NEW BOREL INDEPENDENCE RESULTS

by
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sketch

1. INTRODUCTION.

A considerable amount of interdisciplinary work has been done on countable Borel equivalence relations. See [JKL02].

Many researchers in logic have worked on countable Borel equivalence relations, who have utilized the work of many people in ergodic theory and related areas. These include the following names taken from the list of references of [JKL02]:

S. Adams, W. Ambrose, A. Andretta, H. Becker, R. Camerlo, C. Champetier, J.P.R. Christensen, D.E. Cohen, A. Connes. C. Dellacherie, R. Dougherty, R.H. Farrell, F. Feldman, A. Furman, D. Gaboriau, S. Gao, V. Ya. Golodets, P. Hahn, P. de la Harpe, G. Hjorth, S. Jackson, S. Kahane, A.S. Kechris, A. Louveau,, R. Lyons, P.-A. Meyer, C.C. Moore, M.G. Nadkarni, C. Nebbia, A.L.T. Patterson, U. Krengel, A.J. Kuntz, J.-P. Serre, S.D. Sinel'shchikov, T. Slaman, Solecki, R. Spatzier, J. Steel, D. Sullivan, S. Thomas, A. Valette, V.S. Varadarajan, B. Velickovic, B. Weiss, J.D.M. Wright, R.J. Zimmer.

Many thanks to Simon Thomas for advice on this manuscript.

A Borel equivalence relation is a pair (X,E), where X is a Polish space (complete separable metric space), and E is a Borel equivalence relation on X.

A countable Borel equivalence relation is a Borel equivalence relation in which every equivalence class is countable.

A fundamental situation is that of a Borel function $f:X \to \mathbb{R}$ which is E invariant. This means that for all $x,y \in X$,

$E(x,y) \rightarrow f(x) = f(y)$.

In this note, we will be concerned with the following kind of result.

Let (X,E) be a countable Borel equivalence relation. Every E invariant Borel function $f:X \to \mathbb{R}$ is constant on a "big set".

In the case where X is also equipped with a measure mu, this statement with

"big set" = set of full measure

is known as

(X,E,μ) is ergodic.

Note that because of the countable additivity, ergodicity has these three equivalent forms.

Every E invariant Borel function $f:X \to \mathbb{R}$ is constant on a set of full measure.

Every E invariant Borel function $f:X \to \{0,1\}$ is constant on a set of full measure.

Every E invariant Borel subset of X contains or is disjoint from a set of full measure.

We can fix a very standard probability space like I^n with Lebesgue probability measure or R^n with Lebesgue measure or {0,1} with the usual probability measure. Then we can search for necessary or sufficient conditions for a countable Borel equivalence relation E on I^n, R^n, {0,1} , respectively, to be ergodic.

We mention three well known facts about ergodicity. it won't make any difference which standard Borel space we choose. We will simply choose R for convenience.

ERGODIC 1. There is a countable Borel equivalence relation (R,E) which is ergodic. In particular, E(x,y) iff x-y is rational.

ERGODIC 2. All sufficiently inclusive countable Borel equivalence relations (R,E) are ergodic.

2. BOREL CONSTANCY ON ¬.

We write SUM(a_1,a_2,...) for the set of all sums of infinite subsequences of the a's, provided that the infinite subsequence converges to a real number.

We begin with some Lemmas about SUM(4^{-1},4^{-2},4^{-3},...). Note that SUM(4^{-1},4^{-2},4^{-3},...) is the set of all real numbers whose base 4 expansion is of the form .$a_1a_2a_3$... where each a_i □ {0,1}.

LEMMA 2.1. Let n ≥ 4. Every Turing cone is the set of Turing degrees lying in some translate of SUM(n^{-1},n^{-2},n^{-3},...).

Proof: Let d be a Turing degree. Let a_1,a_2,a_3,... □ {0,2} be of degree d. Let x = .$a_1a_2a_3$... in base n. Let y = .$b_1b_2b_3$... □ SUM(n^{-1},n^{-2},n^{-3},...). The base n digits of x+y are a_1+b_1,a_2+b_2,a_3+b_3,... . We can read a_i+b_i and figure out what a_i and b_i are as follows. If the sum is 0 then $a_i = b_i = 0$. If the sum is 1 then $a_i = 0$ □ $b_i = 1$. If the sum is 2 then $a_i = 2$ □ $b_i = 0$. If the sum is 3 then $a_i = 2$ □ $b_i = 1$. Therefore the degree of x+y is the join of the degrees of x and y. Since the degree of y can be anything, we see that the Turing cone determined by d is the set of Turing degrees lying in SUM(n^{-1},n^{-2},n^{-3},...)+x. QED

LEMMA 2.2. Let n ≥ 4. The Turing degrees lying in any translate of SUM(n^{-1},n^{-2},n^{-3},...) contain a Turing cone. In fact, every SUM(n^{-1},n^{-2},n^{-3},...)+x contains elements of every Turing degree ≥T x.

Proof: Let x □ ¬. We show that the Turing degrees of SUM(n^{-1},n^{-2},n^{-3},...)+x contain a Turing cone. We can assume that x □ (0,1)\Q. Let the base n expansion of x be .$a_1a_2a_3$... .

We first dispense with the case where there exists 0 □ i □ n-3 such that there are infinitely many i's among the a's.

We will modify the digits i,i+1 in x only, by adding various elements of SUM(n^{-1},n^{-2},n^{-3},...) to x. The 1's in these elements of $SUM(n^{-1},n^{-2},n^{-3},...)$ will be added to i's and i+1's only.

Since we only care about the i's and i+1's in x, we let b1,b2,... ∈ {i,i+1} be the listing of the i's and i+1's among the a's, in order of appearance.

Let $c_1,c_2,... \in \{i,i+1\}$. We want to find $y \in SUM(n^{-1},n^{-2},n^{-3},...)$ so that the listing of the i's and i+1's in the base n expansion of x+y is exactly $c_1,c_2,...$.

We start with c_1 and compare it to b_1. If $c_1 = i+1$ then we can change b_1 to c_1, if necessary, by adding 1.

A difficulty occurs if $b_1 = i+1$ and $c_1 = i$. To handle this situation, we look at the longest initial block of i+1's among the b's, and add 1 to all digits in the block. Then the first digit ∈ {i,i+1} is now $c_1 = i$.

We then continue this process indefinitely.

If we choose $c_1,c_2,... \geq_T x$, then this construction yields an element $y+x \in SUM(n^{-1},n^{-2},n^{-3},...)+x$ of degree $c_1,c_2,...$. Thus in this case, SUM(n-1,n-2,n-3,...)+x has elements of every degree $\geq_T x$.

We are left with the case where the a's consist entirely of n-2's and n-1's. We can obviously assume that there are infinitely many of both.

Again, we will modify the digits n-2,n-1 by adding various elements of $SUM(n^{-1},n^{-2},n^{-3},...)$. We have

*) $(n-2)^{s1}\ (n-1)^{t1}\ (n-2)^{s2}\ (n-1)^{t2}\ (n-2)^{s3}\ (n-1)^{t3}\ ...$

Now for each section $n-2\ (n-1)^t$ we have a choice of leaving it alone or modifying it by adding 1 to the rightmost term, resulting in $n-1\ 0^t$.

After we have performed these choices, we can recover *) by looking at all maximal block of the form $n-1\ 0^t$, $t \geq 1$, and replacing it back by $n-2\ 0^t$. We then know what choices we have made.

As before, we use any bit sequence $e_1,e_2,... \geq_T x$ to represent out choices of "leave alone" or "modify". The resulting real number in $SUM(n^{-1},n^{-2},n^{-3},...)+x$ is $=_T$

$e_1,e_2,\ldots$. Thus also in this case, SUM(n-1,n-2,n-3,...)+x has elements of every degree ≥T x. QED

We now use separate results of D.A. Martin (Lemmas 2.3, 2.4), and the present author (see Lemma 2.5).

LEMMA 2.3. Let A □ ¬ be Borel, where x =T y □ (x □ A □ y □ A). Then A contains or is disjoint from a Turing cone of reals. Let f:¬ □ ¬ be a Borel function, where x =T y □ f(x) = f(y). Then f is constant on a Turing cone of reals.

Proof: See [Ma74], [Ma85], [Ke95]. The second claim is easily proved from the first. QED

For x □ N coding a well ordering of type some □$^□$, □ ≥ 1, we define the x-degrees on ¬ as follows. y □x z if and only if y is recursive in the Kleene H set along some proper initial segment of x starting with x,z at the bottom.

Here is a weakening of Lemma 2.3.

LEMMA 2.4. Let x code a well ordering of type some □$^□$, □ ≥ 1. Let A □ ¬ be Borel, where y =x z □ (y □ A □ z □ A). Then A contains or is disjoint from a cone of x degrees. Let f:¬ □ ¬ be a Borel function, where y =x z □ f(y) = f(z). Then f is constant on a cone of x degrees.

LEMMA 2.5. Lemmas 2.3, 2.4 are each provable using uncountably many iterations of the power set operation, but not using any definite countable number of iterations of the power set operation. These are reversals.

Proof: The provability is from [Ma75], [Ma85], [Ke95]. The reversal is from [Fr71]. For related but newer results along these lines, see [Fr05]. QED

THEOREM 2.6. There is a countable Borel equivalence relation on ¬ in which every invariant Borel function f:¬ □ ¬ is constant on a translate of SUM($4^{-1},4^{-2},4^{-3},\ldots$). In fact, this holds if we replace 4 by any larger integer.

Proof: In fact, Turing equivalence works. Let f:¬ □ ¬ be a Borel function such that x =T y □ f(x) = f(y). By Lemma 2.3, f is constant on a Turing cone C. By Lemma 2.1, there is a translate of SUM($n^{-1},n^{-2},n^{-3},\ldots$) that lies entirely in C. Therefore f is constant on this translate of SUM($n^{-1},n^{-2},n^{-3},\ldots$). QED

THEOREM 2.7. In all sufficiently inclusive countable Borel equivalence relations on ¬, every invariant Borel function f:¬ □ ¬ is constant on a translate of SUM(4^{-1},4^{-2},4^{-3},...). In fact, this holds if we replace 4 by any larger integer.

Proof: Immediate from Theorem 2.6 since the statement is preserved under extension of the countable Borel equivalence relation. QED

LEMMA 2.8. Every countable Borel equivalence relation on ¬ is extended by the x degrees for some x □ N coding a well ordering of some type $□^{□}$, □ ≥ 1.

Proof: Let E be a countable Borel equivalence relation on ¬. Then (□x □ ¬)(□ countably many y □ ¬)(E(x,y)). Hence by standard descriptive set theory, there exists a Borel function H:¬ □ ¬ such that for all x □ ¬, H(x) enumerates the y such that E(x,y). The result follows from the coding mechanisms for Borel functions in standard descriptive set theory. QED

THEOREM 2.9. Theorems 2.6, 2.7, 2.9 are each provable using uncountably many iterations of the power set operation, but not using any definite countable number of iterations of the power set operation. These are reversals. The same results hold if we use any individual integer ≥ 4, or for all integers ≥ 4 simultaneously.

Proof: For provability, use the proof given above of Lemma 2.3 □ Theorem 2.6, using Lemma 2.1, and the cited proof of Lemma 2.3.

For the reversal, we now show that Theorem 2.6 □ Lemma 2.4, using Lemmas 2.2, 2.8.

Suppose Theorem 2.6. Let E be the given countable Borel equivalence relation on ¬. By Lemma 2.8, let E be extended by the x degrees.

Now let f:¬ □ ¬ be Borel, where y =x z □ f(y) = f(z). Then f is E invariant. Therefore f is constant on a translate X of SUM(4^{-1},4^{-2},4^{-3},...). By Lemma 2.2, the Turing degrees lying in X contain a Turing cone C. Let u be the base of C. Let v =x w ≥x u. Then (v,x,u) =x (w,x,u) ≥x u. Since (v,x,u),(w,x,u) ≥T u, we have f((v,x,u)) = f((w,x,u)). By the invariance under x degrees, we have f(v) = f(w). This shows that f is constant on the cone of x degrees with base u. QED

The same results hold for the following weak variants of Theorems 2.6, 2.7.

THEOREM 2.10. There is a countable Borel equivalence relation on ¬ in which every invariant Borel subset of ¬ contains or is disjoint from some translate of some SUM($n^{-1},n^{-2},n^{-3},\ldots$), $n \geq 4$.

THEOREM 2.11. In all sufficiently inclusive countable Borel equivalence relations on ¬, every invariant Borel subset of ¬ contains or is disjoint from some translate of some SUM($4^{-1},4^{-2},4^{-3},\ldots$), $n \geq 4$.

What about Lebesgue or Baire measurable functions? Then the statements are refutable.

THEOREM 2.12. There is no countable Borel equivalence relation on ¬ in which every invariant subset of ¬ of measure 0 (or measger) contains or is disjoint from some translate of some SUM($n^{-1},n^{-2},n^{-3},\ldots$), $n \geq 4$.

Proof: Let there be such a countable Borel equivalence relation on ¬, say E. By Lemma 2.8, extend E to the x degrees, where x □ N coding a well ordering □□, □ ≥ 1.

It is well known that there is a cone C of x degrees of measure 0 (or meager). We can use transfinite induction to construct an x degree invariant C′ □ C which neither contains nor is disjoint from any translate of any SUM($n^{-1},n^{-2},n^{-3},\ldots$), $n \geq 4$ as follows.

At the end of < c stages, there are c many elements of any SUM($n^{-1},n^{-2},n^{-3},\ldots$)+x, $n \geq 4$, that lie in C, and so can be thrown into C′. This follows from the fact that any SUM($n^{-1},n^{-2},n^{-3},\ldots$)+x, $n \geq 4$, contains elements of all sufficiently high x degree (see Lemma 2.2 and the proof of Theorem 2.9). Also, at the end of < c stages, there are c many elements of any SUM($n^{-1},n^{-2},n^{-3},\ldots$)+x, $n \geq 4$, that can be banned from C′, simply by cardinality. QED

3. BOREL CONSTANCY ON $¬^n$.

A curve is a homeomorphic image of [0,1].

THEOREM 3.1. There is a countable Borel equivalence relation on $\neg^2$ in which every invariant Borel function f:$\neg^2$ □ ¬ is constant on a line (circle, vertical line, circle about the origin, curve) in $\neg^2$.

THEOREM 3.2. In all sufficiently inclusive countable Borel equivalence relations on $\neg^2$, every invariant Borel function f:$\neg^2$ □ ¬ is constant on a line (circle, vertical line, circle about the origin, curve) in $\neg^2$.

THEOREM 3.3. Theorem 3.1 and 3.2 are each provable using uncountably many iterations of the power set operation, but not using only countably many iterations of the power set operation. This is a reversal.

The same results hold for the following weak variants.

THEOREM 3.4. There is a countable Borel equivalence relation on $\neg^2$ in which every invariant subset of $\neg^2$ contains or is disjoint from a line (circle, vertical line, circle about the origin, curve) in $\neg^2$.

THEOREM 3.5. In all sufficiently inclusive countable Borel equivalence relations on $\neg^2$, every invariant subset of $\neg^2$ contains or is disjoint from a line (circle, vertical line, circle about the origin, curve).

We also have the following.

THEOREM 3.6. There is no countable Borel equivalence relation on $\neg^2$ in which every invariant subset of ¬ of measure 0 (or meager) contains or is disjoint from a curve in $\neg^2$.

These results are proved analogously to those in section 2, using the following.

LEMMA 3.7. Every Turing cone is the set of Turing degrees lying in some vertical line, some horizontal line, and some circle about the origin, in $\neg^2$.

Proof: Let x be a real number of the same Turing degree as the base of the Turing cone. Use the vertical line passing through (x,0), the horizontal line passing through (0,x), and the circle of radius x about 0. QED

For n ≥ 1, let Q[n] be the set of all rationals whose numerator and denominator lie in [-n,n].

Let f:[0,1] □ $¬^2$ be continuous. A code for f is an infinite sequence of functions gn:(Q[n] □ [0,1]) □ $Q[n]^2$ and an h:N □ N such that for all n,m ≥ 1,

m > h(n) □ |f - g_m| < 1/n.

LEMMA 3.8. The Turing degrees lying in any curve in $¬^2$ contain a Turing cone.

Proof: Let □ be a curve in $¬^2$. Let f:[0,1] □ □ be a surjective homeomorphism. Let u code f. Let x □ [0,1], u □T x. Let f(x) = (b,c). Then b,c □T (u,x). Hence (b,c) □T x. Also x □T (b,c,u). Hence x □T (b,c). Therefore x =T (b,c). QED

All of these Theorems hold in $¬^n$, n ≥ 2. In fact, we can use the n-1 dimensional hyperplanes xi = c, 1 □ i □ n, and the n-1 dimensional spheres |x| = c, c > 0.

4. DEGREES INTO FINITELY GENERATED GROUPS.

We use Z for the set of all integers, and N for the set of all nonnegative integers.

We use GROUP for the Borel space of binary operations f:N^2 □ N that form a group. We use the equivalence relation of isomorphism. This does not include the finite groups, but as this is countable, it is inconsequential for our descriptive set theoretic purposes.

We use FGG for the subspace consisting of the finitely generated elements of GROUP. For n ≥ 1, we use FGG(n) for the subspace of , n ≥ 1, for the subspace consisting of the elements of GROUP with a set of at most n generators.

For ordinary descriptive set theory purposes, it suffices to work with these Borel spaces. However, the FGG(n) can be naturally presented as a compact space with various geometric and group theoretic structure. This is commonly done made good use of in geometric group theory. We will have no need to get into this here. E.g., see [Ch00] and [Ch05]. Also wee [Gr84], where the FGG(n) are used to obtain information about the growth of groups.

We sketch a proof that there is an arithmetic map from NN into FGG with certain important properties. The properties that we verify are just the properties that are needed in the proofs given in section 5.

We work throughout with countable groups of permutations of N under composition and function inverse. We use A! for the group of all permutations of the set A.

Let f:A → A, g:B → B. We say that h is a conjugation of f to g if and only if

i. h is a bijection from A onto B.
ii. $g = hfh^{-1}$.

We say that f,g ∈ N! are conjugate by h if and only if h ∈ N! and $g = hfh^{-1}$. We say that f,g are conjugate if and only if there exists h such that f,g are conjugate by h.

We say that f ∈ N! is special if and only if

i. all orbits of f are finite.
ii. f has infinitely many orbits of each nonzero finite cardinality.

Let N!# be the set of all special elements of N!. Note that any two special elements of N!# are conjugate.

Let g ∈ (ZxN)!. For n ∈ Z, define g[n] ∈ ({n}xN)! by $g_n(n,m) = g(m)$. Define g[n]* ∈ (ZxN)! by g[n]*(x) = g[n](x) if x ∈ dom(g[n]); x otherwise.

LEMMA 4.1. There exist arithmetic F_1,F_2,F_3 ∈ (ZxN)! such that for all recursive g ∈ N!# and n ∈ N, the group generated by F_1,F_2,F_3 contains g[n]* ∈ (ZxN)!.

Proof: Let (f_i), i ∈ Z, list all recursive elements of N!#. For all i ∈ Z, let g_i:{i}xZ → {i+1}xZ be a conjugation of $f_i[i]$ to $f_{i+1}[i+1]$. This is obtained by modifying any conjugation of f_i to f_{i+1} in the obvious way. Let g ∈ (ZxN)! be the union of the g_i. We can easily arrange that g is arithmetic.

We claim that g is a conjugation of each $f_i[i]*$ to $f_{i+1}[i+1]*$. To see this,

$f_{i+1}[i+1]*(i+1,n) = g(f_i[i]*(g^{-1}(i+1,n)))$.

$f_{i+1}[i+1]^*(j,n) = g(f_i[i]^*(g^{-1}(j,n)))$, $j \neq i+1$.

The first equation follows immediately from the fact that g|{i}xN is a conjugation of f[i] to f[i+1].

For the second equation, note that the left side is (j,n). Also the first coordinate of $g^{-1}(j,n)$ is not i, and so it is a fixed point of $f_i[i]^*$. Hence the right side is $g(g^{-1}(j,n))$ = (j,n).

It now follows that for all $j \in Z$, g^j is a conjugation of $f_i[i]^*$ to $f_i[i+j]^*$. So $g,f_0[0]^*$ generates all $f_i[i]^*$.

Let $s \in (ZxN)!$ be given by s(n,m) = (n+1,m). From s and the $f_i[i]^*$, we generate all $f_j[i]^*$. Thus we use the three generators, $g,f_0[0]^*,s$. $f_0[0]^*,s$ are recursive, and g can be taken to be arithmetic. QED

LEMMA 4.2. Let $f \in N!$. There exists $g \in N!\#$ such that $fg \in N!\#$. We can take $g \leq_T f$.

Proof: Fix $f \in N!$. We define permutations $g_0 \subseteq g_1 \subseteq \ldots$ of intervals $[0,n_0],[0,n_1],\ldots$, where $n_0 < n_1 < \ldots$. Set $n_0 = 0$ and $g_0(0) = 0$.

Suppose $n_0 < \ldots < n_k$ and $g_0 \subseteq \ldots \subseteq g_k$ have been defined. First extend g_k to a permutation $h:[0,m] \to [0,m]$, $m > k$, so that h has at least k orbits of every nonzero cardinality $\leq k$. Let $fh:A \to hA$. Let h′ be a finite permutation extending fh that maps some [0,r] into [0,r], where h′ has at least k orbits of every nonzero cardinality $\leq k$.

Obviously, $f^{-1}h' \supseteq h$. Extend $f^{-1}h'$ to a permutation $g_k:[0,s] \to [0,s]$. Set $n_{k+1} = s$.

Let g be the union of the g_k's. Then $g,fg \in N!\#$. QED

We write REC(N!) for the group of recursive permutations of N.

LEMMA 4.3. N! is the subgroup of N! generated by N!#. REC(N!) is the subgroup of N! generated by REC(N!) $\cap$ N!#.

Proof: Immediate from Lemma 4.2. QED

LEMMA 4.4. There exist arithmetic $F_1,F_2,F_3 \in (Z\times N)!$ such that the group generated by F_1,F_2,F_3 contains all recursive $g[n]^* \in (Z\times N)!$.

Proof: Immediate from Lemmas 4.1, 4.3. QED

We use Z′ for a disjoint copy of Z.

LEMMA 4.5. There exist arithmetic $G_1,G_2,G_3 \in (Z \cup Z')!$ such that the group generated by G_1,G_2,G_3 contains all recursive $g \in (Z \cup Z')!$ which are the identity on Z′.

Proof: Apply Lemma 4.4, treating Zx{0} as Z and Zx(N\{0}) as Z′. We use only the g[0]*, g recursive. QED

Let $f:N \to N$. We use the machinery of computability relative to f, for functions from Z into Z.

In particular, for each $i \in Z$, let $f\langle i\rangle:N \to N$ be the i-th partial recursive function from N into N if $f\langle i\rangle \in N!$; the identity function on N, otherwise.

Note that $f\langle i\rangle(n)$, as a function of $i \in Z$ and $n \in N$, is arithmetic in f.

LEMMA 4.6. Let $f,g:N \to N$, f =T g. There exists a recursive $h \in Z!$ such that for all $i \in Z$, $g\langle i\rangle = f\langle h(i)\rangle$.

Proof: We can obviously effectively pass from any $i \in Z$ to a $j \in Z$ such that $g\langle i\rangle = f\langle j\rangle$; and from any $i \in Z$ to a $j \in Z$ such that $f\langle i\rangle = g\langle j\rangle$. Build the desired h by a back and forth inductive argument. QED

Now fix $f:N \to N$. We define $\square(f) \in (Z\times N \cup Z')!$ as follows.

$\square(f)(i,n) = (i,f\langle i\rangle(n))$. For $x \in Z'$, $\square(f)(x) = x$.

Note that $\square(f)$ is arithmetic in f, and depends on f.

We define $H_1,H_2,H_3 \in (Z\times N \cup Z')!$ from the $G_1,G_2,G_3 \in (Z \cup Z')!$ given by Lemma 5, as follows. Let j = 1,2,3.

$H_j(i,n) = (G_j(i),n)$. For $x \in Z'$, $H_j(x) = G_j(x)$.

Note that H_1,H_2,H_3 are arithmetic - not just arithmetic in f - and does not depend on f.

Let GRP(□(f),H_1,H_2,H_3) be the subgroup of (ZxN □ Z')! generated by □(f),H_1,H_2,H_3. We use the same notation without □(f).

LEMMA 4.7. GRP(H_1,H_2,H_3) contains all □ □ (ZxN □ Z')! of the form □(i,n) = (h(i),n), □(x) = x for x □ Z', where h □ Z! is recursive.

Proof: If we suppress the second coordinates n, then this is from Lemma 4.5. But the second coordinate is superfluous. QED

LEMMA 4.8. For f =T g, we have GRP(□(f),H_1,H_2,H_3) = GRP(□(g),H_1,H_2,H_3).

Proof: Let f =T g. It suffices to show that □(g) □ GRP(□(f),H_1,H_2,H_3). Then by symmetry, □(f) □ GRP(□(g),H_1,H_2,H_3).

Let h □ Z! be given by Lemma 4.6. We have

□(f)(i,n) = (i,f<i>(n)). For x □ Z', □(f)(x) = x.
□(g)(i,n) = (i,g<i>(n)). For x □ Z', □(g)(x) = x.
□(g)(i,n) = (i,f<h(i)>(n)). For x □ Z', □(g)(x) = x.
□(i,n) = (h(i),n). For x □ Z', □(x) = x.

where □ □ GRP(H_1,H_2,H_3) by Lemma 4.7. Also we have

□(f)□(i,n) = □(f)(h(i),n) = (h(i),f<h(i)>(n)) = (h(i),g<i>(n)).

$□^{-1}$(□(f))□(i,n) = (i,g<i>,n).
$□^{-1}$(□(f))□ = □(g).

QED

LEMMA 4.9. GRP(□(f),H_1,H_2,H_3) includes all □ □ (ZxN □ Z')! of the form □(0,n) = (0,J(n)), □(x) = x, for x □ (ZxN □ Z')\{0}xZ, where J □T f.

Proof: GRP(□(f),H_1,H_2,H_3) contains

(i,n) □ (i,f<i>(n)), identity otherwise.

As in the proof of Lemma 4.8, GRP(□(f),H_1,H_2,H_3) also contains (i,n) □ (i,f<h(i)>(n)), identity otherwise.

where h is any recursive permutation of Z. In particular, for any j □ Z, we can use a recursive h □ Z! given by

f<h(0)> = f<0>$^{-1}$f<j>.
i □ Z\{0} □ f<h(i)> = f<i>$^{-1}$.

By composing these two maps, we obtain

(0,n) □ (0,f<j>(n)), identity elsewhere.

Let J □T f. Choose j so that f<j> = J. QED

LEMMA 4.10. Let f □ NN. For every G □ FGG with G □T f, is embeddable in the group of permutations □T f.

Proof: Let G □ FGG, G □T f. Map n □ N to the permutation G(n,m), as a function of m. This permutation is □T f, and so this defines an embedding from G into the group of permutations □T f. QED

THEOREM 4.11. There is an arithmetic function □:N^N □ FGG(4) such that for all f,g □ N^N,
i. f =T g □ □(f) □ □(g).
ii. For all G □ FGG, G □T f, G is embeddable in □(f).
iii. There is an arithmetic □:FGG(4) □ N^N such that for all G □ N^N, □(□(g)) = g.

Proof: Take □(f) to be GRP(□(f),H_1,H_2,H_3) put into the format of FGG(4). Then □(f) is arithmetic. By Lemma 4.9, the group G of all permutations recursive in f is embeddable in □(f). By Lemma 4.10, clause ii holds. By Lemma 4.8, clause i holds. iii is a consequence of ii, since every finitely generated group is recursive in some f □ A, and hence embeddable in some □(f), f □ A, and every countable group is embeddable in some finitely generated group.

We can make these constructions so that we can recover f appropriately from □(f), but obviously not from GRP(□(f),H_1,H_2,H_3). Hence iii. QED

We still need a further connection between Turing degrees and FGG.

Let G □ FGG. We define the essential Turing degree of G as follows. Let $a_1,...,a_k$ be any set of generators for G. We take the Turing degree of the set of all group equations in $a_1,...,a_k$ that are true in G.

Note that for G,H □ FGG, if G □ H then the Turing degree of the result of the above construction is independent of the choice of generators, and also is the same for G and H.

THEOREM 4.12. The essential Turing degrees of the G □ FGG(3) form an unbounded set of Turing degrees.

Proof: We use the construction made in the proof of Lemma 4.1. Let f □ N^N. We can start the construction with (f_i), i □ Z, listing only two distinct recursive permutations h_1,h_2 □ N!#. We arrange that $\{(i,j): f_i = f_j\}$ =T f. We then proceed with the conjugations g_i:{i}xZ □ {i+1}xZ, and the union g. We can arrange that g □T f. We again let s □ (ZxN)! with s(n,m) = (n+1,m).

If we start with $f_0[0]^*$ and conjugate by g, n times, and then reverse conjugate by s, n times, then we will arrive at h_1^* or h_2^*, depending on whether f_n is h_1 or h_2. Thus the equality relation between the relevant group terms in $f_0[0]^*$,g,s, has Turing degree f. QED

5. BOREL FUNCTIONS ON COUNTABLE GROUPS.

An unbounded set X in GROUP, FGG, FGG(n), is a subset X of GROUP, FGG, FGG(n) such that every element of GROUP, FGG, FGG(n) is embeddable in some element of X.

THEOREM 5.1. Every isomorphically invariant Borel function f:GROUP □ GROUP maps some group to a subgroup.

THEOREM 5.2. Theorem 5.1 can be proved in a weak fragment of third order arithmetic but not in second order arithmetic.

THEOREM 5.3. Every isomorphically invariant Borel function F:FGG(4) □ ¬ is constant on an unbounded set.

Theorem 5.3 does not require an very large amount of set theory. However, we can sharpen Theorem 5.3 as follows.

THEOREM 5.4. Every isomorphically invariant Borel function F:FGG(4) □ ¬ is constant on an unbounded Borel set in FGG(4) of finite Borel rank.

Proof: Let F be as given. Define □:N^N □ FGG(4) as follows. Let □ be as given by Theorem 4.11. Define □(f) = F(□(f)). Then f =T g □ □(f) = □(g). By Lemma 2.3, □ is constant on a cone C with base f □ N^N and constant value c. From the invariance of F, we see that □ is also constant on □[C]. By Theorem 4.11 ii, every group in FGG(4) is embeddable in some element of □[C]. Hence F is constant on an unbounded set in FGG(4). Also □[C] is a Borel set of finite rank by Theorem 4.11 iii. QED

COROLLARY 5.5. In Theorem 5.4, we can use any FGG(n), n ≥ 4, or FGG, or GROUP.

Proof: We can restrict our attention to FGG(4) and apply Theorem 6.1, since any unbounded set in FGG(4) is in fact an unbounded set in FGG, or even in GROUP. QED

The following is a weakening of Lemmas 2.3. We say that E □ ¬ is Turing unbounded if and only if (□x □ ¬)(□y □ E)(x □T y).

THEOREM 5.6. Every Turing invariant Borel subset of ¬ contains or is disjoint from a Turing unbounded Borel set of finite Borel rank.

LEMMA 5.7. Theorem 5.6 is provable using uncountably many iterations of the power set operation.

Proof: The provability comes from the provability of Lemma 2.3, which is given by Theorem 2.7. QED

LEMMA 5.8. The following is provable in ZC. Theorem 5.6 is equivalent to “every Turing invariant Borel subset of ¬ contains or is disjoint from a Turing cone”.

Proof: It suffices to show in ZC that every Turing unbounded Borel set of finite Borel rank has elements of every Turing degree lying in a Turing cone. But this is simply Lemma 2.3 for finite Borel rank, which is provable in ZC. QED

THEOREM 5.9. Theorem 5.6 can be proved using uncountably many iterations of the power set operation, but not using any definite countable number of iterations of the power set operation.

Proof: Immediate from Lemmas 5.7, 5.8, and 2.5. QED

LEMMA 5.10. Theorem 5.4 implies Theorem 5.6.

Proof: Let E □ ¬ be Turing invariant and Borel. Define f:FGG(4) □ {0,1} by f(G) = 1 if the essential Turing degree of G lies in E; 0 otherwise. Obviously, f is isomorphically invariant. By Theorem 5.4, let f be constant on an unbounded Borel set V of finite rank. Then the set of essential Turing degrees of elements of V forms a Borel set of Turing degrees of finite Borel rank.

We claim that the essential Turing degrees of elements of V are unbounded in the Turing degrees. This is immediate from Theorem 4.12 and that if G,H □ FGG, G embeddable in H, then the essential Turing degree of G is □ the essential Turing degree of H. QED

Obviously, we can use any number ≥ 4 in Lemma 5.10, and also FGG.

THEOREM 5.11. Theorem 5.4, even using FGG(n), any n ≥ 4, or using FGG, is provable using uncountably many iterations of the power set operation, but not using any definite number of countable iterations of the power set operation.

Proof: From Theorem 5.9 and Lemma 5.10. QED

THEOREM 5.12. Let f:FGG □ FGG be an isomorphically invariant Borel function. There exists □ □ FGG such that f(□) is embeddable in a term of □.

PROPOSITION 5.13. Let f:FGG □ FGG be an isomorphically invariant Borel function. There exists □ □ FGG such that for all infinite subsequences □ of □, f(□) and f(□) are isomorphic and embeddable in a term of □ (in fact, □).

PROPOSITION 5.14. Let f:FGG □ FGG be an isomorphically invariant Borel function. There exists □ □ FGG and n such that for all tails □ of □, f(□) and f(□) are isomorphic and embeddable in the n-th term of □.

Theorem 5.12 is around ZC. Propositions 5.13, 5.14 are between a Ramsey cardinal and a measurable cardinal.

6. COUNTABLE BOREL QUASI ORDERS.

A quasi order is a reflexive transitive relation. It is said to be countable if and only if the number of predecessors of every point is countable.

We suggest that there should be a major systematic investigation of Borel quasi orders as an expansion of the ongoing major systematic investigation of Borel equivalence relations.

Such an investigation has at least begun. E.g., see [LR05].

Let (X,□) be a quasi order. It induces the equivalence relation x ~ y □ x □ y □ y □ x.

A □ cone is a set of the form {y: x □ y}.

We say that B □ X is □ unbounded if and only if (□x □ X)(□y □ B)(x □ y).

We say that B □ X is □ large if and only if (□x □ X)(□y ≥ x)(□z □ B)(z ~ y).

We say that B □ X is □ small if and only if (□x □ X)(□y ≥ x)(□(□z □ B)(z ~ y)).

A set cannot be both □ small and □ large.

Let S □ X. We say that B is a □ large subset of S if and only if B □ S □ (□x □ S)(□y □ S)(y ≥ x □ (□z □ B)(z ~ y)).

We say that B is a □ small subset of S if and only if B □ S □ (□x □ S)(□y □ S)(y ≥ x □ □(□z □ B)(z ~ y)).

THEOREM 6.1. There exists a Borel countable quasi order □ on ¬ such that every Borel set is □ large or □ small.

THEOREM 6.2. For all sufficiently inclusive Borel countable quasi orders □ on ¬, every Borel set is □ large or □ small.

Let CS(¬) be the space of all countable sets of reals. Borel functions f:CS(¬) □ CS(¬) are defined by lifting to ¬ .

Here is an old theorem of mine.

THEOREM 6.3. Every Borel function from and into CS(¬) maps some set to a subset.

PROPOSITION 6.4. There exists a Borel countable quasi order □ on ¬ such that every Borel function from and into CS(¬) maps some set to a □ small or □ large subset.

PROPOSITION 6.5. For all sufficiently inclusive Borel countable quasi order □ on ¬, every Borel function from and into CS(¬) maps some set to a □ small or □ large subset.

THEOREM 6.6. Theorems 6.1, 6.2 can be proved using uncountably many iterations of the power set operation, but not with any definite countable number of iterations of the power set operation. Theorem 6.3 can be proved in third order arithmetic, but not in second order arithmetic. Propositions 6.4, 6.5 can be proved using uncountably many Woodin cardinals, but not any definite countable number of Woodin cardinals. This uses some work of Woodin on projective Turing degree determinacy without the axiom of choice.

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