ORDER INVARIANT GRAPHS AND INCOMPLETENESS

by
Harvey M. Friedman*
Distinguished University Professor of Mathematics, Philosophy, and Computer Science Emeritus
Ohio State University
July 1, 2014
EXTENDED ABSTRACT

*This research was partially supported by the John Templeton Foundation grant ID #36297. The opinions expressed here are those of the author and do not necessarily reflect the views of the John Templeton Foundation.

Abstract. For all k every order invariant graph on $Q[0,k]^k$ has a maximal clique whose sections at any (i,...,k-1),(i+1,...,k) agree below i. We prove this in an extension of the usual ZFC axioms for mathematics with standard large cardinal hypotheses, and show that ZFC does not suffice (assuming ZFC is consistent).

1. PRELIMINARIES

DEFINITION 1.1. Q is the set of all rationals. We use k,n,m,r,s,t,i,j for positive integers unless otherwise indicated. $Q[0,k] = Q \cap [0,k]$. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. $A \subseteq Q[0.k]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[0.k]^k$, $x \in A \leftrightarrow y \in A$.

DEFINITION 1.2. A graph on V is a pair (V,E), where $E \subseteq V^2$ is irreflexive and symmetric. A clique is an $S \subseteq V$ where for all distinct $x,y \in S$, x E y. A maximal clique is a clique which is not a proper subset of any clique.

DEFINITION 1.3. A graph on $Q[0,k]^k$ is order invariant if and only if $E \subseteq Q[0,k]^k \times Q[0,k]^k = Q[0,k]^{2k}$ is order invariant.

DEFINITION 1.4. Let $S,T \subseteq Q^k$, $x \in Q^n$. The section of S at x is $\{y \in Q^{k-n}: (x,y) \in S\}$. S,T agree below $p \in Q$ if and only if $(\forall x \in Q^k)(\max(x) < p \rightarrow (x \in S \leftrightarrow x \in T))$.

DEFINITION 1.5. Let λ be a limit ordinal. $E \subseteq \lambda$ is stationary if and only if E meets every closed unbounded subset of λ. For $k \geq 1$, λ has the k-SRP if and only if every partition of the unordered k tuples from λ into two pieces has a homogenous set which is stationary in λ.

Here SRP abbreviates "stationary Ramsey property".

DEFINITION 1.6. SRP is the formal system ZFC + $\{(\exists\lambda)(\lambda$ is k-SRP$)\}_k$. SRP^+ is ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-SRP$)$. SRP_k is ZFC + $(\exists\lambda)(\lambda$ is k-SRP$)$. WKL_0 is the second main system of reverse mathematics. See [WIKI].

2. INCOMPLETENESS

THEOREM 2.1. Every countable graph has a maximal clique. Every graph has a maximal clique.

Proof: This is well known. The first statement is proved by a simple sequential construction based on an enumeration of the vertex set. The second statement can be proved by a corresponding transfinite construction, based on a well ordering of the vertex set. The second statement is provably equivalent to the axiom of choice over ZF. QED

PROPOSITION 2.2. For all k, every order invariant graph on $Q[0,k]^k$ has a maximal clique whose sections at any $(i,\ldots,k-1),(i+1,\ldots,k)$ agree below i.

THEROEM 2.3. Proposition 2.2 is provably equivalent to the consistency of SRP over WKL_0. It follows that Proposition 2.2 is
i. provable in SRP^+ but not in SRP (assuming SRP is consistent).
ii. unprovable in ZFC (assuming ZFC is consistent).
iii. neither provable nor refutable in SRP (assuming SRP is 1-consistent).
iv. neither provable nor refutable in ZFC (assuming SRP is 1-consistent).

THEROEM 2.4. For each fixed k, Proposition 2.2 is provable in SRP. This claim is false for any fixed SRP_n (assuming SRP is consistent). There is a fixed k such that Proposition 2.2 is unprovable in ZFC. In fact, there is a fixed k and a fixed graph such that Proposition 2.2 is unprovable in ZFC. We have a target of k = 4.

3. PROOFS

The provability in Theorems 2.3, 2.4 is done almost exactly as in section 9 of [Fr14] (and earlier in section 4 of [Fr11]). The unprovability in Theorems 2.3, 2.4 is essentially done in a rather ponderous way in section 5 of [Fr11]. The section 5 unprovability from [Fr11] has to be substantially upgraded in order to get reasonably sized k for Theorem 2.4, let alone the target of k = 4 which we are presently attempting.

REFERENCES

[Fr11] H. Friedman, Invariant Maximal Cliques and Incompleteness, Downloadable Manuscripts, #71, October 7, 2011, 132 pages.

[Fr14] H. Friedman, Invariant Maximality and Incompleteness, Downloadable Manuscripts, #77, https://u.osu.edu/friedman.8/foundationaladventures/downloadable-manuscripts/, to appear, 2014.

[WIKI] http://en.wikipedia.org/wiki/Reverse_mathematics