Π^0_1 INCOMPLETENESS: finite graph theory 1

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In this abstract, all digraphs are simple; i.e., have no loops or multiple edges. The results remain unaffected if loops are allowed (but not if multiple edges are allowed).

For digraphs G, we write V(G) for the set of all vertices of G, and E(G) for the set of all edges of G. A digraph on a set E is a digraph G where V(G) = E.

A dag is a directed acyclic graph; i.e., a digraph with no cycles.

Let G be a digraph, and $A \subseteq V(G)$. We write GA for the set of all tails of edges in G whose heads lie in A. I.e.,

$$GA = \{y \in V(G): (\exists x \in A)((x,y) \in E(G))\}.$$

We begin by quoting a well known theorem about dags. We call it the complementation theorem.

COMPLEMENTATION THEOREM (finite dags). For all finite dags G there is a unique $A \subseteq V(G)$ such that $GA = V(G)\backslash A$.

We can look at the Complementation Theorem in terms of a large independent set. We say that $A \subseteq V(G)$ is independent in G if and only if there is no edge connecting elements of A.

COMPLEMENTATION THEOREM (finite dags). Every finite dag has a unique independent set A such that $V(G)\backslash A \subseteq GA$.

We will focus on digraphs on sets of the form $[1,n]^k$. Here k,n ≥ 1 and $[1,n] = \{1,2,...,n\}$.

An upgraph on $[1,n]^k$ is a digraph on $[1,n]^k$ such that for all $(x,y) \in E(G)$, $\max(x) < \max(y)$.

The following is an immediate consequence of the Complementation Theorem (finite dags), since upgraphs are obviously dags.

COMPLEMENTATION THEOREM (finite upgraphs). For all $k,n \geq 1$, every upgraph on $[1,n]^k$ has a unique independent set A such that V(G)\A □ GA.

Our development relies on what we call **order invariant digraphs** on $[1,n]^k$. These are the digraphs G on $[1,n]^k$ such that the determination of whether (x,y) is an edge depends only on the relative order of the coordinates of the vector $(x,y) \in [1,n]^{2k}$.

More formally, let $u,v \in \{1,2,3,...\}^p$. We say that u,v are order equivalent if and only if for all $1 \leq i,j \leq p$,

$$u_i < u_j \text{ iff } v_i < v_j.$$

Let G be a digraph on $[1,n]^k$. We say that G is order invariant if and only if the following holds. For all $x,y,z,w \in [1,n]^k$, if (x,y) and (z,w) are order equivalent (as 2k tuples), then

$$(x,y) \in E(G) \leftrightarrow (z,w) \in E(G).$$

Note that an order invariant digraph on $[1,n]^k$ is completely determined, among digraphs on $[1,n]^k$, by its subdigraph induced by $[1,2k]^k$ - regardless of how large n is. Thus the number of order invariant digraphs on $[1,n]^k$ is bounded by $(2k)^k$.

Let G be a digraph and $A,B \subseteq V(G)$. We say that A,B are G isomorphic if and only if the subdigraph of G induced by A is isomorphic to the subdigraph of G induced by B. I.e., there is a bijection $h:A \to B$ such that for all $x,y \in A$,

$$(x,y) \in E(G) \leftrightarrow (hx,hy) \in E(G).$$

Let m be a positive integer. A power of m is a number m^i, where i is a nonnegative integer.

A vector power of m is a vector (of any finite length) all of whose coordinates are powers of m.

We say that c does not appear in a set of vectors if and only if c is not a coordinate of any vector in that set.

PROPOSITION A. For all $n,k,r \geq 1$, every order invariant upgraph G on $[1,n]^k$ has an independent set A such that every $\leq r$ element subset of V(G)\A is G isomorphic to a subset of

GA, with the same vector powers of 2, in which the integer $2^{(4kr)^2} - 1$ does not appear.

Here is a weakening of Proposition A where only a single vector is excluded.

PROPOSITION B. For all $n,k,r \geq 1$, every order invariant upgraph G on $[1,n]^k$ has an independent set A such that every r element subset of V(G)\A is G isomorphic to a subset of GA, with the same vector powers of 2, in which the diagonal vector $(2^{(4kr)^2} - 1,..., 2^{(4kr)^2} - 1)$ does not appear.

Here are two alternative Propositions.

PROPOSITION C. For all $n,k,r \geq 1$, every order invariant upgraph G on $[1,n]^k$ has an independent set A such that every □ r element subset of V(G)\A is G isomorphic to a subset of GA, with the same vector powers of (8kr)!, in which the integer (8kr)!-1 does not appear.

PROPOSITION D. For all $n,k,r \geq 1$, every order invariant upgraph G on $[1,n]^k$ has an independent set A such that every r element subset of V(G)\A is G isomorphic to a subset of GA, with the same vector powers of 2, in which the diagonal vector ((8kr)!-1,..., (8kr)!-1) does not appear.

Propositions A-D can be proved with large cardinals but not in ZFC (assuming ZFC is consistent). Note that Propositions A-D are explicitly □01.

Note that if we remove the exclusionary clauses in Propositions A-D, then we obtain trivial consequences of the Complementation Theorem (finite upgraphs), which is provable in EFA (exponential function arithmetic). What a difference a single integer or vector makes!

Here is more detailed information.

Let MAH = ZFC + $\{$there exists a strongly n-Mahlo cardinal$\}_n$.

Let MAH+ = ZFC + “for all n there exists a strongly n-Mahlo cardinal”.

THEOREM 1. MAH+ proves Propositions A-D. However, none of Propositions A-D are provable in any consistent fragment of MAH that derives Z = Zermelo set theory. In particular, none of Propositions A-D are provable in ZFC, provided ZFC is consistent. These facts are provable in RCA_0.

THEOREM 2. EFA + Con(MAH) proves Propositions A-D.

THEOREM 3. It is provable in ACA that Propositions A-D are each equivalent to Con(MAH).