ON EXPANSIONS OF O-MINIMAL STRUCTURES

PRELIMINARY REPORT

by

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An o-minimal structure is any relational structure in any relational type in the first order predicate calculus with equality, where one symbol is reserved to be a dense linear ordering without endpoints, satisfying the following condition: that every first order definable subset of the domain is a finite union of intervals whose endpoints are in the domain or are $\pm\infty$. First order definability always allows any parameters, unless explicitly indicated otherwise.

Fix $M = (R,<,\ldots)$ to be an o-minimal structure. We say that E has property * over M if and only if $E \subseteq R$ and the following holds:

Let $f_1,\ldots,f_r:I \to R$ be definable over M, where I is an interval with endpoints in R, where each f_i is strictly monotone, and where for all $x \in I$, $f_1(x),\ldots,f_r(x)$ all disagree. Let $a_1,\ldots,a_r \in \{0,1\}$. Then there exists $x \in I$ such that for all $1 \leq i \leq r$,

$$f_i(x) \in E \text{ if and only if } a_i = 1.$$

Let M(E) be the result of expanding M by a unary predicate symbol for membership in E, where E has property * over M. We want to study M(E).

We will now show that M(E) has elimination of quantifiers in the following sense. We assume that M has symbols for every M definable function from every Cartesian power R^k into R, including $k = 0$ (i.e., constants). It is convenient to let 0 be an arbitrary element of R. Thus we will consider only atomic formulas of the form $F(x_1,\ldots,x_k) = 0$ and $F(x_1,\ldots,x_k) \in E$. There is no need to consider any other atomic formulas. We want to prove that every formula is equivalent to a Boolean combination of atomic formulas with no new free variables.

As is customary in quantifier elimination, we need only consider formulas

(∃x)(F(x,y1,…,yk) = 0 & G1(x,y1,…,yk),…,Gr(x,y1,…,yk) ∈ E
& H1(x,y1,…,yk),…,Hs(x,y1,…,yk) ∉ E).

and eliminate the quantifier.

The idea is to think of these k+1-ary functions as providing unary functions Fx,G1x,…,Grx,H1x,…,Hs of x definably in y1,…,yk.

Using the uniform boundedness theorem, we can find t > 0 and M definable functions J1(y1,…,yk),…,Jt(y1,…,yk) such that the following holds. For every y1,…,yk ∈ R,

i) each Ji(y1,…,yk) defines an interval on which the unary functions G1x,…,Grx,H1x,…,Hs are each individually either constant or strictly monotone, and the order type of the r+s-tuple (G1x,…,Grx, H1x,…,Hsx) is constant, and Fx is either constantly 0 or constantly nonzero;

ii) the intervals J1,…,Jt are nonoverlapping nonempty intervals which go from left to right, which partition all of R, and which, if necessary, repeat till the end.

We can then represent the truth value of the above existential formula according to the explicitly finite number of cases of constancy, strict monotonicity, and constant order type involved in i) above. The constantcy cases create questions as to the membership in E of values of M definable functions at y1,…,yk.

In any case, the existential formula gets reduced to a Boolean combination of atomic formulas as required.

So we have shown the following:

THEOREM 1. If E has property * over M then (M,E) admits elimination of quantifiers.

We now want to get some consequences of this.

THEOREM 2. Let E have property * over M and k ≥ 1. If E is not R or ∅, then the (M,E) definable subsets of Rk are exactly the Boolean algebra generated by the inverse images of E under the M-definable functions from Rk into R.

Proof: The (M,E) definable subsets of Rk are defined by Boolean combinations of atomic formulas with k free variables. Now each atomic formula defines an appropriate inverse image of E, or defines an M definable subset of Rk. Now each definable subset of Rk can be written as the inverse image of E.

In order to deal with the cardinality of (M,E) definable sets, we strengthen property * to property ** by simply requiring that there exists |I| many x ∈ I with the displayed property.

THEOREM 3. Let E have property * over M and k ≥ 1. Then every (M,E) definable subset of Rk is either finite or has cardinality that of some interval in R.

Proof: It suffices to prove this for k = 1. We can represent the (M,E) definable subset of R as a disjoint union of sets defined by the conjunction

f1(x),…,fr(x) ∈ E & g1(x),…,gs(x) ∉ E,

where these are M definable functions from R into R. So it suffices to show that this conjunction defines a set which is either finite or of cardinality that of some interval in E.

We again break R up into finitely many nonempty intervals where the functions are well behaved. Each interval gives rise to at most finitely many x's, or is one of the intervals I to which property ** can be applied. This gives rise to |I| x's as is required.

THEOREM 4. Let E have property ** over M, and assume that there exists an M definable function f:R → R obeying f(x) > x. Let U be a binary relation on R that is definable in (M,E). Then there is an M-definable function f:R → R such that for sufficiently large x ∈ R, (∃y)(R(x,y)) ↔ (∃y < f(x))(R(x,y)).

Proof: Represent U(x,y) as an appropriate quantifier free formula with free variables x,y:

U(x,y) is equivalent to a disjunction of conjunctions of the form

f(x,y) = 0 & g1(x,y) = … gr(x,y) □ E & h1(x,y) = … hs(x,y) □ E.

Now for each x □ R we obtain a partition of R into a fixed number of intervals on which the behavior of the M definable functions shown is uniform, and that this happens simultaneously for every one of the disjuncts. By o-minimality, for sufficiently large x, the number of intervals and their behavior is constant.

Now for some of these intervals, it is outright absurd that we can find an appropriate y, and for the remaining, property ** tells us that we can find an appropriate y. For sufficiently large x, in each disjunct, we can uniformly identify the first interval where we can find an appropriate y. Now this first interval might go off to + . In this case, we use the hypothesis function applied to its left endpoint.

We can also obtain the obvious uniform form of Theorem 4 by the same techniques, and also that any (M,E) definable continuous function is M definable.

THEOREM 5. Let M be o-minimal in a relational type of cardinality at most |M|. Then there exists 2|M| sets E with property ** over M.

Proof: By a straightforward transfinite induction.

We now consider the case where (R,<) is the real line. First we need some descriptive set theory.

The following can be proved by standard descriptive set theory techniques.

THEOREM 6. Let M be an o-minimal structure whose domain is the reals (with the usual order), and whose relational type is at most countable. Then there exists continuumly many Borel sets E with property **. In fact, these may be taken to be F-sigma sets.