In section 2.6, we have seen the effect of assuming that $R \subseteq Z^{+k}$ is tame. Existential quantifiers get bounded by exponential type expressions.

Here general $R \subseteq Z^{+k}$ get replaced by $R \subseteq [1,s]^k$, where s is incredibly greater than the parameters.

DEFINITION 2.8.1. We use $[1,s]^k$ as an ambient space, and adhere to the convention that sum base towers for $R \subseteq [1,s]^k$ must live in [1,s]. I.e., the terms are subsets of [1,s].

We begin with the relatively weak Theorems 2.4.1. - 2.4.3.

THEOREM 2.7.1. For all k,p,t there exists r > 1 such that the following holds. Every $R \subseteq [1,r^{p+1}]k$ has a sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$ (starting with some p element set of odd integers).

THEOREM 2.7.2. Let r >> k,p,t. Every $R \subseteq [1,r^{p+1}]^k$ has a sum base tower of length t starting with $\{1,r,r^2,...,r^p\}$ (starting with some p element set containing r).

THEOREM 2.7.3. Theorems 2.7.1 and 2.7.2 are provably equivalent to 1-Con(ATR($<\omega^\omega$) over ACA'. The optimal functions for ∃ and >> eventually dominates all provably recursive functions of ATR($<\omega^\omega$), but is a provably recursive function of ATR(ω^ω).

Theorems 2.4.5 and 2.4.6 become Π^0_4 with a double >>. We will avoid this here.

We now come to FGGT.

PROPOSITION 2.7.4. For all r,p > (8kt)!! there exists s such that the following holds. Every $R \subseteq [1,s]^k$ has a finitary sum base tower of length t, with base r geometric growth, starting with a set of p odd integers.

THEOREM 2.7.5. Proposition 2.7.4 is provably equivalent to 1-Con(SMAH). The optimal function for ∃ eventually dominates all provably recursive functions of SMAH, but is a provably recursive function of $SMAH^+$.

We now come to Theorem 2.5.1.

THEOREM 2.7.6. For all k,t there exists s such that the following holds. Every $R \subseteq [1,s]^k$ has a sum base tower $\{1 < n < m\} = S_1 \subseteq ... \subseteq S_t$, where for all i < n from S_t, $n+i \in S_t \leftrightarrow m+i \in S_t$.

We now treat FST and the remaining similarity statements.

PROPOSITION 2.7.7. For all k,t there exists s such that the following holds. Every $R \subseteq [1,s]^k$ has a sum base tower $\{1 < n < m < r\} = S_1 \subseteq \ldots \subseteq S_t$, where for all i < n from S_t, $n+m+i \in S_t \leftrightarrow m+r+i \in S_t$.

PROPOSITION 2.7.8. For all k,p,t there exists s such that the following holds. Every $R \subseteq [1,s]^k$ has a sum base tower $\{1 < n_1 < \ldots < n_p\} = S_1 \subseteq \ldots \subseteq S_t$, where for all $i < n_1$ from S_t, $n_1+\ldots+n_{p-1}+i \in S_t \leftrightarrow n_2+\ldots+n_p+i \in S_t$.

PROPOSITION 2.7.9. For all k,p,t there exists s such that the following holds. Every $R \subseteq [1,s]k$ has a sum base tower $\{1 < n_1 < \ldots < n_p\} = S_1 \subseteq \ldots \subseteq S_t$, where for all $i < n_{i1} < \ldots < n_{iq}$ and $1 < n_{j1} < \ldots < n_{jq}$, $n_{i\_1}+\ldots+n_{i\_q}+i \in S_t \leftrightarrow n_{j\_1}+\ldots+n_{j\_q}+i \in S_t$.

THEOREM 2.7.10. Theorem 2.7.6 implies 1-Con(ZFC\P), and follows from 1-Con(Z3), over ACA'. Proposition 2.7.7 implies 1-Con(ZFC + $\exists\lambda$)(λ is a subtle cardinal)) and follows from 1-Con(ZFC + ($\exists\lambda$)(λ has the 2-SRP)), over ACA'. Propositions 2.7.8 and 2.7.9 are provably equivalent to 1-Con(SRP) over ACA'. The optimal function for $\exists$ in Theorem 2.7.6 eventually dominates all provably recursive functions of ZFC\P, but is a provably recursive function of Z_3. The optimal function for $\exists$ in Proposition 2.7.7 eventually dominates all provably recursive functions of ZFC + ($\exists\lambda$)(λ is a subtle cardinal), but is a provably recursive function of ZFC + ($\exists\lambda$)(λ is has the 2-SRP). The optimal function for $\exists$ in Propositions 2.7.8 and 2.7.9 eventually dominate all provably recursive functions of SRP but are provably recursive functions of SRP+.

2.8. Length 3 multi sum base towers.

Here we modify the definition of sum base towers to v-sum base towers.

DEFINITION 2.8.1. Let $R \subseteq Z^{+k}$, $1 \leq t \leq \infty$, and $v \in Z^+$. A v-sum base tower of length t for R consists of a sequence of sets $S_1 \subseteq S_2 \subseteq \ldots \subseteq Z^+$ of length t, where $1 \in S_1$, and for all i < t, every $n_1,\ldots,n_v \in S_i$, $n_1+\ldots+n_v$ is either in S_{i+1} or is R related to $S \cap [1,n_1+\ldots+n_v)$, but not both. Again, finitary means all sets are finite, and infinitary means all sets are infinite.

THEOREM 2.8.1. All of the claims in sections 2.2 - 2.7 involving infinitary and finitary sum base towers of length t hold for infinitary and finitary v-sum base towers of length t, where we quantify over $v,t \in Z^+$. We can also quantify over $v \in Z^+$, and fix $t = 3$, without changing any results.

3. Invariant continuations.

We now move from Z^+ to Q^k, where now we use only < on Q, instead of <,+ on Z^+. We shall see that most of the statements in section 3 are provably equivalent to Π^0_1 sentences via Gödel's Completeness Theorem. This was not the case for section 2. For this, in section 2 we needed to use tame $R \subseteq Z+k$.

DEFINITION 3.1. N,Z,Q are the set of all nonnegative integers, integers, and rationals, respectively. A* is the set of all nonempty finite sequences from A. J is a rational interval if and only if
i. $J \subseteq Q$.
ii. If $x < y < z$ and $x,z \in J$, then $y \in J$.
iii. $\inf(J),\sup(J) \in Q \cup \{-\infty,\infty\}$.
The letter J always represents a rational interval.

DEFINITION 3.2. Let $x,y \in Q^*$. lth(x) is the length of x. max(x), min(x) are the largest and smallest coordinates of x, respectively. $x \leq y$, $x < y$, $x \geq y$, $x > y$ if and only if $\max(x) \leq \max(y)$, $\max(x) < \max(y)$, $\max(x) \geq \max(y)$, $\max(x) > \max(y)$, respectively. $\lceil x \rceil$ is the least integer $\geq \max(x)$.

DEFINITION 3.3. Let $p,q \in Q$ and $S \subseteq Q^k$. $S_{<x}$, $S_{\leq x}$, $S_{\geq x}$, $S_{>x}$, S[x,y] are $\{z \in S: z < x\}$, $\{z \in S: z \leq x\}$, $\{z \in S: z \geq x\}$, $\{z \in S: x > x\}$, $\{z \in S: x \leq z \leq y\}$, respectively. $E \subseteq Q^k$ is negative if and only if all coordinates of all elements of E are < 0.

DEFINITION 3.4. Let $x,y \in Q^*$. lth(x) is the length of x. x,y are order equivalent if and only if lth(x) = lth(y), and for all $1 \leq i,j \leq \text{lth}(x)$, $x_i < x_j \leftrightarrow y_i < y_j$. xy is the concatenation of x and y.

DEFINITION 3.4. For $S \subseteq Q^k$, fld(S) is the set of all coordinates of elements of S. $S\# = (\text{fld}(S) \cup \{0\})^k$.

THEOREM 3.1. For $S \subseteq Q^k$, S# is the least superset of S of the form A^k which contains the k dimensional origin.

We will be placing invariance conditions on what we call "continuations", which always sit in a definite ambient space X.

DEFINITION 3.5. Let R be any binary relation. $S \subseteq X$ is R invariant if and only if for all $x,y \in X$, if $R(x,y)$ then $x \in S \leftrightarrow y \in S$. Let $R_1,R_2,\ldots$ be any finite or infinite sequence of binary relations. $S \subseteq X$ is $R_1,R_2,\ldots$ invariant if and only if for all i, $S \subseteq X$ is R_i invariant.

We emphasize the special case of functions, which are treated as binary relations.

DEFINITION 3.6. Let f be any function. $S \subseteq X$ is f invariant if and only if for all $x,f(x) \in X$, $x \in S \leftrightarrow f(x) \in S$. Let $f_1,f_2,\ldots$ be any finite or infinite sequence of functions. $S \subseteq X$ is $f_1,f_2,\ldots$ invariant if and only if for all i, S is f_i invariant.

3.1. Maximal continuations in J^k.

DEFINITION 3.1.1. S is a continuation of $E \subseteq J^k$ if and only if $E \subseteq S \subseteq J^k$, and every element of S^2 is order equivalent to some element of E^2. S is a maximal continuation of $E \subseteq J^k$ if and only if S is a continuation of $E \subseteq J^k$ such that for all $x \in J^k\backslash S$, $S \cup \{x\}$ is not a continuation of $E \subseteq J^k$.

The following is by an obvious inductive argument using any enumeration of J^k.

THEOREM 3.1.1. Every $E \subseteq J^k$ has a maximal continuation.

We now introduce some shift operators on Q*. These add 1 to some of the coordinates but leave the rest fixed.

DEFINITION 3.1.2. The shift of $x \in Q^*$ results from adding 1 to all coordinates of x. The shift of $S \subseteq Q^*$ is the set of all shifts of elements of S.

DEFINITION 3.1.3. The $\geq p$ shift of $x \in Q^*$ results from adding 1 to all coordinates of x that are $\geq p$. The $\geq p$ shift of $S \subseteq Q^*$ is the set of all $\geq p$ shifts of elements of S.

The above shift operators are in a sense too strong for present purposes.

DEFINITION 3.1.4. The $\geq p,N$ shift of $x \in Q^*$ results from adding 1 to all coordinates of x that are $\geq p$, provided they are all integers. The ≥p,N shift of $S \subseteq Q^*$ is the set of all ≥p,N shifts of elements of S.

EXAMPLES. The ≥0,N shift of (-1,0,1/2,4,4,6,8,9,9) is (-1,0,1/2,4,4,6,8,9,9). The ≥1,N shift of (-1,0,1/2,4,4,6,8,9,9) is (-1,0,1/2,5,5,7,9,10,10). Here the coordinates are nondecreasing only for readability.

There is another closely related shift operator.

DEFINITION 3.1.5. The ↑N shift of $x \in Q^*$ results from adding 1 to all nonnegative fractional coordinates of x.

EXAMPLE. The ↑N shift of (-1,0,1/2,4,4,6,8,9,9) is (-1,0,1/2,5,5,7,9,10,10).

There is an important equivalence relation that arises here.

DEFINITION 3.1.6. $x,y \in Q^*$ are upper integral equivalent if and only if
i. x,y are order equivalent.
ii. For all $1 \leq i \leq lth(x)$, $x_i \neq y_i \rightarrow$ every $x_j > x_i$ and every $y_j > y_i$ are in N.
$S \subseteq J^k$ is upper integral invariant if and only if $S \subseteq J^k$ is invariant with respect to upper integral equivalence.

THEOREM 3.1.2. Upper integral equivalence is the least equivalence relation on Q^* containing the graphs of ≥0,N, ≥1,N, ... Upper integral equivalence contains the graph of ↑N shift. Let $S \subseteq J^k$. S is upper integral invariant if and only if for all $n \in N$, $S \subseteq J^k$ is ≥n,N invariant. If $S \subseteq J^k$ is upper integral invariant then $S \subseteq J^k$ is ↑N shift invariant.

INVARIANT MAXIMAL CONTINUATION "THEOREM" (IMC). For all k,n, every finite negative $E \subseteq Q_{\leq n}{}^k$ has a ≥0,N shift invariant maximal continuation.

PROPOSITION 3.1.3. For all k,J, every finite negative $E \subseteq J^k$ has an ↑N shift (upper integral shift) invariant maximal continuation.

THEOREM 3.1.4. IMC and Proposition 3.1.3 (all forms) are provably equivalent to Con(SRP) over WKL_0. If we use "≥0 shift invariant", or if we delete "finite", or if we delete "negative", then IMC and Proposition 3.1.3 become refutable in RCA_0. If we restrict to J where J does not have a right endpoint that is a positive integer element of J, then Proposition 3.1.3 becomes provable in ATR_0. IMC and Proposition 3.1.3 for sup(J) < ∞ are provably equivalent to explicitly Π^0_1 sentences via the Gödel Completeness Theorem.

There is nothing special about pairs in the definition of continuation. We would have the same results if we used m-tuples, for any fixed m. This remark applies throughout the entire abstract.

3.2. Ceiling maximal continuations in Q^k.

We now consider a variant of maximality for continuations. Ceiling maximality is stronger than maximality, and we go beyond ZFC by working in Q^k.

DEFINITION 3.2.1. S is a ceiling maximal continuation of $E \subseteq J^k$ if and only if S is a continuation of $E \subseteq J^k$ such that for all $x \in J^k \setminus S$, $S_{\leq\lceil x\rceil} \cup E \cup \{x\}$ is not a continuation of $E \subseteq J^k$.

THEOREM 3.2.1. Every $E \subseteq Q_{\geq -1}{}^k$ has a ceiling maximal continuation.

INVARIANT CEILING MAXIMAL CONTINUATION THEOREM (ICMC). For all k, every finite negative $E \subseteq Q_{\geq -1}{}^k$ has a ≥0,N shift invariant ceiling maximal continuation.

PROPOSITION 3.2.2. For all k, every finite negative $E \subseteq Q_{\geq -1}{}^k$ has a ↑N shift (upper integral shift) invariant ceiling maximal continuation.

THEOREM 3.2.3. PICMC and Proposition 3.2.2 (all forms) are provably equivalent to Con(SRP) over WKL0. ICMC and Proposition 3.2.2 are provably equivalent to explicitly Π^0_1 sentences via the Gödel Completeness Theorem.

There is nothing special about pairs in the definition of continuation. We would have the same results if we used m-tuples, for any fixed m.

3.3. Inductive continuations in Q^k.

Naïve inductive continuations are a very strong form of ceiling maximal continuations. We call them naïve because they are too strong.

DEFINITION 3.3.1. S is a naive inductive continuation of $E \subseteq Q^k$ if and only if S is a continuation of $E \subseteq Q^k$ such that for all $x \in Q^k \backslash S$, $S\leq x \cup E \cup \{x\}$ is not a continuation of $E \subseteq Q^k$.

THEOREM 3.3.1. $E = \{(-4,-3),(-2,-1)\} \subseteq Q_{\geq -4}{}^2$ has no naïve inductive continuation.

The appropriate notion of inductive continuation is based on the # construction.

DEFINITION 3.3.2. S is an inductive continuation of $E \subseteq Q^k$ if and only if S is a continuation of $E \subseteq Q^k$ such that for all $x \in S\#\backslash S$, $S_{\leq x} \cup E \cup \{x\}$ is not a continuation of $E \subseteq Q^k$.

INVARIANT INDUCTIVE CONTINUATION "THEOREM" (IIC). Every finite negative $E \subseteq Q^k$ has an inductive continuation containing its ≥0 shift.

PROPOSITION 3.3.2. Every finite negative $E \subseteq Q^k$ has an inductive continuation containing its ≥0,N shift.

PROPOSITION 3.3.3. Every finite negative $E \subseteq Q^k$ has an upper integral invariant inductive continuation containing its ≥0,≥1,... shifts.

THEOREM 3.3.4. IIC and Propositions 3.3.2, 3.3.3 are provably equivalent to Con(SRP) over WKL_0. They are provably equivalent to explicitly Π^0_1 sentences via the Gödel Completeness Theorem.

3.4. Inductive ≠-continuations in Q^k.

DEFINITION 3.4.1. $Q^{k\neq} = \{x \in Q^k: x_1,\ldots,x_k$ are distinct|. Let $S \subseteq Q^k$. $S<p> = \{q: (p,\ldots,p,q) \in S\}$.

DEFINITION 3.4.2. S is a ≠-continuation of $E \subseteq Q^k$ if and only if $S \subseteq Q^k$, and for all $x,y \in S$, $y \in Q^{k\neq}$, $x < y$, there exists $z,w \in E$ such that xy and zw are order equivalent.

Thus in ≠-continuations of E, we only continue certaom special aspects of E.

DEFINITION 3.4.3. S is an inductive ≠-continuation of $E \subseteq Q^k$ if and only if S is a ≠-continuation of $E \subseteq Q^k$ such that for all $x \in (S\#\backslash S) \cap Q^{k\neq}$, $S_{<x} \cup E \cup \{x\}$ is not a ≠-continuation of $E \subseteq Q^k$.

INVARIANT INDUCTIVE ≠-CONTINUATION "THEOREM" (II≤C). Every finite negative $E \subseteq Q^k$ has an inductive ≠-continuation S where each $S+1 \cap Q[1,n]^k$ is a subset of S with field $S<(3/2)^n>$.

THEOREM 3.4.2. II≤C is provably equivalent to Con(HUGE) over WKL_s. It is provably equivalent to an explicitly Π^0_1 sentence via the Gödel Completeness Theorem.

3.5. Order theoretic embeddings.

We begin with a discussion of some very tame objects called order theoretic sets and functions.

DEFINITION 3.5.1. $S \subseteq Q^k$ is order theoretic if and only if $S \subseteq Q^k$ can be presented as a Boolean combination of inequalities $x_i < x_j$, $x_i < p$, $p < x_i$

where $1 \le i \le k$, and p is a constant from Q. In actual presentations, it is convenient to use the wider collection of comparisons, $<,>,\le,\ge,=,\ne$, and logical connectives $\neg,\wedge,\vee,\rightarrow,\leftrightarrow$ for Boolean combinations.

Partial functions play a crucial role in the theory.

DEFINITION 3.5.2. Partial $f:Q^k \rightarrow Q^r$ is order theoretic if and only if its graph is order theoretic as a subset of Q^{k+r}.

THEOREM 3.5.1. The order theoretic $S \subseteq Q$ are the finite disjoint unions of rational intervals. Every order theoretic $S \subseteq Q$ is the union of the maximal intervals that it contains. These finitely many maximal intervals are pairwise disjoint rational intervals.

We say that $f:Q \rightarrow Q$ is basic if and only if dom(f) is a nonempty open rational interval, or a singleton, and f is either constant or the identity on dom(f).

THEOREM 3.5.2. Partial $f:Q \rightarrow Q$ is order theoretic if and only if f is a finite union of basic functions. Every order theoretic partial $f:Q \rightarrow Q$ is the union of the maximal basic functions that it contains. These finitely many basic functions have pairwise disjoint domains. Every order theoretic $f:Q \rightarrow Q$ moves finitely many points.

DEFINITION 3.5.3. Let $S \subseteq J^k$. f is an embedding of S if and only if

i. $f:J \to J$ is a one-one partial function.
ii. For all $p_1,\ldots,p_k \in \text{dom}(f)$, $S(p_1,\ldots,p_k) \leftrightarrow S(f(p_1),\ldots,f(p_k))$.

We now treat $\geq p,N$ invariance of $S \subseteq J^k$ in terms of embeddings.

DEFINITION 3.5.4. Let $A \subseteq Q$. sh[J,A] is the partial function from J into J defined by $sh[J,A](p) = p$ if $p \in J_{<\min(A)}$; $p+1$ if $p \in J \cap A$ and $p+1 \in J$; undefined otherwise. By convention, $\min(\emptyset) = -\infty$. If A is finite then we sometimes replace A with an enumeration of A.

THEOREM 3.5.3. sh[J,A] is order theoretic if and only if $J \cap A$ is finite. $S \subseteq J^k$ is $\geq 0,N$ invariant if and only if $S \subseteq J^k$ is sh[J,N] embedded.

We now restate previous results from sections 3.1 - 3.3 using the shifts in Definition 3.5.2.

EMBEDDED MAXIMAL CONTINUATION "THEOREM" (EMC). For all k,n, every finite negative $E \subseteq Q_{\leq n}k$ has a $sh[Q\leq n,0,\ldots,n-1]$ embedded maximal continuation.

EMBEDDED CEILING MAXIMAL CONTINUATION "THEOREM" (ECMC). For all k, every finite negative $E \subseteq Q^k$ has a sh[Q,N] embedded ceiling maximal continuation.

EMBEDDED INDUCTIVE CONTINUATION THEOREM (EIC). Every finite negative $E \subseteq Q^k$ has a sh[Q,N] embedded inductive continuation.

THEOREM 3.5.4. EMC, ECMC, and EIC are provably equivalent to Con(SRP) over WKL_0.

3.6. Conditions on embeddings.

Strictly increasing and (pointwise) continuity are important conditions that we use on embeddings $f:J \to J$. Strictly increasing is necessary. Continuity is not necessary.

THEOREM 3.6.1. Let partial $f:J \to J$ be order theoretic, where J has at least two elements and $k \geq 2$. Suppose every finite $E \subseteq J^k$, all of whose elements are fixed by f, has an f embedded maximal continuation. Then f is strictly increasing. Furthermore, no iterate of f maps some endpoint to a different endpoint.

PROPOSITION 3.6.2. Let partial f:J → J be strictly increasing, continuous, and order theoretic, where J is not closed. For all k, every finite E ⊆ J^k, where all coordinates of elements are fixed by f, has an f embedded maximal continuation.

If J is closed, then we have to put a mild extra condition on f - that dom(f) is infinite.

PROPOSITION 3.6.3. Let partial f:J → J be strictly increasing, continuous, and order theoretic, with infinite domain. For all k, every finite E ⊆ J^k, where all coordinates of elements are fixed by f, has an f embedded maximal continuation.

We can use the following weaker sufficient condition.

PROPOSITION 3.6.4. Let partial f:J → J be strictly increasing, continuous, and order theoretic, where no iterate of f maps one endpoint of J to a different endpoint of J. For all k, every finite E ⊆ J^k, where all coordinates of elements are fixed by f, has an f embedded maximal continuation.

THEOREM 3.6.5. Theorem 3.6.1 is provable in RCA0. Propositions 3.6.2 - 3.6.4 are provably equivalent to Con(SRP) over WKL0.

Since we are not claiming necessary and sufficient conditions, we create the following Template.

EMBEDDED CONTINUATION TEMPLATE. Given order theoretic partial f:J → J. For all k, every finite E ⊆ J^k, where all coordinates of elements are fixed by f, has an f embedded maximal continuation.

EMBEDDED CONTINUATION TEMPLATE CONJECTURE. Every instance of the Embedded Continuation Template is provable or refutable in SRP.

THEOREM 3.6.6. The Embedded Continuation Template Conjecture fails for any single finite fragment of SRP, assuming SRP is consistent.

4. Finite invariant continuations.

Here we give explicitly finite forms of many of the infinitary statements in section 3. These finite forms all assert the existence of certain t successive finite continuations from a starting set E. These successive finite continuations cannot be maximal continuations, as maximality generally forces them to be infinite.

However, finite inductive continuations can always be found. But ≥0,N invariant finite inductive continuations, or inductive cotinuations containing their ≥0 shift cannot always be found, even if the ambient space is J^k, inf(J),sup(J) < ∞.

We use source maximal, source ceiling maximal, source inductive, and source ≠-continuations. These are weaker than maximal, ceiling maximal, inductive, and inflex continuations.

DEFINITION 4.1.1. S is a source maximal continuation of $E \subseteq J^k$ if and only if S is a continuation of $E \subseteq J^k$ such that for all $x \in E\#\backslash S$, $S \cup \{x\}$ is not a continuation of E. S is a source ceiling maximal continuation of $E \subseteq J^k$ and for all $x \in E\#\backslash S$, $S_{\leq\lceil x\rceil} \cup E \cup \{x\}$ is not a continuation of $E \subseteq J^k$.

FINITE INVARIANT MAXIMAL CONTINUATION "THEOREM" (FIMC). For all k,n,t, every finite negative $E \subseteq Q_{\geq n}{}^k$ has t successive finite source maximal continuations that are ≥0,N shift invariant.

PROPOSITIOIN 4.1.1. For all k,J, every finite negative $E \subseteq Q_{\leq n}{}^k$ has t successive finite source maximal continuations that are ↑N shift (upper integral shift) invariant.

FINITE INVARIANT CEILING MAXIMAL CONTINUATION "THEOREM" (FICMC). For all k,t, every finite negative $E \subseteq Q_{\geq -1}{}^k$ has t successive finite source ceiling maximal continuations that are ≥0,N shift invariant.

PROPOSITION 4.1.2. For all k,t, every finite negative $E \subseteq Q_{\geq -1}{}^k$ has t successive finite source ceiling maximal continuations that are ↑N shift (upper integral shift) invariant.

DEFINITION 4.1.3. S is a source inductive continuation of $E \subseteq Q^k$ if and only if S is a continuation of $E \subseteq Q^k$ such that for all $x \in E\#\backslash S$, $S_{\leq x} \cup E \cup \{x\}$ is not a continuation of E.

FINITE INVARIANT INDUCTIVE CONTINUATION "THEOREM" (FIIC). For all k,t, every finite negative $E \subseteq Q^k$ has t successive finite source inductive continuations, where the ≥0 shift of each continuation is a subset of all successive continuations.

PROPOSITION 4.1.3. For all k,t, every finite negative $E \subseteq Q^k$ has t successive finite source inductive continuations, where the ≥0,N shift of each continuation is a subset of all successive continuations.

PROPOSITION 4.1.4. For all k,p,t, every finite negative $E \subseteq Q^k$ has t successive finite source inductive continuations, each of which are upper integral invariant, where the ≥0,...,≥k shifts of each continuation is a subset of all successive continuations.

Note that FIMC, FICMC, FIIC, and Propositions 4.1.1 - 4.1.4 are explicitly $\Pi^0{}_2$.

THEOREM 4.1.5. FIMC, FICMC, FIIC, and Propositions 4.1.1 - 4.1.4 are provably equivalent to Con(SRP) over EFA. An exponential type bound can be placed on the magnitudes of the numerators and denominators of the coordinates used, resulting in explicitly $\Pi^0{}_1$ forms.

DEFINITION 4.1.4. S is a source inductive ≠-continuation of $E \subseteq Q^k$ if and only if S is a ≠-continuation of $E \subseteq Q^k$ such that for all $x \in (E\#\backslash S) \cap Q^{k\neq}$, $S_{<x} \cup E \cup \{x\}$ is not a ≠-continuation of $E \subseteq Q^k$.

FINITE INVARIANT INDUCTIVE ≠-CONTINUATION "THEOREM" (FII≠C). Every finite negative $E \subseteq Q^k$ has t successive finite source inductive ≠-continuations, $S_1,\ldots,S_t$, where $S_t+1 \cap Q(0,t)^k$ is a subset of S_t with field $S_t<(3/2)^t>$.

THEOREM 4.1.6. FII≠C is provably equivalent to Con(HUGE) over EFA. An exponential type bound can be placed on the magnitudes of the numerators and denominators of the coordinates used, resulting in an explicitly $\Pi^0{}_1$ form.

We are checking a proof that if t = 2 then Proposition 4.1.4 is provable in a weak fragment of ZFC, and if t = 3 then Proposition 4.1.4 is provably equivalent to Con(SRP) over WKL_0. Such considerations play a serious role in sections 6 and 7.

5. E followers, E sets, and graphs.

The continuation setup of sections 3 and 4 is the most well motivated way we know to obtain explicitly Π^0_1 independence results.

Here we discuss E followers and E sets, before taking up graph formulations.

E sets are particularly suitable for computer investigations. Both drop the continuation condition $E \subseteq S$.

The E continuation, E follower, E set, and graph approaches are very close, and no substantial differences arise in the developments.

We begin with E followers.

DEFINITION 5.1. Let $E \subseteq J^k$. S is a follower of E if and only if $S \subseteq J^k$, and every element of S2 is order equivalent to some element of E^2. $S \subseteq J^k$ is a maximal follower of E if and only if $S \subseteq J^k$ is a follower of E such that for all $x \in J^k \backslash S$, $S \cup \{x]$ is not a follower of E.

DEFINITIOIN 5.2. S is a ceiling maximal follower of $E \subseteq J^k$ if and only if S is a follower of $E \subseteq J^k$ such that for all $x \in J^k \backslash S$, $S_{\leq \lceil x \rceil} \cup \{x\}$ is not a follower of $E \subseteq J^k$. S is an inductive follower of $E \subseteq Q^k$ if and only if S is a follower of $E \subseteq Q^k$ such that for all $x \in S\# \backslash S$, $S_{\leq x} \cup \{x\}$ is not a follower of $E \subseteq Q^k$.

DEFINITION 5.3. S is a ≠-follower of $E \subseteq Q^k$ if and only if $S \subseteq Q^k$, and for all $x,y \in S$, $y \in Q^{k\neq}$, $x < y$, there exists $z,w \in E$ such that xy and zw are order equivalent. S is an inductive ≠-follower of $E \subseteq Q^k$ if and only if S is a ≠-follower of $E \subseteq Q^k$ such that for all $x \in (S\# \backslash S) \cap Q^{k\neq}$, $S_{<x} \cup \{x\}$ is not a ≠-follower of $E \subseteq Q^k$.

We now restate the Propositions of sections 3.1 - 3.5 in terms of followers.

INVARIANT MAXIMAL FOLLOWER "THEOREM" (IMF). For all k,n, every $E \subseteq Q_{\leq n}{}^k$ has a ≥0,N shift invariant maximal follower.

PROPOSITION 3.1.3 (followers). For all k,J, every $E \subseteq J^k$ has an ↑N shift (upper integral shift) invariant maximal follower.

INVARIANT CEILING MAXIMAL FOLLOWER "THEOREM" (ICMF). For all k, every E ⊆ $Q_{\geq -1}{}^k$ has a ≥0,N shift invariant ceiling maximal follower.

PROPOSITION 3.2.2 (followers). For all k, every E ⊆ $Q_{\geq -1}{}^k$ has a ↑N shift (upper integral shift) invariant ceiling maximal follower.

INVARIANT INDUCTIVE FOLLOWER "THEOREM" (IIF). Every finite negative E ⊆ Q^k has an inductive follower containing its ≥0 shift.

PROPOSITION 3.3.2 (followers). Every E ⊆ Q^k has an inductive follower containing its ≥0,N shift.

PROPOSITION 3.3.3 (followers). Every E ⊆ Q^k has an upper integral invariant inductive follower containing its ≥0,≥1,... shifts.

INVARIANT INDUCTIVE ≠-FOLLOWER "THEOREM" (II≠F). Every E ⊆ Q^k has an inductive ≠-follower S where each S+1 ∩ $Q[1,n]^k$ is a subset of S with field S<$(3/2)^n$>.

Now for E sets.

DEFINITION 5.4 (E sets). Let E ⊆ J^{2k}. S is an E set if and only if S ⊆ J^k, and every element of E2 is order equivalent to an element of E. S ⊆ J^k is a maximal E set if and only if S ⊆ J^k is an E set such that for all x ∈ J^k\S, S ∪ {x] is not an E set.

DEFINITIOIN 5.5 (E sets). Let E ⊆ J^{2k}. S is a ceiling maximal E set if and only if S is an E set such that for all x ∈ J^k\S, $S_{\leq \lceil x \rceil}$ ∪ {x} is not an E set. S is an inductive E set if and only if S is an E set such that for all x ∈ S#\S, $S_{\leq x}$ ∪ {x} is not an E set.

DEFINITION 5.6 (E sets). Let E ⊆ Q^{2k}. S is an E ≠-set, E ⊆ Q^{2k}, if and only if S ⊆ Q^k, and for all x,y ∈ S, y ∈ $Q^{k\neq}$, x < y, there exists z ∈ E such that xy and z are order equivalent. S is an inductive E ≠-set if and only if S is an E ≠-set such that for all x ∈ (S#\S) ∩ $Q^{k\neq}$, $S_{<x}$ ∪ {x} is not an E ≠-set.

INVARIANT MAXIMAL E SET "THEOREM" (IM/Eset). For all k,n, every E ⊆ $Q_{\leq n}{}^{2k}$ has a ≥0,N shift invariant maximal E set.

PROPOSITION 3.1.3 (E sets). For all k,J, every $E \subseteq J^{2k}$ has an ↑N shift (upper integral shift) invariant maximal E set.

INVARIANT CEILING MAXIMAL E SET THEOREM (ICM/Eset). For all k, every $E \subseteq Q_{\geq -1}^{2k}$ has a ≥0,N shift invariant ceiling maximal E set.

PROPOSITION 3.2.2 (E sets). For all k, every $E \subseteq Q_{\geq -1}^{2k}$ has a ↑N shift (upper integral shift) invariant ceiling maximal E set.

INVARIANT INDUCTIVE E SET "THEOREM" (II≠/Eset). Every $E \subseteq Q^{2k}$ has an inductive E set containing its ≥0 shift.

PROPOSITION 3.3.2 (E sets). Every $E \subseteq Q^{2k}$ has an inductive E set containing its ≥0,N shift.

PROPOSITION 3.3.3 (E sets). Every $E \subseteq Q^{2k}$ has an upper integral invariant inductive E set containing its ≥0,≥1,... shifts.

INVARIANT INDUCTIVE E ≠-SET "THEOREM" (II≠/Eset). Every $E \subseteq Q^{2k}$ has an inductive E ≠-set S where each $S+1 \cap Q[1,n]^k$ is a subset of S with field $S<(3/2)^n>$.

Now for graphs. We begin with some basic graph theoretic material.

DEFINITION 5.7. A graph is a pair G = (V,E), where V is the vertex set, $E \subseteq V \times V = V^2$ is the edge set (edge relation, adjacency relation), and E is irreflexive and symmetric. We say that x,y are adjacent in G if and only if E(v,w). A clique in G is an $S \subseteq V$ such that any two distinct elements are adjacent in G. We also consider sequences from V to be cliques in G if and only if its set of terms is a clique in G. A maximal clique in G is a clique in G which is not a proper subset of any clique in G. Note that S is a maximal clique in G if and only if S is a clique in G, where every vertex is nonadjacent, in G, to some element of S.

THEOREM 5.1. Every graph has a maximal clique. Every clique in every graph can be extended to a maximal clique. Both of these statements are provably equivalent to the axiom of choice over ZF. For countable graphs, the first statement is provable in RCA_0, and the second statement is provably equivalent to ACA_0 over RCA_0.

DEFINITION 5.8. $S \subseteq J^k$ is order invariant if and only if for all order equivalent $x,y \in J^k$, $x \in S \leftrightarrow y \in S$. There are finitely many order invariant $S \subseteq J^k$. An order invariant graph on J^k is a graph (Jk,E) such that $E \subseteq J^{2k}$ is order invariant. There are finitely many order invariant graphs on any given J^k.

The graph formulations are essentially the same as the E set formulations. The key observation is that the cliques in an order invariant graph on J^k are the same as the E sets, where E is the union of the edge set of the graph with the diagonal $\{(x,x): x \in J\}$. In the interest of clarity, we will explicitly state the graph theoretic formulations.

DEFINITIOIN 5.9. Let G be a graph on Jk. S is a ceiling maximal clique if and only if S is a clique such that for all $x \in J^k \backslash S$, $S_{\leq\lceil x\rceil} \cup \{x\}$ is not a clique. S is an inductive clique if and only if S is a clique such that for all $x \in S\#\backslash S$, $S_{\leq x} \cup \{x\}$ is not a clique.

DEFINITION 5.1.10. Let G be a graph on Q^k. S is a $\neq$-clique if and only if $S \subseteq Q^k$, and for all distinct $x,y \in S$, $y \in Q^{k\neq}$, $x < y$, x,y are adjacent. S is an inductive $\neq$-clique if and only if S is a $\neq$-clique such that for all $x \in (S\#\backslash S) \cap Q^{k\neq}$, $S<x \cup \{x\}$ is not a $\neq$-clique.

INVARIANT MAXIMAL CLIQUE "THEOREM" (IM/graphs). For all k,n, every order invariant graph on $Q_{\leq n}{}^k$ has a ≥ 0,N shift invariant maximal clique.

PROPOSITION 3.1.3 (graphs). For all k,J, every order invariant graph on J^k has an ↑N shift (upper integral shift) invariant maximal clique.

INVARIANT CEILING MAXIMAL CLIQUE THEOREM (ICM/graphs). For all k, every order invariant graph on $Q_{\geq -1}{}^k$ has a ≥ 0,N shift invariant ceiling maximal clique.

PROPOSITION 3.2.2 (graphs). For all k, every order invariant graph on $Q_{\geq -1}{}^k$ has a ↑N shift (upper integral shift) invariant ceiling maximal clique.

INVARIANT INDUCTIVE CLIQUE "THEOREM" (II/graphs). Every order invariant graph on Q^k has an inductive clique containing its ≥ 0 shift.

PROPOSITION 3.3.2 (graphs). Every order invariant graph on Q^k has an inductive clique containing its ≥0,N shift.

PROPOSITION 3.3.3 (graphs). Every order invariant graph on Q^k has an upper integral invariant inductive clique containing its ≥0,≥1,... shifts.

INVARIANT INDUCTIVE ≠-CLIQUE GRAPH "THEOREM" (II≠/graphs). Every order invariant graph on Q^k has an inductive ≠-clique S where each S+1 ∩ $Q[1,n]^k$ is a subset of S with field $S<(3/2)^n>$.

We also convert the embedding results of section 3.5.

EMBEDDED MAXIMAL FOLLOWER "THEOREM" (EMF). For all k,n, every E ⊆ $Q_{\leq n}{}^k$ has a sh[Q≤n,0,...,n-1] embedded maximal follower.

EMBEDDED CEILING MAXIMAL FOLLOWER "THEOREM" (ECMF). For all k, every E ⊆ $Q_{\geq -1}{}^k$ has a sh[$Q_{\geq -1}$,N] embedded ceiling maximal follower.

EMBEDDED INDUCTIVE FOLLOWER "THEOREM" (EIF). Every E ⊆ Q^k has a sh[Q,N] embedded inductive follower.

EMBEDDED MAXIMAL E SET "THEOREM" (EMF). For all k,n, every E ⊆ $Q_{\leq n2}{}^k$ has a sh[$Q_{\leq n}$,0,...,n-1] embedded maximal follower.

EMBEDDED CEILING MAXIMAL E SET "THEOREM" (ECMF). For all k, every E ⊆ $Q_{\geq -1}{}^{2k}$ has a sh[$Q_{\geq -1}$,N] embedded ceiling maximal follower.

EMBEDDED INDUCTIVE E SET "THEOREM" (EIF). Every E ⊆ Q^{2k} has a sh[Q,N] embedded inductive follower.

EMBEDDED MAXIMAL CLIQUE "THEOREM" (EM/graphs). For all k,n, every order invariant graph on $Q_{\leq n}{}^k$ has a sh[$Q_{\leq n}$,0,...,n-1] embedded maximal clique.

EMBEDDED CEILING MAXIMAL CLIQUE "THEOREM" (ECM/graphs). For all k, every order invariant graph on $Q_{\geq -1}{}^k$ has a sh[$Q_{\geq -1}$,N] embedded ceiling maximal clique.

EMBEDDED INDUCTIVE CLIQUE THEOREM (EI/graphs). Every order invariant graph on Q^k has a sh[Q,N] embedded inductive clique.

THEOREM 5.2. All of these modifications, except II≠F, II≠/Eset, II≠/graphs, are provably equivalent to Con(SRP) over WKL_0. The latter three are provably equivalent to Con(HUGE) over WKL_0.

Finally, we present the modified embedded templates.

EMBEDDED FOLLOWER TEMPLATE. Given order theoretic partial f:J → J. For all k, every E ⊆ J^k has an f embedded maximal follower.

EMBEDDED E SET TEMPLATE. Given order theoretic partial f:J → J. For all k, every E ⊆ J^k has an f embedded maximal E set.

EMBEDDED CLIQUE TEMPLATE. Given order invariant partial f:J → J. For all k, every order invariant graph on J^k has an f embedded maximal clique.

EMBEDDED TEMPLATE CONJECTURE. Every instance of the Embedded Continuation, Follower, E set, and Clique Templates is provable or refutable in SRP.

THEOREM 5.3. The Embedded Template Conjectures fail for any singe finite fragment of SRP, assuming SRP is consistent.

The finite theorems of section 4 can also be converted, without difficulty, obtaining the same results as in section 4. But here there is a distinct advantage to the continuation formulations of section 4. This is because we need towers for the finite versions - i.e., each set is a subset of the next. This inclusion is already present with continuations (a tower of length 2), but not with followers, E sets, and cliques, in which inclusion is dropped.

6. Sequential constructions.

We use the following two Propositions for the sequential construction that drives the computer investigations discussed in section 7.

PROPOSITION 3.3.4 (E sets). Every E ⊆ Q2k has an upper integral invariant inductive E set containing its ≥0,≥1,... shifts.

INVARIANT INDUCTIVE E ≠-SET "THEOREM" (II≠/Eset). Every E ⊆ $Q2^k$ has an inductive E ≠-set S where each S+1 ∩ $Q[1,n]^k$ is a subset of S with field S<$(3/2)^n$>.

We make some adjustments to facilitate computer investigations. The challenge is to control the data enough to allow for exhaustive searches, but keep the process rich enough to reflect the underlying combinatorial structure of high powered set theory. We begin with Proposition 3.3.4.

DEFINITION 6.1. x ∈ $\{1,...,k\}^k$ is normal if and only if the set of terms of every element of E is an initial segment of 1,...,k. E ⊆ $\{1,...,k\}^k$ is normal if and only if every element of E is normal.

THEOREM 6.1. Every x ∈ Q^k order equivalent to a unique normal x ∈ $\{1,...,k\}^k$.

PROPOSITION 6.2. Let k,n and normal E ⊆ $\{1,...,2k\}^{2k}$ be given. E has an upper integral invariant inductive E set S ⊆ $Q[-1,n]^k$ containing its ≥0,...,≤n-1 shifts.

We fix k,n and normal E ⊆ $\{1,...,2k\}^{2k}$. We present a nondeterministic construction Δ(E,n) associated with Proposition 6.1.

Δ(E,n), if successfully implemented for infinitely many stages, creates finite partial functions $f_0 \subseteq f_1 \subseteq$ ... from $Q[-1,n]^k$ into {0,1}, where $f = \cup_i f_i$ is a function from some S into {0,1} such that $f^{-1}(0)$ is the desired inductive E set. I.e., $f^{-1}(0) \subseteq Q[-1,n]^k$ is an inductive E set which is upper integral invariant and contains its ≥0,...,≥n-1 shifts. Thus $f_i(x) = 0$ signifies that x ∈ $Q[-1,n]^k$ is accepted at stage i, and $f_i(x) = 1$ signifies that x ∈ $Q[-1,n]^k$ is rejected at stage i.

First and foremost we must be careful to arrange that each $f_{i-1}(0)$ is an E set in the usual sense. Every element of $f_i^{-1}(0)^2$ is order equivalent to some element of E. We give this condition a name.

DEFINITION 6.2. Let f:Q[-1,n]k → {0,1} be partial. f is E good if and only if
i. f is finite.
ii. $f^{-1}(0)$ is an E set.
f is E bad if and only if f is finite and not E good.

Secondly, at each stage $i+1 \geq 1$ we must resolve a chosen tuple $x \in Q[-1,n]^k \backslash dom(f_i)$ over f_i in the following sense.

DEFINITION 6.3. Let $f:Q[-1,n]^k \to \{0,1\}$ be E good, and $x \in Q[-1,n]^k \backslash dom(f)$. $g:Q[-1,n]^k \to \{0,1\}$ resolves x over f if and only if $f \subseteq g$, g is E good, and

i. $g(x) = 0$; or

ii. $g(x) = 1$ and there exists y such that $g(y) = 0$, $y < x$, where at least one of (x,x),(y,x),(x,y) is not order equivalent to any element of E.

Thirdly, we want each f_i to be suitably invariant in the sense corresponding to Proposition 6.1.

DEFINITION 6.4. Let $f:Q[-1,n]^k \to \{0,1\}$ be E good. f is upper integral invariant if and only if for all upper integral equivalent $x,y \in Q[-1,n]^k$, if $f(x) = 0$ then $f(y) = 0$, and if $f(x) = 1$ then $f(y) = 1$. f is $\geq j$ shift friendly if and only if the following holds. For all $x \in Q[-1,n-1]^k$, if $f(x) = 0$ then f at the $\geq j$ shift of x is 0. If f at the $\geq j$ shift of x is 1 then $f(x) = 1$.

The construction creates finite partial E good upper integral invariant shift friendly $f_0 \subseteq f_1 \subseteq \ldots$ from $Q[-1,n]^k$ into $\{0,1\}$, where for all $i \geq 0$, f_{i+1} resolves x_i over f_i. The additional requirement on f_0 is that $dom(f_0) = (-1,\ldots,-1), (0,\ldots,0), (1,\ldots,1), \ldots, (n,\ldots,n)$.

We have only to specify just what these $x_i \in Q[-1,n]^k \backslash dom(f_i)$ are, $i \geq 0$, that are required to be resolved by f_{i+1}.

To make sure that we satisfy Proposition 6.2, we can take x_i to be any element of $dom(f_j)\#\backslash dom(f_i)$, where j is chosen to be least such that $dom(f_i)\#\backslash dom(f_i)$ is nonempty. For specificity, we can choose x_i among element of $dom(f_i)\#\backslash dom(f_i)$, with least possible j, which is least first by maximum coordinate, and second lexicographically. This guarantees that all relevant x's get resolved. I.e., all $x \in dom(f)\#$ are resolved by f, where $f = \cup_i f_i$. This is what is required for f-10 to be an inductive E set.

However, there is a practical issue with this simple mode of selection. Note that $|fld(dom(f0))| = n+2$, and so $|dom(f_0)\#| = (n+2)^k$. We recommend working things out practically and theoretically, starting with even, say, $k = 2$ and $n = 1$. But we probably need to go at least somewhat further in order to have confidence that we are really challenging the large cardinals inside our computer. So with, say, $k = n = 4$, we are up to $6^4 = 1296$. Now each of these 1296, or at least of them, will themselves be computationally intensive, and so forth.

Furthermore, as a theoretical result, we do know that it is logically weak to have all of your x_i's living in $\text{dom}(f_0)$. We don't get out of weak fragments of ZFC this way. See the remarks at the end of section 4. So in the selection process, we need to get engaged early with coordinates of vectors coming in for resolving earlier vectors. Perhaps alternating with hanging around the early coordinates.

So we will consider a range of heuristic driven interactive selection rules for $x_i \in \text{dom}(f_i)\#\backslash\text{dom}(f_i)$. Here interactive means that the choice of x_i depends perhaps heavily (and deterministically) on what is nondeterministically generated. We can even use pseudorandom number generators as part of the selection procedure, as long as we can rerun its computation.

Given such a selection procedure, we can create a search procedure to find a corresponding $f_0,f_1,\ldots,f_t$, where t is small. In order to do this, we must exploit the fact that we are in the rationals, and we have a purely order theoretic situation, except for the +1 function on nonnegative rationals, and the preferred status of $0,\ldots,n$. Thus we can operate purely order theoretically on fractional parts, and use that n is small.

What remains to be seen is how large the search trees are, and what resources are needed to create them. There should be a deep theory here of how to cut down the size of the relevant search spaces.

Note that we can assume each $\text{fld}(\text{dom}(f_i))$ has at most $kn(i+1)+2$ elements.

PROPOSITION 6.3. Each $\Delta(E,n)$ can be carried out for infinitely many stages.

PROPOSITION 6.4. Each $\Delta(E,n)$ can be carried out for any given finite number of stages.

THEOREM 6.5. Proposition 6.4 are provably equivalent to Con(SRP) over WKL_0. Proposition 6.4 is provably equivalent to Con(SRP) over EFA.

7. Computer Investigations.

We now discuss the partial implementation of the infinite length nondeterministic algorithm $\Delta(E,n)$.

Informally, the goal is to arrange for the computation to be fully practical, and at the same time involve intense engagement with the underlying finite combinatorial structure associated with the SRP hierarchy of large cardinals.

We begin by choosing the dimension k and the integer n to be experimentally adjustable small integers. Also, experimentally pick a size c for a normal $E \subseteq \{1,...,2k\}^{2k}$. Then choose E randomly of this size. Experience here may show that perhaps some heuristics should be used, in addition to the randomness, to get $\Delta(E,n)$ going in an interesting way, and to ensure that no human being would have even the slightest clue as to how to work with it, any more than a completely arbitrary normal $E \subseteq \{1,...,2k\}^{2k}$.

By Proposition 6.4, $\Delta(E,n)$ can be carried out for any finite number of stages, t. But for small t, can we actually construct t stages?

By Theorem 6.5, we know, using Con(SRP), that t stages exist. Yet does this "knowledge" actually help construct t stages?

It k,c,t are suitably small, we can search for such t stages by exhaustion. We build a search tree of height t, but with high width. We need to control the width with an appropriate representation of the "equivalent" f_i's. We have to exploit the fact that the conditions are purely order theoretic, except for the +1 function on $\{0,...,n-1\}$, and the singling out of 0,...,n. Thus the problem is purely order theoretic in fractional parts.

So with care, we can keep control of the width of the search tree. Probably it will not be possible to have the height of such a tree be other than very small, and keep the size of the search tree reasonable.

This search tree approach will in fact accommodate any reasonable automatic, heuristic, pseudo randomized, even interactive selection process for the x_i to be resolved.

If exhaustive search does not come up with a path under the parameters chosen, then we know that SRP is inconsistent.

If exhaustive search does come up with a path under the parameters chosen, then arguably we have confirmed the consistency of at least a fragment of SRP. This would suggest that it more strongly confirms the consistency of ZFC, or even SMAH. Thus we have tested a predication made by Con(SRP) - that exhaustive search will have a positive outcome - something that seems impossible to do by a human, in two senses.

1. It seems impossible to prove by a human alone, in ZFC, or even in SMAH.

2. It seems impossible to actually construct an example by a human alone, even if the human is armed with the "knowledge" of large cardinals.

What is the strength of this kind of confirmation? Bear in mind that if the experiment comes out negative, then we would have a refutation of set theoretic hypotheses.

This issue is difficult to address at this stage of investigation.

Experts in large cardinals have developed a large amount of experience and intuitions that make them comfortable with Con(SRP). How much credence to give this experience and intuition is also difficult to address at this stage.

A possible reservation is that these scholars have a good grasp only of humanly intelligible proofs. It could be that the only inconsistencies in large cardinal hypotheses are not humanly intelligible. Certainly any inconsistency found by one of our proposed exhaustive searches coming up negative is completely humanly unintelligible. There is the remarkable possibility, not altogether absurd, that the proposed exhaustive searches actually do generate wholly humanly unintelligible inconsistencies. Note that we have attempted to arrange for these exhaustive searches to at least feel like they are intensely engaging the large cardinals through their underlying finite combinatorial structure.

8. APPENDIX - FORMAL SYSTEMS USED

EFA Exponential function arithemtid. Based on exponentiation and bounded induction.

RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics.

WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics.

ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics.

ACA' Arithmetic comprehension axiom prime. RCA0 + for every $x \subseteq \omega$ and n, the n-th Turing jump of x exists.

ACA^+ Arithmetic comprehension axiom plus. RCA0 + for every $x \subseteq \omega$, the ω-th Turing jump of x exists.

ATR($<\lambda$) Arithmetic transfinite recursion below λ. Here λ is given by a notation system. ACA0 + for every $x \subseteq \omega$, the Turing jump hierarchy of each length $\alpha < \lambda$ starting with x exists (as a scheme over $\alpha < \lambda$).

ATR_0 Arithmetic transfinite recursion. Our fourth level theory for Reverse Mathematics.

Z_2 Second order arithmetic. A two sorted first order theory. The outer framework for Reverse Mathematics.

ZF(C)\P Zermelo Frankel set theory (with the axiom of choice) without the power set axiom.

Z_n n-th order arithmetic.

TTY Type theory. Essentially $\cup_n Z_n$, a streamlined version of Russell's theory of types.

WZ(C) Weak Zermelo set theory (with the axiom of choice). Same as Z(C) with separation replaced by bounded separation.

Z(C) Zermelo set theory (with the axiom of choice). Essentially ZF(C) without Replacement.

ZF(C) Zermelo set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs.

SMAH ZFC + $(\exists\lambda)(\lambda$ is strongly k-Mahlo), as a scheme in k.

$SMAH^+$ ZRC + $(\forall k)(\exists\lambda)(\lambda$ is a strongly k-Mahlo).

1-SUB ZFC + $(\exists\lambda)(\lambda$ is subtle).

2-SRP ZFC + $(\exists\lambda)(\lambda$ has the 2-SRP).

SRP ZRC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k.

SRP^+ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP).

HUGE ZFC + $(\exists\lambda)(\lambda$ is k-huge), as a scheme in k.

$HUGE^+$ ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-huge).

λ is subtle if and only if for all $R \subseteq \lambda\times\lambda$ and stationary $A \subseteq \lambda$, there exists $\alpha < \beta$ from A such that $R_\alpha \cap \alpha = R_\beta \cap \alpha$.

λ is strongly 0-Mahlo if and only if λ is a strongly inaccessible cardinal. λ is strongly (k+1)-Mahlo if and only if λ is k-Mahlo and every stationary subset of λ has an element which is k-Mahlo.

λ has the k-SRP if and only if for all partitions of the unordered k-tuples from λ into two pieces, there is a stationary $B \subseteq A$, unbounded in λ, all of whose unordered k-tuples lie in the same piece.

λ is k-huge if and only if there exists an elementary embedding $j:V(\alpha) \to V(\beta)$ with critical point λ such that $\alpha = j^{(k)}(\lambda)$. (This hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings).

REFERENCES

[Fr10] H. Friedman, Forty years on his shoulders, in: Kurt Gödel and the Foundations of Mathematics: Horizons of Truth, Cambridge University Press, 2010.

[Fr13] Embedded Maximal Cliques and Incompleteness. Extended Abstract. 18 pages. May 20, 2013. http://www.math.osu.edu/~friedman.8/manuscripts.html

[Fr14] Boolean Relation Theory and Incompleteness, ASL, to appear. http://www.math.osu.edu/~friedman.8/manuscripts.html

[HP93] P. Hajek, P. Pudlak, Metamathematics of First-Order Arithmetic, Perspectives in Mathematical Logic, Springer, 1993.

[Ka94] A. Kanamori, The Higher Infinite, Perspectives in Mathematical Logic, Springer, 1994.

[Pr29] M. Presburger, Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt, Comptes rendus du I”, Congrès des Mathématiciens des pays Slaves, Marszawa, pp. 92-101.

[Si99,09] S. Simpson, Subsystems of Second Order Arithmetic, Springer Verlag, 1999.Second edition, ASL, 2009.