METAMATHEMATICS OF COMPARABILITY

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Abstract. A number of comparability theorems have been investigated from the viewpoint of reverse mathematics. Among these are various comparability theorems between countable well orderings ([2],[8]), and between closed sets in metric spaces ([3],[5]). Here we investigate the reverse mathematics of a comparability theorem for countable metric spaces, countable linear orderings, and sets of rationals. The previous work on closed sets used a strengthened notion of continuous embedding. The usual weaker notion of continuous embedding is used here. As a byproduct, we sharpen previous results of [3],[5].

1. COMPARABILITY OF COUNTABLE WELL ORDERINGS.

In this paper, we assume that the field of all linear orderings is a subset of $\mathbb{N}$, and all linear orderings are reflexive. (This is an official convention that we break at the slightest provocation). We normally write $\leq$, and use the usual notation $<$ for the irreflexive part.

We say that a linear ordering $\leq$ is a well ordering if and only if every nonempty subset of fld($\leq$) has a $<$ least element. A well ordering is a well founded linear ordering.

THEOREM 1.1. The following are provably equivalent in RCA_0.
i) ATR_0;
ii) For any two countable well orderings, there is a comparison map from one to the other;
iii) For any two countable well orderings, there is an order preserving map from one into the other.

Proof: For i) $\leftrightarrow$ ii), see [8], p. 198. By a comparison map we mean an isomorphism from one onto the other or from one onto an initial segment determined by a point in the other. For i) $\leftrightarrow$ iii), see [2]. QED

2. COMPARABILITY OF COUNTABLE METRIC SPACES.

We begin by proving what we call the comparability of countable metric spaces without worrying about what axioms are needed.

Let □ be any metric space. We allow □ to be empty.

Let x □ □ and □ □ □. An □ neighborhood of x is an open subset of □ that contains x. An □ clopen neighborhood of x is a simultaneously open and closed subset of □ that contains x.

An □ limit point of □ is a point in □ all of whose □ neighborhoods contain at least two elements from □. Note that □ limit points of □ do not have to lie in □.

Note that □ □ □ is closed if and only if every □ limit point of □ lies in □.

For each ordinal □, we define □[□] as follows. □[0] = □. □[□+1] is the set of all □ limit points of □[□]. □[□] is the intersection of all □[□], □ < □.

LEMMA 2.1. Let □ be a metric space. For all ordinals □, □[□+1] □ □[□] and □[□] is a closed subset of □. There is an ordinal □ such that □[□] = □[□+1]. If □[□] = □[□+1] then for all □ ≥ □, □[□] = □[□]. For □ □ □, every element of □ is an □ limit point of □ if and only if every element of □ is a □ limit point of □.

Proof: The first claim is proved by transfinite induction on □. The second claim follows from cardinality considerations. For the third claim, let □[□] = □[□+1], and verify by transfinite induction that for all □ ≥ □, □[□] = □[□].

For the final claim, let □ □ □. Observe that for y □ □, every □ neighborhood of y has the same elements from □ as some □ neighborhood of y, and every □ neighborhood of y has the same elements from □ as some □ neighborhood of y. QED

We define rk(□) as the least ordinal □ such that □[□] = □[□+1].

We define the core of □ as c(□) = □[rk(□)]. We can view c(□) as the "final" □[□].

For x □ □, we define rk(x, □) as the greatest ordinal □ such that x □ □[□]. We call this the □ rank of x. (Existence is discussed below).

We use □ to indicate proper inclusion.

LEMMA 2.2. Let □ be a metric space. For all x □ □, rk(x,□) exists if and only if x □ c(X). For all x □ □\c(□), rk(x,□) is the unique □ such that x □ □[□]\□[□+1]. rk(□) is the least ordinal greater than all defined rk(x,□). If □+1 < rk(□) then there are infinitely many x □ □ such that rk(x,□) = □. If □ < rk(□) then there exists x □ □ such that rk(x,□) = □.

Proof: Suppose rk(x) = □ exists. Then x □ □[□+1] ⊇ c(□). Therefore x □ c(□). Conversely, suppose x □ c(X). Write c(□) = □[□]. Then x □ □[□]. There must be a greatest □ < □ such that x □ □[□] since □[0] = □ and at limit stages we take intersections.

For the second claim, rk(x,□) is an □ such that x □ X[□]\X[□+1]. Suppose x □ X[□]\X[□+1]. Since x □ X[□]\X[□+1] we have □ □ □. Since x □ X[□]\X[□+1], we have □ □ □.

For the third claim, suppose rk(x,□) is defined. Write x □ □[□]\□[□+1]. Then □ < rk(□). Now let □ < rk(□). Then □[□] □[□+1]. Let x □ □[□]\□[□+1]. Then rk(x,□) = □. So □ is not greater than all defined rk(x,□).

For the fourth claim, let □+1 < rk(□). By the third claim, let □+1 □ rk(x,□). Let □ be such that x □ □[□]\□[□+1], where □ = rk(x,□). Then x □ □[□+1]\□[□+1], and so □[□+2] □ □[□+1]. Now suppose there are finitely many x such that rk(x,□) = □. Then □[□]\□[□+1] is finite. Therefore □[□] and □[□+1] have the same □ limit points. Hence □[□+1] = □[□+2], which is a contradiction. So there are infinitely many x such that rk(x,□) = □.

For the fifth claim, let □ < rk(□). By the third claim, let □ □ rk(x,□). Let □ be such that x □ X[□]\X[□+1], where □ = rk(x,□). Then x □ X[□]\X[□+1], and so X[□] X[□+1]. But any element of X[□]\X[□+1] is of rank □ in □. QED

LEMMA 2.3. Let □ be an open subspace of the metric space □. For all □, □[□] = □[□] □ □. For all x □ Y, rk(x,Y) = rk(x,X), where both sides are equal or both sides are undefined. rk(□) □ rk(□). c(□) □ c(□). Every element of c(□) is an □ limit point of c(□).

Proof: Let □,□ be as given. We prove the first claim is proved by transfinite induction. □[0] = □[0] □ □ follows from □ □ □.

Suppose □[□] = □[□] □ □. Then x □ □[□+1] □ x is a □ limit point of □[□] □ x is a □ limit point of □[□] □ □. Suppose x is a Y limit point of □[□] □ □. Then x □ □ and every □ neighborhood of x contains at least two points from □[□] □ □ (by intersecting the □ neighborhood with □). Hence x is an □ limit point of □[□], and so x □ □[□+1].

Conversely, suppose x □ □[□+1] □ Y. Then x □ Y, and every □ neighborhood of x contains at least two points from □[□]. Then every □ neighborhood of x contains at least two points from □[□], and these two points must be from □[□] □ □ = □[□]. Therefore x is a □ limit point of Y[□], and so x □ □[□+1].

Finally, suppose for all □ < □, □[□] = □[□] □ □. If x □ □[□] then for all □ < □, x □ □[□] □ □, and so x □ □[□] □ □. If x □ □[□] □ □ then for all □ < □, x □ □[□] □ □ = □[□].

Let x □ □ and □ be the unique ordinal such that x □ □[□]\□[□+1]. Then x □ □[□]\□[□+1] by the first claim. Hence rk(x,Y) = □ = rk(x,X). Let x □ □ and □ be the unique ordinal such that x □ □[□]\□[□+1]. Then again by the first claim, x □ □[□]\□[□+1]. Hence rk(x,X) = □ = rk(x,Y).

By Lemma 2.2, claim 3, rk(□) is the least ordinal > all Y ranks of elements of □. Thus rk(□) is at least as large as the least ordinal > all X ranks of elements of X, since by the second claim, the latter ordinals include the former ordinals.

Let c(□) = □[□], c[□] = □[□]. Then c[□] = □[max(□,□)], c[□] = □[max(□,□)]. Hence c(□) □ c(□) follows from the first claim.

Let c(□) = □[□] = □[□+1]. By definition, every □ limit point of □[□] lies in □[□]. QED

LEMMA 2.4. Let □ be a countable metric space and x □ □. Then every □ neighborhood of x contains an □ clopen neighborhood of x. rk(□) = 0 if and only if no element of □ has an □ rank.

Proof: Let □,x be as given. By cardinality considerations, there exist real numbers 0 < r < s such that {y: d(x,y) < r} = {y: d(x,y) < s}. Hence {y: d(x,y) < r} = {y: d(x,y) □ r}. The left side is open and the right side is closed. Therefore both sides are clopen.

The second claim follows from the third claim of 2.2. QED

We now define the index of □, written ind(□). This is the number of points x □ □ such that rk(x,□) is largest among all defined rk(y,□), y □ □. For countable □, note that ind(□) □ □ □ {□}. We write □ □ □ if and only if either rk(□) < rk(□) or (rk(□) = rk(□) and ind(□) □ ind(□)).

LEMMA 2.5. Let □ be a countable metric space. Ind(□) is the number of points x □ □ such that rk(□) = rk(x,□)+1. Ind(□) = 0 if and only if rk(□) is a limit ordinal or 0. If □ is an open subspace of □ then □ □ □. For every x □ □\c(□) and □ neighborhood S of x, there is an X clopen neighborhood □ □ S of x such that rk(□) = rk(x,□)+1, ind(□) = 1, and c(□) = ∅. □ among metric spaces is reflexive, transitive, and connected (□ □ □ or □ □ □).

Proof: For the first claim, note that by Lemma 2.2, the points x where rk(x,□) is largest must be the points x where rk(x,□)+1 = rk(□). The second claim follows immediately.

For the third claim, by Lemma 2.3, rk(□) □ rk(□). If rk(□) < rk(□) then □ □ □. Assume rk(□) = rk(□). We claim that every element of □ of maximum rank in □ is of maximum rank in □ because such an element must be of rank rk(□)-1 = rk(□)-1, and so of maximum rank in □. (We are using the second claim of Lemma 2.3).

For the fourth claim, let x □ □\c(□), rk(x,□) = □, and S be an □ neighborhood of x. Then x □ □[□+1], and so x is not an □ limit point of □[□]. Let □ be an □ neighborhood of x whose only element in common with □[□] is x. By Lemma 2.4, let □ □ S □ □ be an X clopen neighborhood of x whose only element in common with □[□] is x. By Lemma 2.3, □[□] = {x} and □[□+1] = ∅. Hence rk(x,□) = □+1, ind(□) = 1, and c(□) = ∅.

The final claim is left to the reader. QED

Let □,□ be metric spaces and $n \geq 0$.

An n-approximation from □ into □ is a sequence

$x_1,\ldots,x_n,\square_1,\ldots,\square_n,y_1,\ldots,y_n,C_1,\ldots,C_n$

where

i) $\square_1,\ldots,\square_n$ are pairwise disjoint clopen neighborhoods, respectively, of $x_1,\ldots,x_n$ □ □, with diameters $< 2^{-n}$;
ii) $C_1,\ldots,C_n$ are pairwise disjoint clopen neighborhoods, respectively, of $y_1,\ldots,y_n$ □ □, with diameters $< 2^{-n}$;
iii) for all 1 □ i □ n, $rk(\square_i) = rk(C_i) = rk(x_i)+1 = rk(y_i)+1$, and $ind(\square_i) = ind(C_i) = 1$.

For any n-approximation W, we will speak of x_i of W, $\square_i$ of W, y_i of W, and C_i of W, where 1 □ i □ n. With this terminology, we can avoid introducing hard-to-read notation.

The diameter of a set is the least nonnegative real number at least as large as the distances between all pairs from the set. If there is no such real number, then the diameter is .

LEMMA 2.6. Let □,□ be countably infinite metric spaces with □ □ □, where c(□) = c(□) = ∅. There exists an infinite sequence $W_1,W_2,\ldots$ such that
i) each W_i is an i-approximation;
ii) for all $i \geq 1$, x_i of W_{i+1} equals x_i of W_i, and y_i of W_{i+1} equals y_i of W_i;
iii) for all 1 □ i □ j, $\square_i$ of W_j contains $\square_i$ of W_{j+1}, and C_i of W_j contains C_i of W_{j+1};
iv) for all 1 □ i □ p < q and 1 □ j □ q, either ($\square_i$ of W_p contains $\square_j$ of W_q, and C_i of W_p contains C_j of W_q) or ($\square_i$ of W_p and $\square_j$ of W_q are disjoint);
v) for all x □ □ there exists i such that x is x_i of W_i.

Proof: Suppose we have constructed $W_0,\ldots,W_n$, $n \geq 0$, where clauses i) - iv) hold for $W_0,\ldots,W_n$. Let x be a new element of □ (i.e., not among $x_1,\ldots,x_n$ of W_n). It suffices to construct W_{n+1} so that clauses i) - iv) are preserved, and where x_{n+1} of W_{n+1} is x. Set $x_1,\ldots,x_n$ of W_{n+1} to $x_1,\ldots,x_n$ of W_n, and $y_1,\ldots,y_n$ of W_{n+1} to $y_1,\ldots,y_n$ of W_n.

case 1. There does not exist 1 □ i □ j □ n such that $\square_i$ of W_j includes x. By Lemma 2.5, we set $\square_{n+1}$ of W_{n+1} to be any clopen neighborhood of x of diameter < 2^{-n-1} such that rk($\square_{n+1}$) = rk(x,□)+1, ind($\square_{n+1}$) = 1, and which is disjoint from all $\square_i$ of W_j, 1 □ i □ j □ n. This last requirement is met by demanding that $\square_{n+1}$ be included in the complement of the union of all $\square_i$ of W_j, 1 □ i □ j □ n, which must be an □ neighborhood of x by hypothesis.

Set $\square_1,...,\square_n$ of W_{n+1} to be □ clopen neighborhoods of $x_1,...,x_n$ of W_n of diameter < 2^{-n-1} which are included, respectively, in $B_1,...,B_n$ of W_n. This step preserves rank and index.

To define y_{n+1},C_{n+1} of W_{n+1}, first suppose rk(x,□)+1 < rk(□). By Lemma 2.2, there are infinitely many y of the same Y rank as rk(x,□), and so we can choose y to be distinct from $y_1,...,y_n$ of W_{n+1}, with rk(y,□) = rk(x,□). Set y_{n+1} of W_{n+1} to be y. By Lemma 2.5, set C_{n+1} of W_{n+1} to be any □ clopen neighborhood of y of diameter < 2^{-n-1}, of rank rk(y,□)+1, of index 1, which excludes $y_1,...,y_n$ of W_n.

Secondly, suppose rk(x,□)+1 ≥ rk(□). Since □ □ □, we have rk(x,□)+1 = rk(□) = rk(□). Let p be the number of the $y_1,...,y_n$ of W_{n+1} whose □ rank is rk(□)-1. Then p is also the number of the $x_1,...,x_n$ of W_{n+1} whose □ rank is rk(□)-1. Since x also has X rank rk(X)-1, clearly p < ind(□) □ ind(□). Therefore we can find y of □ rank rk(□)-1 which is distinct from $y_1,...,y_n$ of W_n. Set y_{n+1} of W_{n+1} to be y. By Lemma 2.5, set C_{n+1} of W_{n+1} to be any □ clopen neighborhood of y of diameter < 2^{-n-1}, of rank rk(y,□)+1 = rk(□) = rk(□), of index 1, which excludes $y_1,...,y_n$ of W_n.

Finally, by Lemma 2.5, set $C_1,...,C_n$ of W_{n+1} to be Y clopen neighborhoods of $y_1,...,y_n$ of W_n of diameter < 2^{-n-1} which are included, respectively, in $C_1,...,C_n$ of W_n, and which are disjoint from C_{n+1} of W_{n+1}. This step preserves rank and index.

case 2. Suppose there exists 1 □ i □ j □ n such that $\square_i$ of W_j includes x. Since for a fixed j □ n, the various $\square_i$ of W_j are pairwise disjoint, we see that for each fixed j □ n, there is at most one i such that $\square_i$ of W_j includes x. Let □ be the set of such j □ n. As j increases through □, the corresponding $\square_i$ of W_j form a chain under reverse inclusion by the hypothesis that iii) holds for $W_1,...,W_n$. Furthermore, as j increases through □, the corresponding C_i of W_j also forms a chain under reverse inclusion by iii).

We now fix j to be the largest element of $\square$, and fix i such that $\square_i$ of W_j includes x. By Lemma 2.5, set B_{n+1} of W_{n+1} to be an $\square$ clopen neighborhood of x contained in $\square_i$ of W_j, whose diameter is at most 2^{-n-1}, of rank $rk(x,\square)+1$, and index 1, which excludes $x_1,...,x_n$ of W_n, and also is disjoint from every $\square_r$ of every W_t, $1 \square r \square t \square n$, that does not include x.

Set $\square_1,...,\square_n$ of W_{n+1} to be $\square$ clopen neighborhoods of $x_1,...,x_n$ of W_n of diameter $< 2^{-n-1}$ which are included, respectively, in $\square_1,...,\square_n$ of W_n. This step preserves rank and index.

Note that since $\square_i$ of W_j includes x, its rank is $\geq rk(x,\square)+1$. We claim that $rk(\square_i$ of $W_j) > rk(x,\square)+1$. Suppose this is false. Then $rk(\square_i$ of $W_j) = rk(x,\square)+1 = rk(x_i$ of $W_j)$, and $\square_i$ of W_j has index 1. Therefore x_i of W_j is x, which is a contradiction.

Thus we have $rk(C_i$ of $W_j) = rk(\square_i$ of $W_j) > rk(x,\square)+1$. By Lemma 2.2, C_i of W_j has infinitely many elements of $\square$ rank $rk(x,\square)$. Let $z \square C_i$ of W_j, $rk(z,\square) = rk(x,\square)$, where z is distinct from $y_1,...,y_n$ of W_n. Set y_{n+1} of W_{n+1} to be z. By Lemma 2.5, set C_{n+1} of W_{n+1} to be any $\square$ clopen neighborhood of z contained in C_i of W_j of rank $rk(x,\square)+1$, of index 1, with diameter $< 2^{-n-1}$, which excludes $y_1,...,y_n$ of W_n. Set y_{n+1} of W_{n+1} to be z.

Finally, set $C_1,...,C_n$ of W_{n+1} to be $\square$ neighborhoods of $y_1,...,y_n$ of W_n of diameter $< 2^{-n-1}$ which are included, respectively, in $C_1,...,C_n$ of W_n, and which are disjoint from C_{n+1} of W_{n+1}. This step preserves rank and index.

Note that all of the requirements have obviously been met except iv). Let $1 \square i \square p < n+1$ and $1 \square j \square n+1$. We must verify that either ($\square_i$ of W_p contains $\square_j$ of W_{n+1}, and C_i of W_p contains C_j of W_{n+1}) or ($\square_i$ of W_p and $\square_j$ of W_{n+1} are disjoint). If $j \square n$ then by hypothesis we have ($\square_i$ of W_p contains $\square_j$ of W_n, and C_i of W_p contains C_j of W_n) or ($\square_i$ of W_p and $\square_j$ of W_n are disjoint). But this implies the same statement with W_n replaced by W_{n+1}.

Thus we need only verify that for all $1 \square i \square p < n+1$, either ($\square_i$ of W_p contains $\square_{n+1}$ of W_{n+1}, and C_i of W_p contains C_{n+1} of W_{n+1}) or ($\square_i$ of W_p and $\square_{n+1}$ of W_{n+1} are disjoint).

In case 1 above, $\square_{n+1}$ is chosen so that the second disjunct holds. Suppose case 2 applies, and assume $\square_i$ of W_p meets $\square_{n+1}$ of W_{n+1}. Then $\square_i$ of W_p includes x_{n+1} of W_{n+1}. Hence $\square_i$ of W_p is among the chain under reverse inclusion used in case 2. Therefore $\square_i$ of W_p contains $\square_{n+1}$ of W_{n+1} and C_i of W_p contains C_{n+1} of W_{n+1}. QED

Let $\mathcal{X},\mathcal{Y}$ be metric spaces. We say that $F:\mathcal{X} \to \mathcal{Y}$ is continuous if and only if for all $x \in \mathcal{X}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathcal{X}$, $d(x,y) < \delta \to d(F(x),F(y)) < \varepsilon$. We say that F is a continuous embedding if and only if F is continuous and one-one.

LEMMA 2.7. Let $\mathcal{X},\mathcal{Y}$ be countably infinite metric spaces with $\mathcal{X} \leq \mathcal{Y}$, where $c(\mathcal{X}) = c(\mathcal{Y}) = \emptyset$. Then there exists a continuous embedding from $\mathcal{X}$ into $\mathcal{Y}$.

Proof: Let $W_0,W_1,\ldots$ be given by Lemma 2.6. To define $F:X \to Y$, let $x \in \mathcal{X}$ and assume that x is x_i of W_i. Set F(x) to be y_i of W_i. It is obvious that F is one-one.

We now verify that F is continuous. Let $x \in X$ and $\varepsilon > 0$. Choose $n < m$ such that x is x_n of W_m, and F(x) is y_n of W_m, where $2^{-m} < \varepsilon$. Then $\square_n,C_n$ of W_m are clopen neighborhoods of x,F(x), respectively, of diameter $< \varepsilon$.

We now claim that $F[\square_n$ of $W_m] \subseteq C_n$ of W_m. To see this, let z lie in $\square_n$ of W_m, $z \neq x_n$, where z is x_r of W_r. Then $r \neq n$, and F(z) is y_r of W_r. If $r < m$ then z is the x_r of W_m, and so lies outside $\square_n$ of W_m. Therefore $r > m$.

Hence either ($\square_n$ of W_m contains $\square_r$ of W_r and C_n of W_m contains C_r of W_r) or ($\square_n$ of W_m and $\square_r$ of W_r are disjoint). Therefore $\square_n$ of W_m contains $\square_r$ of W_r and C_n of W_m contains C_r of W_r. Hence F(z) lies in C_n of W_m.

We have shown that $F[\square_n$ of $W_m] \subseteq C_n$ of W_m. To complete the proof, let $\delta > 0$ be such that the open ball in $\mathcal{X}$ with center x is contained in $\square_n$ of W_m. Let $d(x,y) < \delta$. Then y lies in $\square_n$ of W_m, and therefore F(y) lies in C_n of W_m. Since C_n of W_m has diameter $< \varepsilon$ and contains F(x), we have $d(F(x),F(y)) < \varepsilon$. QED

Let $\mathcal{X},\mathcal{Y}$ be metric spaces and $n \geq 0$.

A weak n-approximation from $\mathcal{X}$ into $\mathcal{Y}$ is a sequence

$x_1,\ldots,x_n,\square_1,\ldots,\square_n,y_1,\ldots,y_n,C_1,\ldots,C_n$

where

i) □$_1$,...,□$_n$ are pairwise disjoint □ clopen neighborhoods, respectively, of $x_1,...,x_n$ □ □, with diameters < 2^{-n};
ii) $C_1,...,C_n$ are pairwise disjoint □ clopen neighborhoods, respectively, of $y_1,...,y_n$ □ □, with diameters < 2^{-n}.

LEMMA 2.8. Let □,□ be countably infinite metric spaces where every element of □ is a □ limit point. There exists an infinite sequence $W_0,W_1,...$ such that
i) each W_i is a weak i-approximation;
ii) - v) same as in Lemma 2.6.

Proof: Suppose we have constructed $W_0,...,W_n$, $n \geq 0$, where clauses i) - iv) hold for $W_0,...,W_n$. Let x be a new element of □ (i.e., not among $x_1,...,x_n$ of W_n). It suffices to construct W_{n+1} so that clauses i) - iv) are preserved, and where x_{n+1} of W_{n+1} is x. Set $x_1,...,x_n$ of W_{n+1} to $x_1,...,x_n$ of W_n, and $y_1,...,y_n$ of W_{n+1} to $y_1,...,y_n$ of W_n.

case 1. There does not exist 1 □ i □ j □ n such that □$_i$ of W_j includes x. Then we set □$_{n+1}$ of W_{n+1} to be any □ clopen neighborhood of x of diameter < 2^{-n-1} which is disjoint from all □$_i$ of W_j, 1 □ i □ j □ n.

Set □$_1$,...,□$_n$ of W_{n+1} to be □ clopen neighborhoods of $x_1,...,x_n$ of W_n of diameter < 2^{-n-1} which are included, respectively, in □$_1$,...,□$_n$ of W_n.

Set y_{n+1} of W_{n+1} to be distinct from $y_1,...,y_n$. Set C_{n+1} of W_{n+1} to be any □ clopen neighborhood of y of diameter < 2^{-n-1}, which excludes $y_1,...,y_n$ of W_n.

Finally, set $C_1,...,C_n$ of W_{n+1} to be □ clopen neighborhoods of $y_1,...,y_n$ of W_n of diameter < 2^{-n-1} which are included, respectively, in $C_1,...,C_n$ of W_n, and which are disjoint from C_{n+1} of W_{n+1}.

case 2. Suppose there exists 1 □ i □ j □ n such that □$_i$ of W_j includes x. Since for a fixed j, the various □$_i$ of W_j are pairwise disjoint, we see that for each fixed j, there is at most one i such that □$_i$ of W_j includes x. Let □ be the set of such j. As j increases through □, the corresponding □$_i$ of W_j form a chain under reverse inclusion by iii). Furthermore, as j increases through □, the corresponding C_i of W_j also form a chain under reverse inclusion by iii). Let j be the largest element of □, and let i be such that □$_i$ of W_j includes x. Set

$\Box_{n+1}$ of W_{n+1} to be an □ clopen neighborhood of x contained in $\Box_i$ of W_j, whose diameter is at most 2^{-n-1}, which excludes $x_1,\ldots,x_n$ of W_n, and which is disjoint from every $\Box_r$ of every W_t, $1 \Box r \Box t \Box n$, that does not include x.

Set $\Box_1,\ldots,\Box_n$ of W_{n+1} to be □ clopen neighborhoods of $x_1,\ldots,x_n$ of W_n of diameter $< 2^{-n-1}$ which are included, respectively, in $\Box_1,\ldots,\Box_n$ of W_n.

Set y_{n+1} of W_{n+1} to be any element of C_i of W_j distinct from $y_1,\ldots,y_n$ of W_n. Note this this uses the hypothesis that every element of □ is a □ limit point. Set C_{n+1} of W_{n+1} to be any □ neighborhood of y_{n+1} of W_{n+1} with diameter $< 2^{-n-1}$.

Finally, set $C_1,\ldots,C_n$ of W_{n+1} to be □ neighborhoods of $y_1,\ldots,y_n$ of W_n of diameter $< 2^{-n-1}$ which are included, respectively, in $C_1,\ldots,C_n$ of W_n, and which are disjoint from C_{n+1} of W_{n+1}. QED

LEMMA 2.9. Let □,□ be countably infinite metric spaces, where every element of □ has a limit point. Then there exists a continuous embedding from □ into □.

Proof: The proof of Lemma 2.7 did not use that the W_i are i-approximations, but rather only that the W_i are weak i-approximations. Thus we can merely repeat the proof of Lemma 2.7. QED

THEOREM 2.10. Let □,□ be countable metric spaces. There is a continuous embedding from □ into □ or vice versa.

Proof: If □,□ are finite then this is obvious. It is also obvious if one of □,□ is finite and the other is infinite. Thus we assume that □,□ are both infinite. Suppose c(□) = c(□) = ∅. Obviously |□| □ |□| or |□| □ |□|. Apply Lemma 2.5.

We are left with the case where c(□) or c(□) is nonempty. By symmetry, assume c(□) is nonempty. By Lemma 2.3, every element of c(□) is a □ limit point of c(□). By Lemma 2.9, let F be a continuous embedding from □ into c(□). Obviously F is also a continuous embedding from □ into □. QED

We now show that ATR_0 suffices to prove Theorem 2.10.

For the purposes of formalization, we take a countable metric space to be a function of the form $d:\Box^2 \Box$ R, where □ □ □ (d is the metric), satisfying the usual inequalities. (This is an official convention that we break on the slightest provocation).

Let □ be a countable metric space. A ranking of □ consists of a countable linear ordering □□ (with field □ □) together with a relation R □ fld(□□)□fld(X), such that the following holds. Let x □ fld(□□).

1. If x is the least element of □□ then R_x = □.
2. If x is a limit point of □□ then R_x is the intersection of the R_y, y <□ x.
3. If x is the immediate successor of y in □□ then R_x is the set of all limit points of R_y.
4. □□ has a least element.
5. Every element that is not greatest has an immediate successor.
6. If x <□ y then $R_x \supseteq R_y$.

A ranking is said to be full if it additionally satisfies the following condition:

7. There exists x such that $R_x = R_y$, where y is the immediate successor of x in □□.

LEMMA 2.11. The following is provable in ACA_0. Let □,□ be countable metric spaces with rankings and □□ be a well ordering. Then
i) there is at most one full ranking of □ using □□;
ii) if there is a full ranking of □ using □□ and a ranking of □ using □□, then there is a continuous embedding from □ into □ or vice versa;
iii) Lemma 2.9.

Proof: By straightforward verification. QED

LEMMA 2.12. The following is provable in ATR_0. Let □ be a countable metric space and □□ be a countable well ordering. Then there is a unique ranking of □ using □□.

Proof: The first three clauses constitute a definition by arithmetic transfinite recursion. QED

LEMMA 2.13. The following is provable in ATR_0. Let □ be a countable metric space such that there does not exist a full ranking whose linear ordering is a well ordering. Then □ has a ranking based on a non-well-founded countable linear ordering that is not full.

Proof: Otherwise, using Lemma 2.12, we would have a $\square^1_1$ definition of countable well ordering, which is impossible by [8], p. 172. QED

LEMMA 2.14. The following is provable is ATR_0. Let □ be a countable metric space. Either □ has a full ranking using a countable well ordering, or there exists a nonempty □ □ □ such that every element of □ is an □ limit point in □.

Proof: Let □ be a countable metric space which does not have a full ranking using a countable well ordering. By Lemma 2.13, let R be a ranking of □ based on the non-well-founded countable linear ordering □□ that is not full. Let S □ fld(□□) have no □□ least element. Let □ be the union of the R_i, i □ S. Since R is not full, Y must be nonempty. We claim that every element of □ is an □ limit point of □. To see this, let x □ R_i, i □ S. Let j <□ i, j □ S, and let k be the immediate successor of j in □□. Then x □ R_k. But also x is an □ limit point of R_j. Hence x is an □ limit point of □. QED

THEOREM 2.15. The following is provable in ATR_0. Let □,□ be countable metric spaces. There is a continuous embedding from □ into □ or vice versa.

Proof: Let □,□ be given. Without loss of generality, we can assume that □,□ are both infinite.

First suppose that □,□ have full rankings using well orderings. By using the comparability of countable well orderings, we see that □,□ have full rankings using a common countable well ordering. By Lemma 2.11, there is a continuous embedding from □ into □ or vice versa.

Now suppose that □ does not have a full ranking using a well ordering (the other case being symmetric). By Lemma 2.14, there exists a nonempty W □ □ such that every element of W is a □ limit point of W. By the third claim of Lemma 2.11, there is a continuous embedding from □ into W, and therefore from □ into □. QED

From Lemmas 2.7 and 2.9, we see that a countable metric space □ is continuously embeddable into a countable metric space Y if and only if either |□| □ |Y| or c(Y) ≠ ∅. Here the only if part is easily verified.

From this we conclude that the ordering of countable metric spaces under continuous embeddability forms a well founded connected transitive relation, which, when factored out by its equality relation, is of order type $\square_1$+1. The maximum element corresponds to all of the countable metric spaces with nonempty core.

This raises a question in descriptive set theory. How many countable metric spaces are there up to homeomorphism?

For countable compact metric spaces, our rank and index give a complete set of invariants for homeomorphism, and so the answer is $\square_1$ in this case. A similar complete set of invariants can be given for countable closed sets of real numbers up to homeomorphism, also giving the answer $\square_1$ in that case.

We now give an explicit construction of $2^{\square_0}$ many nonhomeomorphic countable metric spaces, even with empty cores (c(□) = ∅). Note that having an empty core is equivalent to having no nonempty subspace in which every point is a limit point.

THEOREM 2.16. There is an uncountable Borel set of sets of rationals with empty cores, no two of which are homeomorphic.

Proof: In a metric space □, we say that X is compact at x □ □ if and only if there is a neighborhood of x whose closure is compact.

For each n ≥ 1, let $\square_n$ □ [2n,2n+1] be a closed set of rationals of order type $\square^n$+1 whose greatest element is 2n+1. Note that 2n+1 is of rank n in $\square_n$. We can destroy the compactness of $\square_n$ at 2n+1 by adding new points to $\square_n$ as follows. For each p ≥ 2, add a set of points to (2n+1-(1/p+1),2n+1-(1/p)) of order type □ that do not lie in $\square_n$ and whose sup does not lie in $\square_n$. The resulting space $\square_n$□ is not compact at 2n+1. However, $\square_n$□ is compact at every point in $\square_n$□\{2n+1}.

For each S □ □\{0} we define S* to be the union of the $\square_n$, n □ S, together with the union of the $\square_n$□, n □ S. Note that S = {n ≥ 1: every point in S* is compact at every point of rank n}. The result follows since this set is a topological invariant. QED

Su Gao has investigated countable metric spaces under homeomorphism. He has shown that graph isomorphism is Borel reducible to homeomorphism of countable metric spaces (unpublished), greatly extending Theorem 2.16. The proof is a variation of the corresponding result for countable Boolean algebras in [1]. Gao also notes that by Stone duality, countable compact metric spaces under homeomorphism is appropriately equivalent to superatomic countable Boolean algebras under isomorphism, whose classification is well known and can be found in [7].

3. COMPARABILITY OF COUNTABLE LINEAR ORDERINGS.

In this section, we give a purely order theoretic version of Theorem 2.10, and show that it implies ATR_0 over RCA_0. The results of sections 2 and 3 are then used in section 4 to obtain the main results of the paper. We follow the terminology used in section 1.

Let □□ be a countable linear ordering. We say that x is a topological limit point in □□ if and only if

i) x is not the left endpoint and there is no greatest y <□ x; or
ii) x is not the right endpoint and there is no least y >□ x.

We say that $x_1,x_2,\ldots$ approaches y in □□ if and only if

i) for all z <□ y, the x's are eventually >□ z;
ii) for all z >□ y, the x's are eventually <□ z.

We say that F is an order continuous map from □□ into □* if and only if

i) F:fld(□□) □ fld(□*);
ii) if $x_1,x_2,\ldots$ approaches y in □□, then $F(x_1),F(x_2),\ldots$ approaches F(y) in □*.

We say that F is an order continuous embedding from □□ into □* if and only if F is an order continuous map from □□ into □* which is one-one.

LEMMA 3.1. Let F be an order continuous embedding from □□ into □*. Then F maps topological limit points to topological limit points.

Proof: Let F be as given, and x be a topological limit point in □□. By symmetry, assume x is not the left endpoint and there is no greatest y <□ x. By primitive recursion, define y_1 <□ y_2 <□ ... <□ x approaching x. Then $F(y_1),F(y_2),...$ approaches F(x), and the $F(y_i)$ are distinct. Hence we can find an infinite subsequence which are either all < F(x) or all > F(x). Then by using the definition of approaches, we see that F(x) is a topological limit point. QED

LEMMA 3.2. The following are equivalent over RCA_0.
i) ACA_0;
ii) For all one-one $F:\mathbb{N} \to \mathbb{N}\setminus\{0\}$, rng(F) exists.

Proof: See [8], p. 106. Compose values with +1. QED

LEMMA 3.3. The following implies ACA_0 over RCA_0. Let □□,□* be countable well orderings with greatest elements. There is an order continuous embedding from □□ into □* or vice versa.

Proof: Working in RCA_0, we assume the given comparability statement. We will use the usual lexicographic ordering □□ on $\{(n,m): n,m \in \mathbb{N}\} \cup \{(\infty,\infty)\}$. (Technically speaking, we need to use codes for pairs in order to adhere to the convention that the fields of linear orderings are subsets of $\mathbb{N}$). It is well known that RCA_0 proves that □□ is a well ordering.

Let $F:\mathbb{N} \to \mathbb{N}$ be one-one. We will use a one-one sequence numbering. We say that n is a good sequence number if and only if n is the index of a nonempty sequence $(a_1,...,a_p)$ such that for all $1 \leq i \leq p$, $a_i = F^{-1}(i)$ if it exists; 0 otherwise. We say that n is an m-good sequence number if and only if n is the index of a sequence $(a_1,...,a_p)$ such that for all $1 \leq i \leq p$, either $a_i = F^{-1}(i)$ or $a_i = 0$, and if $F^{-1}(i) \leq m$ then $a_i = F^{-1}(i)$. Note that n is a good sequence number if and only if for all $m \geq 0$, n is an m-good sequence number. Also, using Σ^0_1-induction (available in RCA_0), there is exactly one good sequence number of each length ≥ 0.

We define □* to have field $D = \{(i,n,m): 0 \leq i \leq 1 \wedge n$ is an m-good sequence number $\wedge\ m \in \mathbb{N}\} \cup \{(i,n,\infty): 0 \leq i \leq 1 \wedge n \in \mathbb{N}\} \cup \{(i,\infty,\infty): 0 \leq i \leq 1\}$, where the triples are ordered lexicographically. Obviously □* is also a well ordering with a greatest element.

We can characterize the topological limit points of □* as follows. They are the (i, ,), and the (i,n,), where n is a good sequence number.

Let F be an order continuous embedding from □□ into □* or vice versa. We first assume that F is an order continuous embedding from □′ into □*.

Since each (n,0) is a topological limit point in □□, each F((n,0)) is a topological limit point in □*. Fix n ≥ 0. Then F((0,0)),...,F((2n+3,0)) is a list of 2n+4 distinct topological limit points of □*. According to the characterization of the topological limit points of □* above, at least 2n+2 of the entries in this list are of the form (i,p,), where p is a good sequence number. Hence we can assume that at least n+1 of the entries in this list are of the form (0,p,), where the p's are of different lengths. (We could require 1 instead of 0, but that case is symmetric). Therefore the first coordinate of at least one entry in F((0,0)),...,F((2n+3,0)) is a good sequence number of length ≥ n. Therefore n is a value of F if and only if there exists an entry in in F((0,0)),...,F((2n+3,0)) whose first coordinate is of length ≥ n and whose n-th term is nonzero. This is an appropriate definition of the set of values of F. Hence rng(F) exists.

We now assume that F is an order continuous embedding from □* into □□ and derive a contradiction. Since F is one-one, either F((0, ,)) or F((1, ,)) is distinct from (,). By symmetry, we can let F((0, ,)) = (n,m), n,m □ □. By Lemma 3.1, (n,m) is a limit point, and hence n ≥ 1 and m = 0. Note that (0,0,),(0,1,),(0,2,),... approaches (0, ,). Hence by Lemma 3.1, F((0,0,),F((0,1,)),... approaches (n,0).

For each good sequence number p, (0,p,) is a topological limit point. Hence by Lemma 3.1, there are arbitrarily large p such that F((0,p,)) is a topological limit point. But there are no topological limit points between (n-1,1) and (n,1) except for (n,0). This is a contradiction. QED

Let □□ be a countable well ordering. We define the countable linear ordering $□^{□□}$ which is the usual lexicographic ordering on the set of all finite sequences $(x_1,n_1),...,(x_k,n_k)$, where

$k \geq 0$, $n_1,\ldots,n_k \geq 1$, and $x_1 >_\alpha x_2 >_\alpha \ldots >_\alpha x_k$. I.e., we first order the ordered pairs in fld(α) $\times \mathbb{N}\backslash\{0\}$ lexicographically, and then order such tuples of pairs lexicographically. This amounts to looking at Cantor normal forms "below" ω^α. (Technically speaking, we need to use sequence numbers in order to adhere to our convention that the field of any linear ordering is a subset of $\mathbb{N}$).

LEMMA 3.4. The following is provable in ACA_0. If α is a countable well ordering then ω^α is a countable well ordering.

Proof: This is proved in [4], but for the sake of completeness, we sketch a proof. Let S be a nonempty subset of ω^α with no ω^α least element. Form the infinite sequence $(x_1,n_1),(x_2,n_2),\ldots$ in ω^α, where (x_1,n_1) is the α least first term among the elements of S, (x_2,n_2) is the α least second term among the elements of S whose first term is (x_1,n_1), etcetera. This construction can be carried out within ACA_0 (but not within RCA_0). Clearly $x_1 >_\alpha x_2 >_\alpha \ldots$, which contradicts that α is a well ordering. QED

For $x \in$ fld(ω^α), we write x# for the first coordinate of the last term of x. This is undefined for the empty sequence.

LEMMA 3.5. The following is provable in RCA_0. Let α be a countable well ordering, and $x_1,x_2,\ldots$ approach y in ω^α, where the x_i are all distinct. Then for all sufficiently large i, $x_i < y$ in ω^α. For all sufficiently large i, $x_i\# <_\alpha y\#$.

Proof: Let $\alpha,x_1,x_2,\ldots,y$ be as given. Since the x_i are distinct, α is nonempty. Hence y has an immediate successor. Hence for all sufficiently large i, $x_i \leq y$ in ω^α. The first claim follows by distinctness. For the second claim, it suffices to show that for some $z < y$ in ω^α, every w strictly between z and y must have $z\# <_\alpha y\#$. To construct z, first note that y is nonempty (otherwise it would be the left endpoint), and y# is not the left endpoint, 0, of α (otherwise y would have an immediate predecessor). Let (y#,p) be the last term of y. Take z to be the result of replaceing (y#,p) by (y#,p-1),(0,1) if $p \geq 2$; (0,1) if $p = 1$. QED

LEMMA 3.6. The following is provable in RCA_0. Let α be a countable well ordering, $x <_\alpha y\#$, $y \in$ fld(ω^α). Then there exists $w_1 < w_2 < \ldots < y$ in ω^α approaching y in ω^α, where each $w_i\# = x$.

Proof: Let (y#,p) be the last term of y. First assume that y# is a limit. Let x <□ b_1 <□ b_2 <□ ... approach y#. Set w_i to be the result of replacing (y#,p) by (y#,p-1),(b_i,1),(x,1) if p ≥ 2; (b_i,1),(x,1) if p = 1.

Now assume that y# is the immediate successor of b ≠ x. Set w_i to be the result of replacing (y#,p) by (y#,p-1),(b,i),(x,1) if p ≥ 2; (b,i),(x,1) if p = 1.

Finally, assume that y# is the immediate successor of x. Set w_i to be the result of replacing (y#,p) by (y#,p-1),(x,i) if p ≥ 2; (x,i) if p = 1. QED

LEMMA 3.7. The following is provable in RCA_0. Let F be an order continuous embedding from □[□□] into □[□*], where □□ and □* are countable well orderings. Let y □ fld(□[□□]) and x <□ y#. Then there exists z □ fld(□[□□]) such that z# = x and F(z)# <* F(y)#.

Proof: Let F,□□,□*,y,x be as given. By Lemma 3.6, let w_1 < w_2 < ... < y approach y in □[□□], where each w_i# = x. Then F(w_1),F(w_2),... approaches F(y) in □[□*], where the F(w_i) are distinct. By Lemma 3.5 applied to □*, for all sufficiently large i, F(w_i)# <□ F(y)#. Set z = w_i, for any particular sufficiently large i. QED

LEMMA 3.8. The following is provable in ACA_0. Let F be an order continuous embedding from □[□□] into □[□*], where □□ and □ are countable well orderings. Then there exists an order preserving map from □□ into □*.

Proof: Let F,□□,□* be as given. We define G:fld(□□) □ fld(□*) by G(u) = the <* least element of {F(y)#: y# = u}. Let x <□ u. We wish to show that G(x) <* G(u). Let G(u) = F(y)#, where y# = u. Then x <□ y#, and so by Lemma 3.7, let z □ fld(□[□□]), z# = x, F(z)# <* F(y)#. Then G(x) = the <* least element of {F(y)#: y# = x} □* F(z)# <* F(y)# = G(u). QED

THEOREM 3.9. The following implies ATR_0 over RCA_0. Let □□ and □* be countable well orderings with greatest elements. Then there is an order continuous embedding from □□ into □* or vice versa.

Proof: Assume the comparability statement. According to Lemma 3.3, we have ACA_0. Now let □□ and □* be well orderings. By Lemma 3.4, □[□□]+1 and □[□*]+1 are well orderings. Hence there is an order continuous function from □□□+1 into □□*+1 or vice versa. By symmetry, let F be an order continuous function from □□□+1 into □□*+1.

We use   for the greatest elements of □□□+1 and □□*+1. If F does not take on the value   then F is an order continuous embedding from □□□ into □□*. Assume F(x) = . Then F is an order continuous embedding from □□□\x into □□*.

We claim that there is an order isomorphism h from □□□ one-one onto □□□\x. To see this, let x = $(x_1,n_1),\ldots,(x_k,n_k)$, where k ≥ 0, $n_1,\ldots,n_k$ ≥ 1, and x_1 >□ x_2 >□ ... >□ x_k. Define h to be the identity below x. We can explicitly define x+0,x+1,..., which form an interval in □□□ starting with x+0 = x. Take h(x+i) = x+i+1, and set h to be the identity higher up.

It is now clear that Foh is an order continuous embedding from □□□ into □□*. Hence by Lemma 3.8, there exists an order preserving map from □□ into □*. We have thus verified iii) of Theorem 1.1. Hence by Theorem 1.1, we have ATR_0. QED

4. COMPARABILITY OF SETS OF RATIONALS.

In this section, we tie everything together and derive our main results.

Note that a set of rationals can be looked at in two relevant ways. Firstly, as a metric space, inherited from the usual metric on the rationals, given by |x-y|. Secondly, as a linear ordering, inherited from the usual ordering on the rationals. Thus the notions of continuity (using the metric) and order continuity (using the order) are both to be considered.

Let $x_1,x_2,\ldots,y$ □ Q. We say that $x_1,x_2,\ldots$ converges to y if and only if for all □ > 0 there exists n such that m ≥ n □ $|x_m - y|$ < □.

LEMMA 4.1. The following is provable in RCA_0. Let □,□ □ Q and F:□ □ □. Then F is continuous at y □ □ if and only if for all $x_1,x_2,\ldots,y$ □ □, if $x_1,x_2,\ldots$ converges to y then $F(x_1),F(x_2),\ldots$ converges to F(y).

Proof: Suppose F is continuous at y and $x_1,x_2,\ldots$ converges to y. Let □ > 0. Since F is continuous, let □ > 0 be such that for all z □ □, |z-y| < □ □ |F(z)-F(y)| < □. Let n be such that m ≥ n □ $|x_m - y|$ < □. Then m ≥ n □ $|F(x_m)-F(y)|$ < □.

Suppose for all $x_1,x_2,\ldots,y$ □ □, if $x_1,x_2,\ldots$ converges to y then $F(x_1),F(x_2),\ldots$ converges to F(y). Let □ > 0. Choose a positive rational □□< □. Suppose F is not continuous at y. Then for each n there exists x_n □ A such that $|x_n - y| < 1/n$ and $|F(x_n)-F(y)|$ > □□. In RCA_0, we can form the infinite sequence $x_1,x_2,\ldots$, which obviously converges to y. But also $F(x_1),F(x_2),\ldots$ does not converge to F(y). QED

We say that □ □ Q is compact if and only if for all infinite sequences of nonempty open intervals (p_i,q_i) with rational endpoints which cover A, some finite initial segment covers □.

LEMMA 4.2. The following is provable in RCA_0. Let □ □ Q be compact and $x_1,x_2,\ldots,y$ □ □. Then $x_1,x_2,\ldots$ converges to y if and only if $x_1,x_2,\ldots$ approaches y in □ as a linear ordering.

Proof: Let □,$x_1,x_2,\ldots,y$ be as given. It is easy to see that □ is bounded, and so let □ □ (p,q). The forward direction is obvious. Now assume $x_1,x_2,\ldots$ approaches y in A as a linear ordering.

Suppose $x_1,x_2,\ldots$ does not converge to y. Let □ > 0 be rational, where there are arbitrarily large n such that $|x_n - y|$ > □. We thus find $z_1,z_2,\ldots$ approaching y where each $|z_n - y|$ > □. By a primitive recursion, we can find $w_1 > w_2 > \ldots > y$ approaching y, or $w_1 < w_2 < \ldots < y$ approaching y, where each $|w_n - y|$ > □. But in either case, we get a violation of compactness as follows. In the second case, $(p,w_2),(y-($□$/2),q),(w_1,w_3),(w_2,w_4),\ldots$ covers □, yet no finite initial segment covers □. The first case is symmetric. QED

LEMMA 4.3. The following are provably equivalent over RCA_0. In addition, each of the following implies ACA_0 over RCA_0.

i) Let □,□ □ Q be a compact well ordering. There is an order continuous embedding from □ into □;

ii) Let □,□ □ Q be a compact well ordering. There is a continuous embedding from □ into □.

Proof: By Lemma 4.1 and 4.2, i) and ii) are equivalent. We now assume i) and derive ACA_0. Look at the proof of Lemma 3.3. It suffices to show that the well orderings □□,□* used in the proof of Lemma 3.3 are order isomorphic to compact well orderings □,□ □ Q. Note that any order isomorphism from a well ordering one-one onto a linear ordering must be an isomorphism onto a well ordering. Hence we have only to construct order isomorphisms from □□,□* one-one onto compact □,□ □ Q.

For □□ we can use {n-(1/m): n,m □ □ □ m ≥ 2} □ □ □ { }, where in order to make this a subset of Q, we use the transformation -1/x+1. To verify compactness, one of the intervals in the open cover by rational open intervals must cover , and so it suffices to find a finite subcover of {n-(1/m): n,m □ □ □ m ≥ 2 □ n □ r} □ {0,1,...,r}, for any r. Choosing r intervals in the cover that cover {0,1,...,r} reduces this to finding a finite subcover of a finite set, which is immediate.

For □* we can use two copies of the above, one on top of the other, where n is required to be an m-good sequence number in order for n-(1/m) to appear. It suffices to verify compactness for just one copy. The same argument works. QED

LEMMA 4.4. The following is provable in ACA_0. Every well ordering with a greatest element is order isomorphic to a compact well ordered set of rationals.

Proof: We are now armed with the full power of ACA_0. Let □□ be a well ordering. We can assume that fld(□□) = □. Define J:□ □ R by J(n) = □{2^{-i}: i <□ n}, J(0) = 0, where 0 is the left endpoint of □□. Let □ = rng(J). Clearly J is an order isomorphism from □□ one-one onto □. In ACA_0, we see that □ is closed and bounded in the normal sense, and therefore compact. And in ACA_0, we can construct an order isomorphism from one-one onto which maps □ onto a set □ of rationals. Since □ is compact and well ordered, □ is compact and well ordered. QED

THEOREM 4.5. The following are provably equivalent over RCA_0.
i) ATR_0;
ii) For any two countable metric spaces, there is a continuous embedding from one into the other;
iii) For any two sets of rationals, there is a continuous embedding from one into the other;
iv) for any two compact well ordered sets of rationals, there is a continuous embedding from one into the other;

v) for any two countable well orderings with greatest elements, there is an order continuous embedding from one into the other.
vi) for any two countable well orderings, there is an order continuous embedding from one into the other.

Proof: By Theorem 2.15, i) □ ii). The implication ii) □ iii) □ iv) is immediate, as is vi) □ v). By Theorem 3.10, v) □ i). Also i) □ vi) since comparison maps are order continuous. Hence we have i) □ ii) □ iii) □ iv), and v) □ vi) □ i).

Assume iv). By Lemma 4.3, we have ACA_0, and also for any two compact well ordered sets of rational, there is an order continuous embedding from one into the other. Let □□,□* be countable well orderings with greatest elements. By Lemma 4.4, they are order isomorphic to compact well ordered sets of rationals, □,□. Hence there is an order continuous embedding from one of □,□ into the other. By diagram chasing, there is an order continuous embedding from one of □□,□* into the other. QED

In [3] and [5], stronger statements than iv) above are reversed to ATR_0, involving stronger notions of continuous embedding. Thus Theorem 4.5 sharpens these results from [3] and [5].

We remark that RCA_0 proves every compact set of rationals is closed and bounded. Therefore we can replace “compact” in Theorem 4.5 by “closed and bounded.” In [6] it is shown that “closed and bounded sets of rationals are compact” is provably equivalent to WKL_0 over RCA_0.

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