CONCRETE MATHEMATICAL INCOMPLETENESS: BASIC EMULATION THEORY

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ABSTRACT. By the modern form of Gödel's First Incompleteness Theorem, we know that there are sentences (in the language of ZFC) that are neither provable nor refutable from the usual ZFC axioms for mathematics (assuming, as is generally believed, that ZFC is free of contradiction). Yet it is clear that the usual examples are radically different from normal mathematical statements in several glaring ways such as the mathematically remote subject matter and the essential involvement of uncharacteristically intangible objects. Starting in 1967, we embarked on the Concrete Mathematical Incompleteness program with the principal aim of developing readily accessible thematic mathematical research areas with familiar mathematical subject matter replete with examples of such incompleteness involving only characteristically tangible objects. The many examples developed over the years represent Concrete Mathematical Incompleteness ranging from weak fragments of finite set theory through ZFC and beyond. The program has reached a mature stage with the development of Emulation Theory. Emulation Theory, in its present basic developed form, involves finite length tuples of rational numbers. Only the usual ordering of rationals is used, and there is no use of even addition or multiplication. The basics are fully accessible to early undergraduate mathematics majors and gifted high school mathematics students, who will be able to engage with some simple nontrivial examples in two and three dimensions, with illustrations. In this paper, we develop the positive side of the theory, using various levels of set theory for systematic development. Some of these levels lie beyond ZFC and include familiar large cardinal hypotheses. The

necessity of the various levels of set theory will be established in a forthcoming book [Fr18].

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1. INTRODUCTION

According to the modern form of Gödel's First Incompleteness Theorem, there are sentences in the language of ZFC that are neither provable nor refutable from the usual ZFC axioms for mathematics, assuming that ZFC is consistent (going back to [Go31]). This seminal result already puts an end to one important facet of the multifaceted Hilbert's Program. Since then, attention has naturally and inevitably focused on the nature of the examples of this incompleteness from ZFC.

The first clear specific example of incompleteness from ZFC is already given by the modern form of Gödel's Second Incompleteness Theorem: the consistency of ZFC, Con(ZFC), is neither provable nor refutable from ZFC (going back to [Go31]). The second clear specific example is Cantor's continuum hypothesis, CH, that every uncountable set of real numbers can be mapped onto all real numbers. That CH is not refutable in ZFC is by [Go40], and that CH is not provable in ZFC is by [Co63,64].

Attention naturally focuses on the nature of the examples of incompleteness from ZFC - and in particular, their subject matter. This move to the consideration of the underlying subject matter in examples of incompleteness is entirely natural and inevitable. Both of these examples are easily recognized to be profoundly different from normal mathematical propositions in vividly important ways that are instantly recognized by the general mathematical community. Con(ZFC) is a statement about provability in a certain formal system, and thus involves mathematically (but not philosophically) remote subject matter. CH is a statement in abstract set theory, involving uncontrolled sets of real numbers, which are immediately recognized as uncharacteristically intangible mathematical objects.

Starting in 1967, we embarked on the Concrete Mathematical Incompleteness program with the principal aim of developing readily accessible mathematical research areas with familiar mathematical subject matter replete with examples of such incompleteness involving only characteristically tangible objects. The many examples developed over the years represent Concrete Mathematical Incompleteness ranging from weak fragments of finite set theory through ZFC and beyond. A detailed discussion and presentation of the major results in Concrete Mathematical Incompleteness before this Emulation Theory can be found in [Fr14], Introduction.

Concrete Mathematical Incompleteness has reached a mature stage with the development of Emulation Theory. Emulation Theory, in its present basic developed form, involves finite length tuples of rational numbers. We use the usual ordering of rational numbers, but not addition, subtraction, or multiplication. The basics are fully accessible to early undergraduate mathematics majors and gifted high school mathematics students, who will be able to engage with some simple nontrivial examples in two and three dimensions, with illustrations. In this paper, we develop the positive side of the theory, using various levels of set theory for systematic development. Some of these levels lie beyond ZFC and include familiar large cardinal hypotheses. The large cardinal hypotheses used here are given by the SRP hierarchy (stationary Ramsey property), which are beyond strongly inaccessible, strongly Mahlo, weakly compact, and indescribable cardinals, are intertwined with the subtle and ineffable cardinal hierarchy, which lives well below $\kappa \rightarrow \omega$, and thus is compatible with ZFC + V = L (see [Ka94]). Emulation Theory has also been extended involving necessary uses of the HUGE cardinal hierarchy, which are stronger than measurable cardinals, supercompact cardinals, and Vopenka's Principle, but lie below nontrivial $j:V(\kappa) \to V(\kappa)$. See [Ka94]. Use of the HUGE cardinal hierarchy and the necessity of the various levels of set theory throughout Emulation Theory will appear in a forthcoming book.

In section 1.1, we start with the most general form of Basic Emulation Theory presented relative to any given relational structure in the usual sense of elementary logic. We successively lower this generality in several steps down to the particular relational structure M = (Q[0,1],<) that drives section 3.

In section 1.2 we give a systematic account of the major results in the paper.

Throughout the paper, we follow the convention that any free variables are implicitly universally quantified in front. Until section 3.5, we mostly see implicit free variables for dimension k, for a relational structure M, for a subset E of M^k, for a subset E of $Q[0,1]^k$, and for a relation $R \subseteq Q[0,1]^k \times Q[0,1]^k$. In section 3.5, we also see the free variable r used for r-emulations.

In this paper we focus on the use of certain large cardinal hypotheses to develop Emulation Theory. In fact, we only use the consistency of these large cardinal hypotheses, and in particular, Con(SRP). For the presentation of SRP, see Appendices A,B. We have shown that the basic results of Emulation Theory proved here from Con(SRP) are in fact provably equivalent to Con(SRP) over WKL_0 (see Appendix B). These so called reversals will appear in [Fr18].

The reversal asserting that RCA_0, or even ZFC proves A → Con(SRP), guarantees that A cannot be proved in SRP (unless SRP is inconsistent). Also, ZFC proves A → Con(ZFC) tells us that A cannot be proved in ZFC (unless ZFC is inconsistent). The reason for this is Gödel's Second Incompleteness Theorem, that no reasonable system can prove its own consistency (unless it be inconsistent).

Already in the currently available [Fr18], Boolean Relation Theory, reversals of combinatorial statements are fully worked out (in that case, 1-Con(MAH) is used with the somewhat weaker MAH). In both cases, the general method is the same. We start with the statement A (from Boolean Relation Theory or from Emulation Theory). We then build a series of structures, rather explicitly, which become more and more like models of set theory with large cardinals, until finally they really are. However, as can be seen with Boolean Relation Theory, [Fr18], there are many obstacles that have to be overcome while traveling down that long path.

1.1. BASIC EMULATION THEORY

Basic Emulation Theory starts with a given relational structure M in the usual sense of elementary logic. Thus M = (D,...), where the domain D is a nonempty set, and ... are the components, as in Definition 2.1.

It is convenient to use M^k for D^k with the understanding that the working space is the relational structure M. Tuples x,y are M equivalent if and only if x,y obey the same unnested atomic formulas. See Definition 2.2.

Now comes the crucial definition which we first give in its most compact form.

MAXIMAL EMULATION DEFINITION/1. ME/DEF/1. S is a maximal emulator of $E \subseteq M^k$ if and only if $S \subseteq M^k$ and every element of S^2 is M equivalent to an element of E^2, where this is false if S is replaced by any proper superset of S.

Here E^2,S^2 are viewed as sets of 2k-tuples. The where clause is equivalent to saying that if we add a new point $x \in M$ to S then this ruins S^2 having all of its elements M equivalent to an element of E^2.

We can equivalently break this definition into the following two parts.

MAXIMAL EMULATION DEFINITION/2. ME/DEF/2. S is an emulator of $E \subseteq M^k$ if and only if $S \subseteq M^k$ and every element of S^2 is M equivalent to an element of E^2. S is a maximal emulator of $E \subseteq M^k$ if and only if S is an emulator of $E \subseteq M^k$ which is not a proper subset of an emulator of $E \subseteq M^k$.

In Basic Emulation Theory, we investigate basic properties of maximal emulators of $E \subseteq M^k$.

MAXIMAL EMULATION/1. ME/1. Every $E \subseteq M^k$ has a maximal emulator.

There is an important sharper version.

MAXIMAL EMULATION/2. ME/2. Every $E \subseteq M^k$ has a maximal emulator containing any given emulator.

In section 2, we prove that ME/1,2 are equivalent to the full axiom of choice over ZF.

We also show that if M is countable (i.e., D = dom(M) is countable) with finitely many components, then ME/1 is provable in RCA_0 and ME/2 is provable in ACA_0 by ordinary recursion along the nonnegative integers. In fact, for such M, the sharper form is provably equivalent to ACA_0 over RCA_0.

We now start with the most general formulation of Basic Emulation Theory as a Template. We then take the generality down five steps to BETA/6, which is where the current development of Basic Emulation Theory resides.

MAXIMAL EMULATION USE DEFINITION. MEU/DEF. $R \subseteq M^k \times M^k$ is ME usable if and only if for all subsets of M^k, some maximal emulator contains its R image.

The R image of S is the forward image $\{y: (\exists x \in S)(R(x,y))\}$.

This is a convenient place to make a purely expositional point. The reader might question the wisdom of the particular Engish construction we have used to present MEU/DEF. In particular, it might appear more natural, grammatically, to write

"if and only if every subset of M^k has a maximal emulator containing its R image".

However, this introduces an ambiguity. Might this be an emulator that, among the emulators containing its R image, is maximal? That is quite different, and in fact easily seen to be true on general grounds. What we of course mean is that we have a maximal emulator that just happens to also contain its R image. Note how the formulation of MEU/DEF completely avoids this ambiguity.

BASIC EMULATION THEORY AIM/1. BETA/1. Investigate the ME usable $R \subseteq M^k \times M^k$.

The following necessary condition for ME usability is used throughout the paper.

MAXIMAL EMULATION/4. ME/4. If $R \subseteq M^k \times M^k$ is ME usable then R is M preserving in the sense that $(\forall x,y)(R(x,y) \rightarrow x,y$ are M equivalent).

We have yet to place any restrictions on the relation R or the subset E of M^k. We will naturally want to focus on reasonably well behaved R and subsets of M^k. The M elementary sets are the subsets of M^k that are defined by a quantifier free formula over M, with parameters allowed. E.g., the M elementary sets, where M is the ordered field of real numbers, are the semi algebraic subsets of $\Re^k$.

If M has finitely many components, then in the Maximal Emulation Use Definition, we can replace "subsets of M^k" with "finite subsets of M^k" and have an equivalent definition.

BASIC EMULATION THEORY AIM/2. BETA/2. Investigate the ME usable $R \subseteq M^k \times M^k$, where M is a relational structure with finitely many components and $R \subseteq M^k \times M^k$ is elementary.

It is also natural to focus on well behaved M. We propose three main rich sources of such M. For uncountable M, there are the structures that are definable over the ordered field of real numbers. For countable M, there are the structures that are definable over the ordered field of real algebraic numbers or over the structure (Q,Z,<,+).

BASIC EMULATION THEORY AIM/3. BETA/3. Investigate the ME usable $R \subseteq M^k \times M^k$ where M has finitely many components and is definable over the ordered field of real numbers, and $R \subseteq M^k \times M^k$ is M elementary.

BASIC EMULATION THEORY AIM/4. BETA/4. Investigate the ME usable $R \subseteq M^k \times M^k$ where M has finitely many components and is definable over the ordered field of real algebraic numbers, and $R \subseteq M^k \times M^k$ is M elementary.

BASIC EMULATION THEORY AIM/5. BETA/5. Investigate the ME usable $R \subseteq M^k \times M^k$ where M has finitely many components and is definable over (Q,Z,<,+), and $R \subseteq M^k \times M^k$ is M elementary.

We now arrive at the particular case in Basic Emulation Theory that is discussed in this paper, which uses only M = (Q[0,1],<). Here Q[0,1] = Q ∩ [0,1], where Q is the set of all rational numbers with the usual numerical <.

BASIC EMULATION THEORY AIM/6. BETA/6. Investigate the ME usable $R \subseteq M^k \times M^k$, where M = (Q[0,1],<) and R is order theoretic.

Here the order theoretic R are exactly the elementary R, and, in fact, the definable R by the well known quantifier elimination for dense linear orderings.

For the rest of the paper, we focus on BETA/6 for (Q[0,1],<). We use Q[0,1] as an abbreviation for (Q[0,1],<), as the < is tacitly understood.

1.2. ON Q[0,1]

Section 2 presents proofs of the results discussed in section 1.1. Section 3 is the heart of the paper, and treats Basic Emulation Theory on Q[0,1]. We use ME/DEF/1,2 and MEU/DEF with M = (Q[0,1],<).

In section 3.1, we present eight illustrative examples of $E \subseteq Q[0,1]^2$ and their emulators and maximal emulators to familiarize the reader with these notions.

We show that the ME usability of any given order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ forms a sentence φ which is implicitly Π^0_1 over WKL_0 (Corollary 3.1.7). I.e., is provably equivalent to a Π^0_1 sentence over WKL_0. This is shown by use of the Gödel Completeness Theorem. A consequence of φ being implicitly Π^0_1 over WKL_0 is that φ is WKL_0 falsifiable, in the sense of Definition 3.1.6. Falsifiability resonates with the idea in physical science that in order for a statement to be physically meaningful, it must be refutable by experimentation. According to Theorem 3.1.4, implicitly Π^0_1 and falsifiable are nearly the same notions.

Thus the usability of a given order theoretic R forms a statement that is, at least implicitly, of the most concrete level of complexity for mathematical statements involving infinitely many objects. The highlight of this paper is how this leads to independence from the usual ZFC axioms for mathematics. These statements still involve the use of an infinite object, namely the maximal emulator. This is a feature that is much stronger than merely having infinitely many objects. So although these statements are implicitly Π^0_1, they are not explicitly Π^0_1 or even explicitly finite. Thus it would be of great interest to have similarly interesting and strategic mathematical examples of independence from ZFC that are explicitly finite or even explicitly Π^0_1. We have arguably achieved this in a particularly satisfying way, and this work will appear elsewhere in [Fr18].

Recall our basic necessary condition for ME usability in ME/4. For M = (Q[0,1],<), we call this necessary condition order preserving. I.e., $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is order preserving if and only if $(\forall x,y)(R(x,y) \rightarrow x,y$ are order equivalent). (E.g., (.5,.7,.6) and (.2,1,.9) are order equivalent).

We illustrate the depth of ME usability through two simple examples.

MAXIMAL EMULATION EXAMPLE/1. MEX/1. For finite subsets of $Q[0,1]^2$, some maximal emulator is equivalent at (1/2,1/3),(1/3,1/4).

MAXIMAL EMULATION EXAMPLE/2. MEX/2. For finite subsets of $Q[0,1]^2$, some maximal emulator is equivalent at (1,1/2),(1/2,1/3).

I.e., we demand, for some maximal emulator S, that $(1/2,1/3) \in S \leftrightarrow (1/3,1/4) \in S$ in MEX/1 and $(1,1/2) \in S \leftrightarrow (1/2,1/3) \in S$ in MEX/2. Thus MEX/1 asserts the ME usability of the symmetric R = {((1/2,1/3),(1/3,1/4)),((1/3,1/4),(1/2,1/3))} of cardinality 2 in dimension 2, and analogously for MEX/2.

We show that MEX/1 is rather superficial in that it is merely a consequence of the fact that isomorphic copies of maximal emulators are maximal emulators. MEX/2 cannot be proved this way because of the use of the right endpoint 1, and so there is something deeper going on here.

In section 3.2, we address the problem of finding a necessary and sufficient condition for finite R to be ME usable. Two key notions are “appearance” and “alteration” in Definition 3.2.1, which we repeat here for the reader's convenience.

DEFINITION 3.2.1. Let $(x,y) \in Q[0,1]^k \times Q[0,1]^k$ and $R \subseteq Q[0,1]^k \times Q[0,1]^k$. p is present in (x,y) if and only if p is a coordinate of x or y. p is altered in (x,y) if and only if there exists i such that $p = x_i \neq y_i$ or $p = y_i \neq x_i$. p is present in R if and only if there exists i such that $p = x_i$ or $p = y_i$. p is altered by R if and only if p is altered in some element of R. We also write "p appears in (x,y)", "(x,y) alters p", "p appears in R", and "R alters p".

EXAMPLE. In ((.5,.7,.5),(.6,.7,.5)), 0 is not present, .7 is present but not altered, .5,.6 are present and altered.

We start with a particularly easy result, which generalizes MEX/1.

MAXIMAL EMULATION FINITE USE/1. MEFU/1. Any finite order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ in which neither 0 nor 1 appear, is ME usable.

More difficult is

MAXIMAL EMULATION FINITE USE/2. MEFU/2. Any finite order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ in which not both 0,1 appear, is ME usable.

which generalizes MEX/2. The full result along these lines reads

MAXIMAL EMULATION FINITE USE/3. MEFU/3. Any finite order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ that does not alter both 0 and 1, is ME usable.

which is the form that we prove. It immediately implies MEX/1,2, MEFU/1,2. The proof of MEFU/3 is given in ACA', and we don't have a proof in ACA_0. We conjecture that ACA_0 does not suffice to prove MEU/2,3, although RCA_0 suffices for MEFU/1.

If finite R alters both 0,1, then open issues arise. However, we do have the following.

MAXIMAL EMULATION FINITE USE/4. MEFU/4. Any order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ of cardinality 1 is ME usable.

We show that MEFU/4 is false with cardinality 2, even for R = {((0,1/2),(1/2,1)),((1/2,1),(0,1/2))}, which is order preserving and symmetric of cardinality 2 in dimension k = 2.

We present a complete determination of the symmetric ME usable R of cardinality 2 in dimension $k \leq 2$. However, we do not have a complete determination of the symmetric ME usable R of cardinality 2 in any dimension $k \geq 3$.

In sections 3.3, and 3.4, we address ME usability for infinite order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$. In section 3.3 we focus on the "large" R where infinitely many numbers are altered, and in section 3.4 we focus on the "small" R where finitely many numbers are altered.

We show that the large $Q(0,1)^{2<} \times Q(0,1)^{2<}$ is not ME usable. However, we show that the large relation $Q[1/2,1)^{2<} \times Q[1/2,1)^{2<}$ is ME usable. On the other hand, we show that for dimension $k \geq 3$, the large relation $Q[1/3,1/2]^{k<} \times Q[1/3,1/2]^{k<}$ is not ME usable. There are many issues left open for the ME usability of large order theoretic relations R, even in dimension 2. So far, the results about large relations have not even come close to challenging ZFC.

The highlight of the paper is in section 3.4 where we focus on the ME usability for small order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$. It is here that it is necessary and sufficient to go well beyond the usual ZFC axioms in order to obtain basic information.

We start with the obvious parameterization of finite $R \subseteq Q[0,1]^k \times Q[0,1]^k$ obtained by merely adding a new dimension given by R'(x,y) if and only if $R((x_1,...,x_k),(y_1,...,y_k))$ and $x_{k+1} = y_{k+1}$. This is a crude false start because generally this R' is not even order preserving - the basic necessary condition for ME usability.

But we can easily and very naturally recover by going to lower parameterizations. Here we use R'(x,y) if and only if $R((x_1,...,x_k),(y_1,...,y_k))$ and $x_{k+1} = y_{k+1} < x_1,...,x_k,y_1,...,y_k$. This is promising because the lower parameterization of an order preserving $R \subseteq Q[0,1]^k \times$

$Q[0,1]^k$ is order preserving.

MAXIMAL EMULATION SMALL USE/1. MESU/1. The lower parameterization of any order preserving finite $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable.

This lower parameterization idea is familiar from indiscernibles in set theory. Let $R \subseteq \lambda^k$, where λ is a suitable large cardinal. A strong kind of SOI (set of indiscernibles) often considered is an $I \subseteq \lambda$ such that for all $\alpha_1 < \ldots < \alpha_{k-1}$ and $\beta_1 < \ldots < \beta_{k-1}$ from I, and for all $\gamma < \min(\alpha_1,\beta_1)$, we have $(\gamma,\alpha_1,\ldots,\alpha_{k-1}) \in R \leftrightarrow (\gamma,\beta_1,\ldots,\beta_{k-1}) \in R$. Various forms of this lower parameterization idea requires that λ be a large cardinal in what is called the SRP hierarchy, Appendix A. Thus from the results of this paper, we can view Emulation Theory as a particularly natural discrete form of the SRP hierarchy.

MESU/1 is the first of the paper's two most immediately transparent statements independent of ZFC. MED/1 is more specific but more specialized. We say that S is drop equivalent at x,y if and only if $x,y \in Q[0,1]^k \wedge x_k = y_k \wedge (\forall p \in [0,x_k))(S(x_1,\ldots,x_{k-1},p) \leftrightarrow S(y_1,\ldots,y_{k-1},p))$.

We think of x,y as raindrops in the space $Q[0,1]^k$, at the same height $x_k = y_k$ over the ground. As they fall to the ground in tandem, they generally go in and out of a given set $S \subseteq Q[0,1]^k$. Drop equivalence says that as they fall in tandem, one is in S if and only if the other is in S.

MAXIMAL EMULATION DROP/1. MED/1. For finite subsets of $Q[0,1]^k$, some maximal emulator is drop equivalent at $(1,1/2,\ldots,1/k),(1/2,\ldots,1/k,1/k)$.

We derive MED/1 easily from MESU/1. We also present two strengthenings MESU/2,3 of MESU/1, and two strengthenings MED/2,3 of MED/1.

MESU/2 is based on the equivalence relations $R_k(A)$, on $Q[0,1]^k$, associated with each $A \subseteq Q[0,1]$. MESU/2 asserts that for finite $A \subseteq Q(0,1]$, $R_k(A)$ is ME usable. MESU/3 is based on a natural finiteness condition on $R \subseteq Q[0,1]^k \times Q[0,1]^k$, and asserts that every order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ with the finiteness condition, not altering 0, is ME usable.

MED/2 strengthens MED/1 by giving a necessary and sufficient condition (droppable) for a pair of k-tuples to work for MED/1. MED/3 strengthens MED/2 by asserting that we can simultaneously use any finite list of pairs of k-tuples for MED/1 if and only if we can use any one of the pairs for MED/1. The necessity of these conditions is provable in RCA_0.

We derive all six statements from MESU/2 and derive MED/1 from all six statements. We have shown that MED/1 implies Con(SRP) over RCA_0. This reversal will appear elsewhere in [Fr18].

In section 3.5, we derive MESU/2 in dimension k = 2 using a transfinite construction of uncountable length. This puts MESU/2 in dimension k = 2 well within ZFC, and, with some modification, even in Z and even Z_3. We then derive full MESU/2 in WKL_0 + Con(SRP). Thus, with the help of the reversal to appear elsewhere ([Fr18]), we have established that MESU/1,2,3, MED/1,2,3 are all provably equivalent to Con(SRP) over WKL_0. It follows that MESU/1,2,3, MED/1,2,3 are all independent of ZFC, assuming SRP is consistent.

We conjecture that MESU/2 for dimension k = 2 is provable already in RCA_0. However, we also conjecture that such a proof would be much more difficult than the proof given in section 3.5 using the transfinite construction of uncountable length. It would require essentially a complete analysis of the sets of maximal emulators of finite subsets of $Q[0,1]^2$. We know that there are finitely many such sets of maximal emulators in any dimension k, so an exhaustive analysis is theoretically possible. But see below for a sharpened form of MESU/2 for dimension k = 2.

In section 3.6, we introduce r-emulators, where the emulators are the 2-emulators. Most of the earlier results go through without serious modification for r-emulators, with noted exceptions. We conjecture that MESU/1 in dimension k = 1 and MESU/2,3, MED/1,2,3 in dimension k = 2, sharpened with r-emulators, are not provable in ZFC\P, or equivalently, not in Z_2.

We also conjecture that MESU/1 in dimension k = 2 and MESU/2,3, MED/1,2,3 in dimension k = 3, sharpened with r-emulators, are not provable in ZFC (assuming ZFC is consistent). In fact, we conjecture that they are provably equivalent to Con(ZFC + "there exists a subtle cardinal") over WKL_0. Note that MED/1 has the "raindrops falling in tandem" interpretation, which is particularly vivid in 3 dimensions.

In section 4, we discuss a number of General Conjectures. These do not specifically pertain to the statements discussed in section 3. The first of these strategic conjectures is

GENERAL CONJECTURE 1. GC1. There is an algorithm for determining whether a given order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is usable. For inputs, use a standardly digitized form of quantifier free formulas over (Q[0,1],<) with parameters.

about which we know essentially nothing. However we establish that its sharpening,

GENERAL CONJECTURE 2. GC2. There is a Turing machine with at most $2^{2^{\wedge}1000}$ states/symbols each, for determining whether a given order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is usable. For inputs, use a standardly digitized form of quantifier free formulas over (Q[0,1],<) with parameters.

is not provable in ZFC, assuming SRP is consistent.

2. GENERAL MAXIMAL EMULATION

Here we work in the most general context of relational structures M.

DEFINITION 2.1. A relational structure is a system M = (D,...), where the domain D is a nonempty set, and ... are the components, consisting of named constants from D, named relations on D of finite arity, and named functions from and into D of finite arity. In full generality, the number of components is arbitrary, although most commonly there are finitely many components. Equality is considered implicitly present, and does not have to be a component. M is countable if and only if its domain and number of components is countable.

We use M^k for the set D^k with the understanding that the environment is M.

DEFINITION 2.2. $x,y \in M^k$ are M equivalent if and only if x,y obey the same unnested atomic formulas in the sense that

$x_i = x_j \leftrightarrow y_i = y_j$

$x_i = c \leftrightarrow y_i = c$

$R(x_{i1},\ldots,x_{in}) \leftrightarrow R(y_{i1},\ldots,y_{in})$

$F(x_{j1},\ldots,x_{jm}) = x_b \leftrightarrow F(y_{j1},\ldots,y_{jm}) = y_b$

where $1 \leq i,j,i_1,\ldots,i_n,j_1,\ldots,j_m,b \leq k$, c is a constant, R is an n-ary relation, and F is an m-ary function of M. $EQR(M,k) \subseteq M^k \times M^k = M^{2k}$ is the equivalence relation of M equivalence on M^k.

MAXIMAL EMULATION DEFINITION/1. ME/DEF/1. S is a maximal emulator of $E \subseteq M^k$ if and only if $S \subseteq M^k$ and every element of S^2 is M equivalent to an element of E^2, where this conjunction is false if S is replaced by any proper superset of S.

Here ME is read "maximal emulator". Also S^2,E^2 are viewed as sets of 2k-tuples.

We rely on presenting E as a subset of M^k in order to specify the environment in which we are operating. Thus only supersets of S that are subsets of M^k are relevant.

We can equivalently break this definition into the following two parts.

MAXIMAL EMULATION DEFINITION/2. ME/DEF/2. S is an emulator of $E \subseteq M^k$ if and only if $S \subseteq M^k$ and every element of S^2 is M equivalent to an element of E^2. S is a maximal emulator of $E \subseteq M^k$ if and only if S is an emulator of $E \subseteq M^k$ which is not a proper subset of an emulator of $E \subseteq M^k$.

MAXIMAL EMULATION/1. ME/1. Every $E \subseteq M^k$ has a maximal emulator.

There is an important sharper form.

MAXIMAL EMULATION/2. ME/2. Every $E \subseteq M^k$ has a maximal emulator containing any given emulator.

This general situation is nicely clarified as follows.

THEOREM 2.1. (Z) The following are equivalent.
i. The axiom of choice.

ii. ME/2.
iii. ME/1.
iv. Every finite $E \subseteq M^2$ has a maximal emulator, where M is an equivalence relation.

Proof: For i → ii, let $E \subseteq M^k$ and S be an emulator of $E \subseteq M^k$. Let $<_D$ be a well ordering of dom(M) = D. Perform the usual greedy transfinite algorithm where S' is unique such that S' = {x: (S' ∩ {y: y $<_D$ x}) ∪ S ∪ {x} is an emulator of $E \subseteq M^k$}. ii → iii → iv is immediate. For iv → i, we use the axiom of choice in the form that for every equivalence relation (D,R), there is a set containing exactly one element from each equivalence class. Let (D,R) be any equivalence relation. We can assume that D has at least two elements with are not R related. Let E = {x,y}, where x,y are not R related. The emulators of $E \subseteq (D,R)^2$ are exactly the subsets of D that contain at most one element from each equivalence class under R. Clearly any maximal emulator of $E \subseteq (D,R)^2$ contains exactly one element from each equivalence class of R. QED

FINITE SUBSET EMULATION. Assuming M has finitely many components, every $E \subseteq M^k$ is an emulator of some finite $E' \subseteq M^k$ with $E' \subseteq E$. This is provable in RCA_0 for countable M with finitely many components.

Proof: For the first claim, note that there are finitely many equivalence classes in any given dimension under M equivalence. Pick a representative from each M equivalence class of elements of E^2, and take E' to be the set of all elements of E that are used. Working within RCA_0, we first construct the set V of elements of E^2 that are not M equivalent to any lessor element of E^2 (where lessor refers to an ad hoc enumeration of E^2 based on the enumeration of D). Then using finite Σ^0_1 separation, available in RCA_0, we form the set of all elements of E that are used. QED

EMULATION TRANSITIVITY. If S is an emulator of $E \subseteq M^k$ and E is an emulator of $E' \subseteq M^k$, then S is an emulator of $E' \subseteq M^k$. Let E be an emulator of $E' \subseteq M^k$ and E' be an emulator of $E \subseteq M^k$. The emulators of $E \subseteq M^k$ are the same as the emulators of $E' \subseteq M^k$. The maximal emulators of $E \subseteq M^k$ are the same as the maximal emulators of $E' \subseteq M^k$. This is provable in RCA_0 for countable M.

Proof: The first claim follows immediately from M equivalence being an equivalence relation. Now let E,E' be mutual emulators in M^k. The second claim follows from the first claim.

For the third claim, first suppose S is a maximal emulator of $E \subseteq M^k$. Then S is an emulator of $E' \subseteq M^k$. Let $S' \supseteq S$ be an emulator of $E' \subseteq M^k$. Then S' is an emulator of $E \subseteq M^k$, and so S' = S. Suppose S is a maximal emulator of $E' \subseteq M^k$. By the analogous argument, S is a maximal emulator of $E \subseteq M^k$. QED

MAXIMAL EMULATION/3. ME/3. (RCA_0) Let M be countable with finitely many components. Every subset of M^k has a maximal emulator. The following are equivalent.
i. ACA_0.
ii. If M is countable then every subset of M^k has a maximal emulator containing any given emulator.
iii. In every equivalence relation M on N, every finite subset of N has a maximal emulator containing any given emulator.

Proof: For the first claim, let $E \subseteq M^k$. By Finite Subset Emulation and Emulation Transitivity (second claim), we can assume that E is finite. Then use the usual greedy algorithm. For i → ii, use the same argument, starting with the given emulator, which creates the need for ACA_0 instead of just RCA_0 (because the given emulator may be infinite). ii → iii is obvious. Assume iii, and we derive ACA_0. It suffices to show that the range of every one-one $f:2N \rightarrow 2N+1$ exists. Let R be the equivalence relation on N given by R(n,m) if and only if $f(n) = m \vee f(m) = n \vee n = m$.

For iii → i, we use the structure M = (N,R). The emulators of {0,2} are the subsets of N that contain at most one element from each equivalence class of R. We use the emulator 2N of $\{0,2\} \subseteq N$. Thus iii implies that there is a set S containing exactly one element from each equivalence class under R and also contains 2N. Thus S consists of 2N together with the odd numbers that are outside the range of f. From S, we obtain the range of f. QED

We now present the most general formulation of Basic Emulation Theory as a Template.

DEFINITION 2.3. Let $R \subseteq M^k \times M^k$. The R image of S, or (forward) image of S under R, is $R[S] = \{y: (\exists x \in S)(R(x,y))\}$.

MAXIMAL EMULATION USE DEFINITION. MEU/DEF. $R \subseteq M^k \times M^k$ is ME usable if and only if for all subsets of M^k, some maximal emulator contains its R image.

In the countable case, actually forming images (as sets) requires ACA_0, and is not available in RCA_0. However, the notion "S contains its R image" is viewed as not presupposing that we actually form the image (as a set). So RCA_0 can be used to investigate ME usability in the countable context.

THEOREM 2.2. Let M have finitely many components. $R \subseteq M^k \times M^k$ is ME usable if and only if for all finite subsets of M^k, some maximal emulator contains its R image. RCA_0 proves this for countable M with finitely many components.

Proof: Let M be as given, and suppose that for all finite $E \subseteq M^k$, some maximal emulator of $E \subseteq M^k$ contains its R image. Let $E \subseteq M^k$. By Finite Subset Emulation, let E be an emulator of finite $E' \subseteq M^k$, where $E' \subseteq E$. By Emulation Transitivity, the maximal emulators of $E \subseteq M^k$ are the same as the maximal emulators of $E' \subseteq M^k$. Hence some maximal emulator of $E \subseteq M^k$ contains its R image. For the second claim, RCA_0 is enough because it was enough for Finite Subset Emulation and Emulation Transitivity. QED

When verifying ME usability, we will generally use finite $E \subseteq M^k$ in accordance with Theorem 2.2, in order to emphasize the concrete aspects of Emulation Theory.

We now give a clarifying necessary condition for ME usability.

MAXIMAL EMULATION/4. ME/4. If $R \subseteq M^k \times M^k$ is ME usable then R is M preserving in the sense that $(\forall x,y)(R(x,y) \rightarrow x,y$ are M equivalent). RCA_0 proves this for countable M.

Proof: Let $R \subseteq M^k \times M^k$ be ME usable. We claim that every M equivalence class $[x] \subseteq M^k$ is the unique maximal emulator of $[x] \subseteq M^k$. To see this, let S be an emulator of [x]. For $y \in S$, (y,y) is M equivalent to some $(x',x') \in [x]^2$. Hence for y

$\in$ S, x,y are M equivalent. So S $\subseteq$ [x]. Since [x] is an emulator of [x], clearly the unique maximal emulator of [x] $\subseteq M^k$ is [x].

Now let R(x,y). [x] has a maximal emulator S containing its R image. Hence [x] contains its R image. Hence y $\in$ [x], and so x,y are M equivalent. QED

DEFINITION 2.4. Let R $\subseteq M^k \times M^k$. R is symmetric if and only if $(\forall x,y)(R(x,y) \leftrightarrow R(y,x))$. $R^{-1} = \{(y,x): R(x,y)\}$. The inverse image of S under R is $R^{-1}[S] = \{x: (\exists y \in S)(R(x,y))\}$. S is R invariant if and only if $(\forall x,y)(R(x,y) \rightarrow (x \in S \leftrightarrow y \in S))$. S is equivalent at x,y if and only if $x \in S \leftrightarrow y \in S$.

THEOREM 2.3. Let R $\subseteq M^k \times M^k$. S is R invariant if and only if S contains its image and inverse image under R. If R is symmetric, then S is R invariant if and only if S contains its R image. RCA_0 proves this for countable M.

Proof: Suppose $(\forall x,y)(R(x,y) \rightarrow (x \in S \leftrightarrow y \in S))$. If y is in the forward image of S under R then let R(x,y), x $\in$ S. Then y $\in$ S. If x is in the inverse image of S under R then let R(x,y), y $\in$ S. Then x $\in$ S. Conversely, suppose $S \supseteq R[S] \cup R^{-1}[S]$, and let R(x,y). If x $\in$ S then y $\in$ R[S] and so y $\in$ S. If y $\in$ S then $x \in R^{-1}[S]$, and so x $\in$ S. The rest is left to the reader. QED

MAXIMAL EMULATION INVARIANT USE DEFINITION. MEIU/DEF. R $\subseteq M^k \times M^k$ is ME invariantly usable if and only if for all subsets of M^k, some maximal emulator is R invariant.

THEOREM 2.4. R $\subseteq M^k \times M^k$ is ME invariantly usable if and only if $R \cup R^{-1}$ is ME usable. If R $\subseteq M^k \times M^k$ is symmetric then R is ME invariantly usable if and only if R is ME usable. RCA_0 proves this for countable M.

Proof: Left to the reader. QED

THEOREM 2.5. (RCA_0) Let M be countable with finitely many components, and x,y $\in M^k$. The following are equivalent.
i. For finite subsets of M^k, some maximal emulator is equivalent at x,y $\in M^k$.
ii. {(x,y)} is ME invariantly usable.

iii. {(x,y),(y,x)} is ME usable.

Proof: Left to the reader. QED

We shall see in section 3.2 that the particularly simple form i, with its point equivalence, is already delicate.

The following provides basic tools for establishing results concerning ME usability and ME invariant usability.

THEOREM 2.6. The following hold.
i. Every subset of a (invariantly) ME usable $R \subseteq M^k \times M^k$ is (invariantly) ME usable.
ii. Let $f:M \to M'$ be an isomorphism from M onto M', $E,S \subseteq M^k$, and $R \subseteq M^k \times M^k$. S is an (maximal) emulator of $E \subseteq M^k$ if and only if f[S] is an (maximal) emulator of $f[E] \subseteq M'^k$. $R \subseteq M^k \times M^k$ is ME (invariantly) usable if and only if $f[R] \subseteq M'^k \times M'^k$ is (invariantly) usable.
iii. Let $g:M \to M$ be an automorphism of $EQR(M,k) \subseteq M^{2k}$. S is an (maximal) emulator of $E \subseteq M^k$ if and only if g[S] is an (maximal) emulator of $g[E] \subseteq M^k$. $R \subseteq M^k \times M^k$ is ME (invariantly) usable if and only if $g[R] \subseteq M^k \times M^k$ is ME (invariantly) usable.
Here f,g always act coordinatewise.

Proof: i,ii are left to the reader. For iii, let M,g,E,S,R be as given. Suppose S is an emulator of E. Then every $(x,y) \in S^2$ is M equivalent to some $(z,w) \in E^2$. Let $(x,y) \in f[S]^2$. Write $(x,y) = (g(x'),g(y'))$, $x',y' \in S$. Let $(z,w) \in E^2$ be such that $(x',y'),(z,w)$ are M equivalent. Then $(g(x'),g(y')),(g(z),g(w))$ are M equivalent. Hence $(x,y),(g(z),g(w))$ are M equivalent, and $(g(z),g(w)) \in g[E]^2$. Hence g[S] is an emulator of $g[E] \subseteq M^k$. The previous argument holds with the automorphism g^{-1}, and so $g^{-1}[g(S)]$ is an emulator of $g^{-1}[g[E]] \subseteq M^k$. Hence S is an emulator of $E \subseteq M^k$.

Now suppose S is a maximal emulator of $E \subseteq M^k$. Let S' be an emulator of $g[E] \subseteq M^k$ containing g[S]. Then $g^{-1}[S']$ is an emulator of $g^{-1}[g[E]] \subseteq M^k$ containing S, and so $g^{-1}[S'] \supseteq S$ is an emulator of $E \subseteq M^k$. Hence $g^{-1}[S'] = S$, and so $S' = g[S]$. Hence S' is a maximal emulator of $g[E] \subseteq M^k$. Finally suppose g[S] is a maximal emulator of $g[E] \subseteq M^k$. The previous argument holds for the automorphism g^{-1} of

EQR(M,k), and so $g^{-1}[g[S]]$ is a maximal emulator of $g^{-1}[g[E]] \subseteq M^k$. Hence S is a maximal emulator of $E \subseteq M^k$.

Next suppose $R \subseteq M^k \times M^k$ is ME usable. We now show that $g[R] \subseteq M^k \times M^k$ is ME usable. Let S be a maximal emulator of $g^{-1}[E] \subseteq M^k$, where S contains its R image. By the previous claim, g[S] is a maximal emulator of $E \subseteq M^k$. We claim that g[S] contains its g[R] image. To see this, let g[R](x,y), where $x \in g[S]$. Then $R(g^{-1}(x),g^{-1}(y))$, where $g^{-1}(x) \in S$, and since S contains its R image, we have $g^{-1}(y) \in S$, $y \in g[S]$. Now suppose $g[R] \subseteq M^k \times M^k$ is ME usable. The previous argument holds for the automorphism g^{-1} of EQR(M,k), and so $g^{-1}[g[R]] = R$ is ME usable. The claim for ME invariantly usable is verified by applying the previous claims to symmetric R. QED

This completes our brief development of general Basic Emulation Theory.

3. MAXIMAL EMULATION ON Q[0,1]

We now develop Basic Emulation Theory in our special context Q[0,1] = (Q[0,1],<), where $Q[0,1] = Q \cap [0,1]$. We view the MESU/1 and MED/1 of section 3.4 as the most immediately transparent statements independent of ZFC in this paper.

In section 3.1, we present some background material concerning Maximal Emulation on Q[0,1]. We present eight illustrative examples of $E \subseteq Q[0,1]^2$ and their emulators and maximal emulators in order to help orient the reader.

In section 3.2, we work with finite relations $R \subseteq Q[0,1]^k \times Q[0,1]^k$, with Theorems MEFU/1,2,3, MEOU/1,2. MEFU/1 and MEOU/1,2 are proved in RCA_0, whereas MEFU/2,3 are proved in ACA'. We conjecture that MEFU/2,3 cannot be proved in ACA_0.

In sections 3.3 and 3.4, we work with infinite relations $R \subseteq Q[0,1]^k \times Q[0,1]^k$, with a focus on the order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$. Here we view order theoretic R as "large" or "small" according to whether infinitely many or finitely many numbers are altered. In section 3.3, we prove MELU/1,2,3,4 in RCA_0.

In section 3.4, we present MESU/1,2,3, and MED/1,2,3.

MESU/1 and MED/1 are the most immediately transparent statements. All six are derived from MESU/2 and shown to imply MED/1, all within RCA_0. MESU/2 for dimension k = 2 is proved using a transfinite construction of uncountable length. Full MESU/2 is then proved in SRP^+. The proof is then modified to take place in WKL_0 + Con(SRP). In a reversal that will appear elsewhere in [Fr18], we prove Con(SRP) in RCA_0 + MED/1. This establishes that all six statements MESU/1,2,3, MED/1,2,3 are provably equivalent to Con(SRP) over WKL_0. This also establishes that all six statements are independent of ZFC, assuming SRP is consistent.

We conjecture that MESU/2 for dimension 2 can be proved in RCA_0. We also conjecture that such a proof will be much more complicated than the proof given here using ω_1. However, see below.

In section 3.6, we present a very natural extension of the notion of emulator to r-emulator. An emulator is just a 2-emulator. S is an emulator of E if and only if every element of S^2 is order equivalent to an element of E^2. S is an r-emulator of E if and only if every element of S^r is order equivalent to an element of E^r. Thus an emulator is just a 2-emulator. We revisit all statements quantifying over all r, and also for fixed r. Most of the previous results hold as long as r is not fixed to 1. In particular, MESU/1,2,3, MED/1,2,3 remain provably equivalent to Con(SRP) over WKL_0 when extended to r-emulators, r universally quantified.

We conjecture that MESU/1,2,3, MED/1,2,3 in dimension k = 2, using r-emulators, quantifying over r, is not provable in ZFC\P or, equivalently in Z_2. We also conjecture that MESU/1,2,3, MED/1,2,3 in any fixed dimension k ≥ 3, using r-emulators, quantifying over r, are all equivalent to Con(ZFC + "there exists a k-subtle cardinal") over WKL_0. In particular, we conjecture that MESU/1,2,3, MED/1,2,3 in dimension k = 3, using r-emulators, quantifying over r, are all independent of ZFC, assuming ZFC + "there exists a subtle cardinal" is consistent.

From Theorem 3.1.6 and Corollary 3.1.7, it is clear that MEX/2, MEFU/2,3, MELU/3, MELU/5, MESU/1,2,3, MED/1,2,3 are all implicitly Π^0_1 over WKL_0, and WKL_0 falsifiable via Gödel's Completeness Theorem (MESU/3 stated for order theoretic R). This is also clear if we use r-emulators, either quantifying over all $r \geq 1$, or fixing $r \geq 1$.

3.1. ME USABILITY

DEFINITION 3.1.1. $Q,Z,Z^+,N,\Re$ are the sets of all rational numbers, integers, positive integers, nonnegative integers, and real numbers, respectively. We use variables p,q, with or without subscripts, over rationals, unless indicated otherwise. We use variables i,j,k,n,m,r,s,t, with or without subscripts, over positive integers, unless indicated otherwise. We use inequality chaining in the sense that, e.g., $p\ \alpha\ q\ \beta\ b \leftrightarrow p\ \alpha\ q \wedge q\ \beta\ b$, where $\alpha,\beta \in \{=,\neq,<,>,\leq,\geq\}$. $Q[(p,q)] = Q \cap [(p,q)]$, covering all four endpoint possibilities. Let $x \in Q^k$ and $y \in Q^n$. $(x,y) \in Q^{k+n}$ is the concatenation of x and y. min(x),max(x) are the least and greatest coordinates of x, respectively. For $1 \leq i \leq k$, x_i is the i-th coordinate of x. Thus x_i exists if and only if x is a tuple (finite sequence) of length $\geq i$. Let $S \subseteq Q^k$. $S|{<}p$, $S|{\leq}p$, $S|{>}p$, $S|{\geq}p$ is $S \cap (-\infty,p)^k$, $S \cap (-\infty,p]^k$, $S \cap (p,\infty)^k$, $S \subseteq [p,\infty)$, respectively. $Q^{k<} = \{x \in Q^k: x_1 < \ldots < x_k\}$, $Q^{k>} = \{x \in Q^k: x_1 > \ldots > x_k\}$.

We use the crucial order equivalence relation on $Q[0,1]^k$. For example, (.5,.7,.6) and (.2,1,.9) are order equivalent. This is the same as (Q,<) equivalence in the sense of M equivalence in Definition 2.2. By default, we have the order equivalence relation on $Q[0,1]^k$, which is also the same as (Q[0,1],<) equivalence. However, order equivalence plays such a fundamental role that we give the following equivalent definition.

ORDER EQUIVALENCE DEFINITION. $x,y \in Q[0,1]^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

DEIFNITION 3.1.2. S is a maximal emulator of $E \subseteq Q[0,1]^k$ if and only if $S \subseteq Q[0,1]^k \wedge$ every element of S^2 is order equivalent to an element of $E^2 \wedge$ this conjunction is false with S replaced by any proper superset of S.

Note that here E^2,S^2 are sets of 2k-tuples. We can equivalently break Definition 3.1.2 into two parts.

DEFINITION 3.1.3. S is an emulator of $E \subseteq Q[0,1]^k$ if and only if $S \subseteq Q[0,1]^k$ and every element of S^2 is order equivalent to an element of E^2. S is a maximal emulator of $E \subseteq Q[0,1]^k$ if and only if S is an emulator of $E \subseteq Q[0,1]^k$ which is not a proper subset of an emulator of $E \subseteq Q[0,1]^k$.

Note that Definitions 3.1.2, 3.1.3 are special cases of ME/DEF/1,2 of section 2, as here we are using M = (Q[0,1],<).

We now give some illustrative examples of emulators and maximal emulators. Obviously, ∅ is vacuously an emulator of any $E \subseteq Q[0,1]^k$, and is a maximal emulator of $\emptyset \subseteq Q[0,1]^k$. EX1 is in dimension k = 1 and EX2-8 are in dimension k = 2.

EX1. $E \subseteq Q[0,1]$. If E = ∅ then ∅ is the only emulator and it is the maximal emulator. If |E| = 1 then the emulators are subsets of Q[0,1] of cardinality at most 1, and the ones of cardinality 1 are the maximal emulators. If |E| ≥ 2 then the emulators are all of the subsets of Q[0,1], and Q[0,1] is the unique maximal emulator.
EX2. $E = \{(0,0)\} \subseteq Q[0,1]^2$. The emulators are ∅ and singletons {(p,p)}, 0 ≤ p ≤ 1. The maximal emulators are these singletons.
EX3. $E = \{(0,1)\} \subseteq Q[0,1]^2$. The emulators are ∅ and singletons {(p,q)}, 0 ≤ p < q ≤ 1. The maximal emulators are these singletons.
EX4. $E = \{(0,0),(1,1)\} \subseteq Q[0,1]^2$. The emulators are the subsets of {(p,p): 0 ≤ p ≤ 1}. Exactly one is maximal, {(p,p): 0 ≤ p ≤ 1}.
EX5. $E = \{(0,0),(0,1)\} \subseteq Q[0,1]^2$. The emulators are the sets that are contained in some {p} × Q[p,1], 0 ≤ p < 1. The maximal emulators are the sets {p} × Q[p,1], 0 ≤ p ≤ 1.
EX6. E = {(0,2/5),(1/5,3/5),(2/5,4/5),(3/5,1)}. The emulators are the graphs of strictly increasing partial f:Q[0,1) → Q(0,1], where each defined f(x) > x. There are continuumly many maximal emulators of E.
EX7. $E = \{(p,q) \in Q[0,1]^2: p < 1/2 < q\} \subseteq Q[0,1]^2$. The emulators and maximal emulators are calculated in a self contained way in the proof of Lemma 3.2.2.
EX8. $E = \{(1/6,1/4),(1/7,1/3),(0,1/5),(1/2,1)\} \subseteq Q[0,1]^2$. The emulators and maximal emulators are calculated in a self contained way in the proof of MELU/2 in section 3.3.

DEFINITION 3.1.4. Since we are using Q[0,1] throughout section 3, we make the following convention. If the dimension k of E or S has been given, then we can write E or S instead of $E \subseteq Q[0,1]^k$ or $S \subseteq Q[0,1]^k$. If the dimension k of R has been given, then we can write R instead of $R \subseteq Q[0,1]^k \times Q[0,1]^k$. These conventions also apply where E,S,R appear with superscripts or subscripts. Exceptions to this convention occur inside proofs where we use linear orderings other than Q[0,1].

Here is some background information on the crucial order equivalence relation on the $Q[0,1]^k$.

THEOREM 3.1.1. (RCA_0) There are finitely many equivalence classes under order equivalence on $Q[0,1]^k$. The number, ot(k) is the same as the number of preferential arrangements, up to isomorphism, in the sense of [Gr62]. ot(1) = 1, ot(2) = 3, ot(3) = 13, ot(4) = 75, ot(5) = 541, ot(6) = 4,683, ot(7) = 47,293, ot(8) = 545,835, ot(9) = 7,087,261, ot(10) = 102,247,563, ot(11) = 1,622,632,573, ot(12) = 28,091,567,595, ot(13) = 526,858,348,381, ot(14) = 10,641,342,970,443. (Q[0,1],<) can be replaced by any dense linear ordering.

Proof: Here ot is read "order type". Note that $x,y \in Q[0,1]^k$ are order equivalent if and only if $\{(i,j): x_i < x_j\} = \{(i,j): y_i < y_j\}$, and the number of sets of this form is obviously finite, trivially bounded by 2^{k^2}. A preferential arrangement is commonly defined to be a connected and transitive relation R on a set V. By connectivity, R must be reflexive on V. The derived relation $x \sim y \leftrightarrow x\ R\ y \wedge y\ R\ x$ is an equivalence relation on V. Given $[x] \neq [y]$, we have $x\ R\ y \vee y\ R\ x$. Hence we have trichotomy for all [x],[y], namely exactly one of x R y, y R x, x = y holds. Thus R can be viewed as a reflexive linear ordering on the equivalence classes of ~.

With an underlying domain of k elements, the isomorphism types of these R's are in one-one correspondence with k-tuples from Q[0,1] (or any given dense linear ordering) under order equivalence. To see this, let R be connected and transitive on {1,...,k}. By the previous paragraph, list the equivalence classes under ~ by $A_1,\ldots,A_n$, in increasing R order. The corresponding k-tuple is $t_1,\ldots,t_k$, where each t_i is the index of [i] in the list $A_1,\ldots,A_n$.

The displayed 14 values of ot are from [Gr62]. QED

For additional work on ot(k), see [Slxx].

THEOREM 3.1.2. (RCA_0) Every $E \subseteq Q[0,1]^k$ is an emulator of a finite subset. E has a recursive maximal emulator.

Proof: See Finite Subset Emulation and Theorem 2.2 and the first claim of ME/3, all in section 2. QED

It is clear that we obtain a maximal emulator in Theorem 3.1.2 of low computational complexity. It would be interesting to carefully investigate these from a computational complexity perspective.

DEFINITION 3.1.5. $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable if and only if for finite subsets of $Q[0,1]^k$, some maximal emulator contains its R image. $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME invariantly usable if and only if for finite subsets of $Q[0,1]^k$, some maximal emulator is R invariant.

Note that Definition 3.1.5 is the special case of MEU/DEF and MEIU/DEF of section 2, for M = (Q[0,1],<). Here we use finite subsets as is justified by Theorem 2.5.

Throughout the rest of this paper, ME usable and ME invariantly usable will refer only to our Q[0,1] context (i.e., M = (Q[0,1],<)). Thus for $X \subseteq Q[0,1]^k \times Q[0,1]^k$, we write "$R \subseteq X$ is ME (invariantly) usable" to mean "$R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME (invariantly) usable and $R \subseteq X$". Thus this use of X in no way invalidates the intention that we are always working in the space Q[0,1] = (Q[0,1],<).

We will show that for the important class of order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$, "R is ME usable" is implicitly Π^0_1 over WKL_0 and WKL_0 falsifiable as defined below. But first we discuss the very robust notion of order theoretic, which is a special case of the notion of elementary subset that we introduced right after the discussion of ME/4 in section 1.1.

ORDER THEORETIC DEFINITION. $S \subseteq Q[0,1]^k$ is order theoretic if and only if S is elementary in (Q[0,1],<). I.e., S is of the form $\{x \in Q[0,1]^k: \varphi\}$, where φ is a finite propositional combination of formulas $x_i < x_j$, $x_i < p$, $p < x_i$, with $1 \leq i,j \leq k$ and $p \in Q[0,1]$.

THEOREM 3.1.3. $S \subseteq Q[0,1]^k$ is order theoretic if and only if S is first order definable in (Q[0,1],<) with parameters.

Proof: By the usual quantifier elimination for dense linear orderings. QED

DEFINITION 3.1.6. Let T be a first order theory whose language includes 0,S,+,x,< (on a sort for the nonnegative integers), which is recursively axiomatized and proves PFA. φ is implicitly Π^0_1 over T if and only if there is a Π^0_1 sentence ψ such that T proves φ ↔ ψ. φ is T falsifiable if and only if T proves "¬φ → T proves ¬φ".

Note that the two outermost T proves, involves only the ordinary mention of prove. The single innermost T proves ¬φ refers to a formalization of proofs within T.

THEOREM 3.1.4. (RCA_0) Let T be a recursively axiomatized theory that proves PFA. Every sentence implicitly Π^0_1 over T is T falsifiable. Furthermore assume T is finitely axiomatized. Every T falsifiable sentence is implicitly Π^0_1 over T augmented with induction for all formulas. In fact, induction only for formulas of quantifier complexity at most the maximum of T,φ is needed.

Proof: Let T be as given. Let φ be implicitly Π^0_1 over T. Let ψ be Π^0_1 where T proves φ ↔ ψ. Arguing in T, assume ¬φ. Then ¬ψ, and so T proves ¬ψ. Now T sees that T proves φ ↔ ψ. Therefore T sees that T proves ¬φ.

Now suppose φ is T falsifiable. We claim φ ↔ Con(T+φ) is provable in T with induction. To see this, argue in T with induction. We see if φ is false then φ is refutable in T. Hence ¬φ → ¬Con(T+φ). Now suppose ¬Con(T+φ). By cut elimination, we get a refutation of φ in T with a proof of quantifier complexity at most that of the maximum of the quantifier complexities of T,φ. We then perform an induction in T to derive ¬φ. QED

According to Theorem 3.1.4, implicitly Π^0_1 and falsifiable are essentially the same notions.

The featured statements in section 3 are implicitly Π^0_1 over

WKL_0 via Gödel's Completeness Theorem, and therefore WKL_0 falsifiable. This can be easily seen through the following general result.

LEMMA 3.1.5. (EFA) Fix $k \geq 1$, finite $E \subseteq Q[0,1]^k$, and (an order theoretic presentation of) order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$. The statement φ = "some maximal emulator S of E has $R[S] \subseteq S$" is implicitly Π^0_1 over WKL_0. Furthermore, the associated Π^0_1 forms and the equivalence proofs in WKL_0 can be constructed effectively from k,E and the order theoretic presentation of R, in a way that RCA_0 can verify.

Proof: Fix k,E,R as given. The associated Π^0_1 sentence will be Con(T[k,E,R]), where T(k,E,R) is a finitely axiomatized system associated with k,E and the given order theoretic presentation of R.

The parameters used to present R can be taken to be exactly the t+2 rationals $0 < p_1 < \ldots < p_t < 1$. For any finite $E \subseteq Q[0,1]^k$, we form the theory T[k,E,R] in first order predicate calculus with equality, the binary relation symbol <, constants $0,1,p_1,\ldots,p_t$, and k-ary relation symbol S. The finitely many axioms of T[k,E,R] are

i. < is a strict dense linear ordering with left endpoint 0 and right endpoint 1.
ii. $0 < p_1 < \ldots < p_t < 1$.
iii. If $x \in S$ and $R(x,y)$, then $y \in S$.
iv. Any k-tuple x lies in S if and only if every element of $(S \cup \{x\})^2$ is order equivalent to an element of E^2.

For iii, we use the definition of R in $<,=,0,1,p_1,\ldots,p_t$. For iv, we don't use actual elements of E, but simply use the enumerated order types of elements of E^2.

Arguing in RCA_0, suppose there is a model of T[k,E,R] with domain a subset of N. There is no problem formulating "there is a model of T[k,E,R]" as T has only finitely many axioms. Then there is a model of T[k,E,R] of the form (Q[0,1],<,...), by isomorphism. Also by isomorphism, we can arrange that the constants $p_1,\ldots,p_t$ are actually $p_1,\ldots,p_t$. So we have the model of T[k,E,R], $(Q[0,1],<,0,1,p_1,\ldots,p_t,S)$. It is now clear that by iv, S is an emulator of $E \subseteq Q[0,1]^k$. By iv, S is a maximal emulator of $E \subseteq Q[0,1]^k$. By iii we see that $R[S] \subseteq S$, where this inclusion is formulated without actually forming R[S] as a set. Hence φ holds.

Arguing in RCA_0, suppose φ holds, and let S be a maximal emulator of E ⊆ $Q[0,1]^k$, where R[S] ⊆ S. Then clearly $(Q[0,1],<,0,1,p_1,...,p_t,S)$ is a model of T[k,E,R], which we can make official via an isomorphism onto N. We want to derive Con(T[k,E,R]). If we do this in the most obvious way, we are going to be using induction with respect to a formula that involves the satisfaction predicate for the model of T, and that isn't even arithmetic. However, we can apply cut elimination for predicate calculus, which is available in RCA_0. Noting that the axioms of T are (universally quantified) Σ^0_1 formulas, we will only need the satisfaction relation for Σ^0_1 formulas and hence Σ^0_1 induction, which is available in RCA_0.

Now simply cite the formalized completeness theorem in WKL_0, which tells us that WKL_0 proves "if Con(T[k,E,R]) then T = T[k,E,R] has a model with domain a subset of ω". Thus we have a proof in WKL_0 of φ ↔ Con(T[k,E,R]). Thus we are using Con(T[k,E,R]) as our Π^0_1 sentence. The construction of Con(T[k,E,R]) and the equivalence proof is obviously effective from k,E and (the given order theoretic presentation of) R. QED

THEOREM 3.1.6. (EFA) Consider the statement φ(k,E,R) = "For finite E ⊆ $Q[0,1]^k$, some maximal emulator S of E ⊆ $Q[0,1]^k$ has R[S] ⊆ S".
i. If k,E,R are fixed in advance, where R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then φ(k,E,R) is implicitly Π^0_1 over WKL_0.
ii. If k,R are fixed in advance, where R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then (∀E ⊆ $Q[0,1]^k$)(φ(k,E,R)) is implicitly Π^0_1 over WKL_0.
iii. In the equivalence proofs for implicitly Π^0_1 in i,ii, the forward direction can be taken to be in RCA_0 and the backward direction can be taken to be in WKL_0 (where the tree axiom has no set parameters).
iv. In the falsifiability proofs, the first T can be taken to be WKL_0 (where the tree axiom has no set parameters) and the second T can be taken to be RCA_0.
Furthermore, the associated Π^0_1 forms and equivalence proofs can be constructed effectively from the fixed parameters,

in such in a way that EFA can verify.

Proof: Apply Lemma 3.1.5 and its proof. This establishes i-iii. For iv, using WKL_0 for the first T, argue in WKL_0. Suppose φ(k,E,R) is false. Then Con(T[k,E,R]) is false. So Con(T[k,E,R]) is refutable in RCA_0 (even in EFA). Hence φ(k,E,R) is refutable in RCA_0, since RCA_0 proves φ(k,E,R) → Con(T[k,E,R]). QED

COROLLARY 3.1.7. (EFA) Consider the statement φ(k,R) = "R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is ME usable".
i. If k,R are fixed in advance, where R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then φ(k,R) is implicitly Π^0_1 over WKL_0.
iii,iv and the last sentence of Theorem 3.1.6 also apply here.

Proof: The statement in quotes is the same as "For all finite E ⊆ $Q[0,1]^k$, some maximal emulator S of E ⊆ $Q[0,1]^k$ has R[S] ⊆ S" and so we can apply Theorem 3.1.6. QED

3.2. FINITE RELATIONS

We now address the ME usability of finite R ⊆ $Q[0,1]^k \times Q[0,1]^k$. The trivial case k = 1 is dispensed with later by MELU/1 in section 3.3, even for arbitrary R.

ORDER PRESERVING DEFINITION. R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is order preserving if and only if (∀x,y)(R(x,y) → x,y are order equivalent).

MAXIMAL EMULATION NECESSARY USE. MENU. If R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is ME usable then R is order preserving.

Proof of MENU: In RCA_0. By ME/4 in section 2. QED

In light of MENU, we need be concerned only with order preserving R ⊆ $Q[0,1]^k \times Q[0,1]^k$ throughout the rest of section 3.

Are we already done in the sense that every finite order preserving R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is ME usable? We shall see that this is close to being true, but the endpoints 0,1 create issues. See MEFU/3 below.

Let's first look at two very simple examples of ME usability. These examples are so simple that we conveniently state them directly without using the ME usability terminology.

MAXIMAL EMULATION EXAMPLE/1. MEX/1. For finite subsets of $Q[0,1]^2$, some maximal emulator is equivalent at (1/2,1/3),(1/3,1/4).

MAXIMAL EMULATION EXAMPLE/2. MEX/2. For finite subsets of $Q[0,1]^2$, some maximal emulator is equivalent at (1,1/2),(1/2,1/3).

By Theorem 2.5, MEX/1,2 assert, respectively, that the two element relations {((1/2,1/3),(1/3,1/4)),((1/3,1/4),(1/4,1/3))} and {((1,1/2),(1/2,1/3)),((1/2,1/3),(1,1/2))} are ME usable, and equivalently, that the one element relations {((1/2,1/3),(1/3,1/4))} and {((1/3,1/2),(1/2,1))} are ME invariantly usable. The first pair of relations are of cardinality 2, and the second pair of relations are of cardinality 1.

Proof of MEX/1: In RCA_0. Let $E \subseteq Q[0,1]^2$ be finite, and let S be a maximal emulator of E. We now find $1 > p > q > r > 0$ such that $(p,q) \in S \leftrightarrow (q,r) \in S$.

case 1. $(1/2,1/3) \in S$. We can assume $(1/3,1/4)(1/3,1/5) \notin S$. We can assume $(1/4,1/5),(1/5,1/6) \in S$. Then we are done.

case 2. $(1/2,1/3) \notin S$. We can assume $(1/3,1/4),(1/3,1/5) \in S$. We can assume $(1/4,1/5),(1/5,1/6) \notin S$. Then we are done.

Now we map S onto S' via any increasing bijection of Q[0,1] onto Q[0,1] mapping p,q,r to 1/2,1/3,1/4, This preserves being a maximal emulator of E. QED

Note that this argument will not work for MEX/2 because of the endpoint 1. Before addressing MEX/2, we first give the appropriate general form of MEX/1.

MAXIMAL EMULATION FINITE USE/1. MEFU/1. Any finite order preserving $R \subseteq Q(0,1)^k \times Q(0,1)^k$ is ME usable.

Proof of MEFU/1: In RCA_0. Let R be as given and let $E \subseteq Q[0,1]^k$ be finite. Let $p_1 < ... < p_n \in Q(0,1)$ be the rationals appearing in R. Let S be a maximal emulator of E.

We now apply the usual finite Ramsey theorem from [Ra30] as follows. Let $V \subseteq Q(0,1)$ be finite and sufficiently large. Let $V' \subseteq V$, $|V'| = n$, where membership in S of k-tuples from V' depend only on their order type. Let $f:Q[0,1] \to Q[0,1]$ be a strictly increasing bijection mapping V' onto $\{p_1,...,p_n\}$. By Theorem 2.6, f[S] is a maximal emulator of E. Note that membership in f[S] of k-tuples from $\{p_1,...,p_n\}$ depends only on their order type. We now claim that f[S] contains its R image. To see this, let $R(x,y)$, $x \in S$. Then $x,y \in \{p_1,...,p_n\}^k$, and x,y are order equivalent. Hence $y \in S$. This argument will also show that R is ME invariantly usable, although we can also get this by using the relation $R'(x,y) \leftrightarrow R(x,y) \vee R(y,x)$, which must also be finite and order preserving. QED

We now give a proof of MEX/2 before taking up much more general results. Additional ideas are required, and the proof is not given in RCA_0.

Proof of MEX/2: In ACA_0. Let $E \subseteq Q[0,1]^2$ be finite. Construct sets $S_1 \subseteq ... \subseteq S_5$ in five steps, where each $S_i \subseteq Q[0,i]^2$ is a maximal emulator of $E \subseteq Q[0,i]^2$. We claim that each S_i,S_{i+1} agree on $Q[0,i]^2$. Let $x \in Q[0,i]^2$. If $x \notin S_i$ then $S_i \cup \{x\}$ is not an emulator of $E \subseteq Q[0,i]^2$, and hence $S_{i+1} \cup \{x\}$ is not an emulator of $E \subseteq Q[0,i+1]^2$, and therefore $x \notin S_{i+1}$. (Here the same E is viewed as a subset of the various $Q[0,i]^2$).

By the argument by cases in the proof of MEX/1, let $a > b > c$ be from $\{1,2,3,4,5\}$, where $(a,b) \in S_5 \leftrightarrow (b,c) \in S_5$. By the above claim, $(a,b) \in S_a \leftrightarrow (b,c) \in S_a$. Now map $S_a \cap Q[0,a]^2$ onto $S' \cap Q[0,1]^2$ by an increasing bijection mapping a,b,c to 1,1/2,1/3. Then S' is a maximal emulator of $E \subseteq Q[0,1]^2$ and $(1,1/2) \in S \leftrightarrow (1/2,1/3) \in S$. QED

Here is our first generalization of MEX/2.

DEFINITION 3.2.1. DEFINITION 3.2.1. Let $(x,y) \in Q[0,1]^k \times Q[0,1]^k$ and $R \subseteq Q[0,1]^k \times Q[0,1]^k$. p is present in (x,y) if and only if p is a coordinate of x or y. p is altered in (x,y) if and only if there exists i such that $p = x_i \neq y_i$ or $p = y_i \neq x_i$. p is present in R if and only if there exists i such that $p = x_i$ or $p = y_i$. p is altered by R if and only if p is altered in some element of R. We also write "p appears in (x,y)”, “(x,y) alters p”, “p appears in R”, and “R alters p”.

MAXIMAL EMULATION FINITE USE/2. MEFU/2. Any finite order preserving $R \subseteq Q(0,1]^k \times Q(0,1]^k$ is ME usable.

Here is a second, stronger, generalization of MEX/2, which we prove.

MAXIMAL EMULATION FINITE USE/3. MEFU/3. Any finite order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ not altering both of 0,1 is ME usable.

Obviously MEFU/3 → MEFU/2 → MEFU/1 in RCA_0.

LEMMA 3.2.1. (RCA_0) Suppose MEFU/3 holds with "not alter 0". Then MEFU/3 holds with "not alter 1".

Proof: Assume MEFU/3 holds with "not alter 0". Let finite order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ not alter 1, and finite $E \subseteq Q[0,1]^k$ be given. We use the bijection $f:Q[0,1] \to Q[0,1]$ given by $f(p) = 1-p$. We claim that f is an automorphism of the order equivalence relation $EQR(Q[0,1],k)$ on $Q[0,1]^k$ (as a subset of $Q[0,1]^{2k}$). To see this, we need to verify that $(p_1,...,p_k),(q_1,...,q_k)$ are order equivalent if and only if $(1-p_1,...,1-p_k),(1-q_1,...,1-q_k)$ are order equivalent. Suppose $(p_1,...,p_k),(q_1,...,q_k)$ are order equivalent. Let $1 \leq i,j \leq k$. We want $(1-p_i < 1-p_j \leftrightarrow 1-q_i < 1-q_j)$. I.e., $(p_j < p_i \leftrightarrow q_j < q_i)$, which follows from $(p_1,...,p_k),(q_1,...,q_k)$ being order equivalent. The converse is proved analogously.

By Theorem 2.6, we see that R is ME usable if and only if 1-R is ME usable. We claim that 1-R is order preserving and does not alter 0. Suppose 1-R(x,y). Then R(1-x,1-y), and so 1-x,1-y are order equivalent and x,y are order equivalent. Suppose 1-R alters 0, and let 1-R(x,y), where 0 lies in the two element set $\{x_i,y_i\}$. Then 1 lies in the two element set $\{1-x_i,1-y_i\}$ and $R(1-x_i,1-y_i)$. This contradicts that R does not alter 1.

Now since MEFU/3 holds with "not alter 0", we see that 1-R is ME usable. Hence R is ME usable. QED

Proof of MEFU/3. In ACA'. By Lemma 3.2.1, it suffices to prove MEFU/3 with "does not alter 0". Let finite $R \subseteq Q[0,1]^k$

× $Q[0,1]^k$ not alter 0, and finite $E \subseteq Q[0,1]^k$ be given. Let the rationals appearing in R be among $0 < p_1 < \ldots < p_n < 1$. Construct $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_m$ in m steps where each S_i is a maximal emulator of $E \subseteq Q[0,i]^k$. Here m is so large relative to k,n that we can use it here with the usual finite Ramsey theorem, [Ra30]. As in the proof of MEX/2, each S_i,S_{i+1} agree on $Q[0,i]^k$. By the finite Ramsey theorem, let V be an n+1 element subset of {1,...,m} such that membership of k-tuples from V ∪ {0} in S_m depends only on their order type with constant 0. Hence membership of k-tuples from V ∪ {0} in $S_{max(V)}$ depends only on their order type with constant 0.

Now let $h:V \cup \{0\} \to \{0,p_1,\ldots,p_n,1\}$ be the unique increasing bijection. Extend h to an increasing bijection $h':Q[0,\max(V)] \to Q[0,1]$. It is clear that membership of k-tuples from $\{0,p_1,\ldots,p_n,1\}$ in $h[S_{max(V)}]$ depends only on their order type with constant 0. Also, by Theorem 2.6, $h'[S_{max(V)}]$ is a maximal emulation of $h'[E] \subseteq Q[0,1]$.

We claim that $S_{max(V)}$ contains its $h'^{-1}[R]$ image. To see this, let $h'^{-1}[R](x,y)$, $x \in S_{max(V)}$. Then $R(h'(x),h'(y))$, and so $h'(x),h'(y)$ are order equivalent with constant 0. Hence x,y are order equivalent with constant 0. Therefore $x \in S_{max(V)} \leftrightarrow y \in S_{max(V)}$, and since $x \in S_{max(V)}$, we have $y \in S_{max(V)}$.

It now follows that $h'[S_{max(V)}]$ contains its R image. So $h'[S_{max(V)}]$ is a maximal emulation of $h'[E] \subseteq Q[0,1]^k$ containing its R image. Now h'[E] is an emulation of $E \subseteq Q[0,1]^k$ and E is an emulation of $h'[E] \subseteq Q[0,1]^k$. So by Emulation Transitivity in section 2, $h'[S_{max(V)}]$ is also a maximal emulation of $E \subseteq Q[0,1]^k$. QED

We conjecture that MEFU/2,3 are not provable in ACA_0. Note that we have proved MEFU/1 in RCA_0.

If finite R alters both 0,1, then open issues arise. However, we have the following.

MAXIMAL EMULATION SINGLETON USE/1. MEOU/1. Any order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ of cardinality 1 is ME usable.

Proof of MEOU/1: In RCA_0. Let R = {(x,y)} be order preserving. I.e., x,y are order equivalent. Let $E \subseteq Q[0,1]^k$ be finite.

case 1. E has an element order equivalent to y. Then {y} is an emulator of $E \subseteq Q[0,1]^k$, and so extends to a maximal emulator S of $E \subseteq Q[0,1]^k$. Since $y \in S$, S contains its R image, which is {y} if $x \in S$ and $\emptyset$ if $x \notin S$.

case 2. E has no element order equivalent to y. Let S be a maximal emulator of $E \subseteq Q[0,1]^k$. Then $x,y \notin S$ since x,y are order equivalent. Hence S contains its R image, which is $\emptyset$.

QED

MEOU/1 is false for cardinality 1 even in dimension k = 2 and R of cardinality 2. In fact, we have the following.

LEMMA 3.2.2. (RCA_0) {((0,1/2),(1/2,1))} is not ME invariantly usable.

Proof: Let $E = \{(p,q) \in Q[0,1]^2: p < 1/2 < q\}$. We claim that S is an emulator of $E \subseteq Q[0,1]^2$ if and only if $S \subseteq Q[0,1]^{2<}$ and the first coordinate of every element of S is less than the second coordinate of every element of S. To see this, suppose S is an emulator of $E \subseteq Q[0,1]^2$. If $x \in S$ then (x,x) is order equivalent to some (y,y) in E^2, and so $x \in Q[0,1]^{2<}$. Hence $S \subseteq Q[0,1]^{2<}$. If $x,y \in S$ then (x,y) is order equivalent to some (z,w) in E^2, and so $x_1 < y_2$. Conversely, suppose $S \subseteq Q[0,1]^{2<}$, where the first coordinate of every element of S is less than the second coordinate of every element of S. Let $x,y \in S$. Note that $x_1 < y_1 = x_2 < y_2$.

Let $0 \leq p_1,q_1 < 1/2 < p_2,q_2 \leq 1$, where $(x_1,y_1),(p_1,q_1)$ are order equivalent, and $(x_2,y_2),(p_2,q_2)$ are order equivalent. Then (x,y) and $((p_1,q_1),(p_2,q_2))$ are order equivalent.

Let S be a maximal emulator of E. Obviously S is nonempty. Let α be the sup of the first terms of the pairs in S and β be the inf of the second terms of the pairs in S. Clearly $\alpha \leq \beta$.

If $\alpha < \beta$ then $S \cup \{(p,q)\}$ is an emulator of E where $\alpha < p < q < \beta$, violating the maximality of S. Hence $\alpha = \beta$.

case 1. α is irrational. Then all first terms are < and all second terms are > α. Hence $S = \{(p,q): p < \alpha < q\}$.

case 2. $\alpha = \beta$ is rational. If $\alpha = 0$ then $S = \{(0,q): 0 < q \leq 1\}$, and so $(0,1/2) \in S \wedge (1/2,1) \notin S$, violating R invariance. If $\alpha = 1$ then $S = \{(p,1): 0 \leq p < 1\}$, and so $(0,1/2) \notin S \wedge (1/2,1) \in S$, violating R invariance. Now suppose $0 < \alpha < 1$. If $0 \leq p < \alpha < q \leq 1$ then $S \cup \{(p,q)\}$ is an emulator of E, and so $(p,q) \in S$. What is not clear is which (α,q) and (p,α) lie in S. If none of these lie in S then $S \cup \{(\alpha,p)\}$ is an emulator of E. Hence at least one of these lie in S. Suppose some (α,q) lies in S. Then no (p,α) lies in S. Hence all (α,q), $\alpha < q \leq 1$, lie in S. Alternatively, suppose some (p,α) lies in S. Then no (α,q) lies in S, and hence all (p,α), $0 \leq p < \alpha$, lie in S.

This determines exactly what the maximal emulators of E are. They are the sets S =
i. $\{(p,q) \in Q[0,1]^2: 0 \leq p \leq \alpha < q \leq 1\}$, where α is a real number in $[0,1)$.
ii. $\{(p,q) \in Q[0,1]^2: 0 \leq p < \alpha \leq q \leq 1\}$, where α is a real number in $(0,1]$.

case i. $(0,1/2) \in S$ if and only if $\alpha < 1/2$, and $(1/2,1) \in S$ if and only if $1/2 \leq \alpha < 1$ if and only if $1/2 \leq \alpha$. Hence S is not equivalent at $(0,1/2),(1/2,1)$.

case ii. $(0,1/2) \in S$ if and only if $0 < \alpha \leq 1/2$ if and only if $\alpha \leq 1/2$, and $(1/2,1) \in S$ if and only if $1/2 < \alpha$. Hence S is not equivalent at $(0,1/2),(1/2,1)$.

QED

We now present a method for establishing that $\{(x,y)\}$ is ME invariantly usable.

LEMMA 3.2.3. (RCA_0) Suppose $x,y \in Q[0,1]^k$ are order equivalent and there exists $z \in Q[0,1]^k$ such that $(x,y),(x,z),(z,y)$ are order equivalent (as 2k-tuples). Then $\{(x,y)\} \in Q[0,1]^2 \times Q[0,1]^2$ is ME invariantly usable.

Proof: Let x,y,z be as given. Let $E \subseteq Q[0,1]^2$ be finite. We find a maximal emulator of E that is equivalent at x,y.

case 1. There exist $x',y' \in E$ such that $(x',y'),(x,y)$ are order equivalent. Then $\{x,y\}$ is an emulator of E which extends to a maximal emulator of E, which is obviously equivalent at x,y.

case 2. Otherwise. If E does not have an element order equivalent to x, then any maximal emulator of E is equivalent at x,y. Suppose otherwise. Clearly x,y,z are order equivalent and so {z} is an emulator of E. Let S be a maximal emulator of E with $z \in S$. Since (x,z),(z,y) are not order equivalent to any element of E^2, clearly $x,y \notin S$. Hence S is equivalent at x,y.

QED

We now have the following complete determination of the ME invariant usability of singletons in dimension k = 2.

MAXIMAL EMULATION SINGLETON USE/2. MEOU/2. Let $x,y \in Q[0,1]^2$ be order equivalent. The following are equivalent.
i. {(x,y)} is ME invariantly usable.
ii. {(x,y),(y,x)} is ME usable.
iii. For finite subsets of $Q[0,1]^2$, some maximal emulator is equivalent at x,y.
iv. {x,y} is not any {(0,p),(p,1)} and not any {(1,p),(p,0)}, 0 < p < 1.

Proof: In RCA_0. The equivalence of i,ii,iii is immediate (or use Theorems 2.3,2.4). Assume i. By Lemma 3.2.2, {((0,1/2),(1/2,1))} is not ME invariantly usable. By Theorem 2.6, ME invariant usability (and ME usability) remains unchanged under increasing automorphisms of Q[0,1], and so {((0,p),(p,1))}, 0 < p < 1, is not ME invariantly usable. Also note that coordinate reversal is an automorphism of EQR(Q[0,1]). Hence by Theorem 2.6, ME invariant usability (and ME usability) remains unchanged under coordinate reversal, and so {((1,p),(p,0))}, 0 < p < 1, is not ME invariantly usable. This establishes iv.

Assume iv. If not both 0,1 are altered in {(x,y)} then by MEFU/3, {(x,y)} is ME invariantly usable. So we assume that both 0,1 are altered in (x,y). We derive i. We now split according to the fact that 0 is among x_1,x_2,y_1,y_2.

case 1. (x,y) = ((0,a),(b,c)). Suppose b = 0. Since 0,1 are both altered, {a,c} = {0,1}. This, however, violates that x,y are order equivalent. If b = 1 then (x,y) = ((0,a),(1,c)), and by order equivalence, (x,y) = ((0,0),(1,1)). Since

((0,0),(1,1)),((0,0),(1/2,1/2)),((1/2,1/2))(1,1)) are order equivalent, {(x,y)} is ME invariantly usable by Lemma 3.2.3. So we can assume 0 < b < 1. 1 is altered in (x,y), we have a = 1 ∨ c = 1.

case 1.1. a = 1. Then (x,y) = ((0,1),(b,c)), where 0 < b < c ≤ 1 by order equivalence. Since 1 is altered, we have 0 < b < c < 1. Since ((0,1),(b,c)),((0,1),(b′,c′)),((b,c),(b′,c′)) are order equivalent, assuming b < b′ < c′ < c, we see that {(x,y)} is ME invariantly usable by Lemma 3.2.3.

case 1.2. c = 1. Then (x,y) = ((0,a),(b,1)), where 0 < a,b < 1 by order equivalence. By iv, a ≠ b. If a < b then ((0,a),(b,1)),((0,a),(c,d)),((c,d),(b,1)) are order equivalent for a < c < d < b. If a > b then ((0,a),(b,1)),((0,a),(c,d)),((c,d),b,1)) are also order equivalent for 0 < c < b < a < d < 1. Hence {(x,y)} is ME invariantly usable by Lemma 3.2.3.

case 2. (x,y) = ((a,0),(b,c)). Let (x′,y′) be the result of flipping the two coordinates in both x and y. Then both 0,1 are altered in (x′,y′), and (x′,y′) = ((0,a),(c,b)). Apply case 1 to obtain that {(x′,y′)} is ME invariantly usable. As explained earlier, flipping the two coordinates in both x′,y′ preserves ME invariant usability (using Theorem 2.6), so we see that {(x,y)} is ME invariantly usable.

case 3. (x,y) = ((b,c),(0,a)). Obviously {(x,y)} is ME invariantly usable if and only if {(y,x)} is ME invariantly usable. Apply case 1 to (y,x).

case 4. (x,y) = ((b,c),(a,0)). Obviously {(x,y)} is ME invariantly usable if and only if {(y,x)} is ME invariantly usable. Apply case 2 to (y,x).

QED

We conjecture that a complete determination of the ME invariant usability of singletons in higher dimensions can be accomplished in RCA_0 (in our Q[0,1] context). We further conjecture that a complete determination of the ME usability (and hence the ME invariant usability) of the finite $R \subseteq Q[0,1]^k \times Q[0,1]^k$ can be carried out in ACA'.

3.3. LARGE RELATIONS

We begin with the easy complete determination of the ME usable $R \subseteq Q[0,1] \times Q[0,1]$. Note that every $R \subseteq Q[0,1] \times Q[0,1]$ is trivially order preserving.

LEMMA 3.3.1. (RCA_0) The maximal emulators of $\emptyset \subseteq Q[0,1]$ is just $\emptyset$. The maximal emulators of any $\{p\} \subseteq Q[0,1]$ are all of the singletons. The maximal emulator of $E \subseteq Q[0,1]$, $|E| \geq 2$, is just $Q[0,1]$.

Proof: Left to the reader. QED

MAXIMAL EMULATION LARGE USE/1. MELU/1. $R \subseteq Q[0,1] \times Q[0,1]$ is ME usable if and only if there is some $p \in Q[0,1]$ not altered by R.

Proof: In RCA_0. Suppose R does not alter $p \in Q[0,1]$. Let $E \subseteq Q[0,1]$ be finite. By Lemma 3.3.1, there is a maximal emulator of $E \subseteq Q[0,1]$ among the three sets $\emptyset,\{p\},Q[0,1]$, each of which contain their R image. Hence R is ME usable.

Suppose R is ME usable. Let S be an emulator of $\{0\} \subseteq Q[0,1]$ which contains its R image. By Lemma 3.3.1, let $S = \{p\}$. So any q with $R(p,q)$ must be p. I.e., p is not altered by R. QED

In particular, MELU/1 tells us that the "large" relations $Q[0,1) \times Q[0,1)$ and $Q(0,1] \times Q(0,1]$ are ME usable. That's why we use "large" in the header of MELU/1.

More generally, we consider order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ to be "large" if infinitely many numbers are altered, and "small" if finitely many numbers are altered. We are less clear about what we want to mean by "large" and "small" for general R, but the development in this paper is driven by the order theoretic R.

Here we consider only the particularly simple "large" relations of the form $V \times V \subseteq Q[0,1]^k \times Q[0,1]^k$.

THEOREM 3.3.2. (RCA_0) Let $V \subseteq Q[0,1]^k$. The following are equivalent.
i. $V \times V \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME invariantly usable.
ii. $V \times V \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable.
iii. For subsets of $Q[0,1]^k$, some maximal emulator contains or is disjoint from V.

If any of these hold then all x,y ∈ V are order equivalent.

Proof: i → ii is immediate. Assume ii and let $E \subseteq Q[0,1]^k$ be finite. Let S be a maximal emulator of $E \subseteq Q[0,1]^k$ that contains its V × V image. If S meets V then E contains V. Hence S contains or is disjoint from V. Now suppose iii and let $E \subseteq Q[0,1]^k$ be finite. Let S be a maximal set that either contains or is disjoint from V. In the first case, S is V × V invariant. In the second case, S is V × V invariant, vacuously.

For the last claim, if any of i-iii hold then V × V is order preserving (MENU of section 3.1), and hence any x,y ∈ V are order equivalent. QED

MELU/1 fails badly in the highly nontrivial environment of dimension k = 2.

MAXIMAL EMULATION LARGE USE/2. MELU/2. $Q(0,1)^{2<} \times Q(0,1)^{2<}$ is not ME usable. It is order preserving, order theoretic, and 0,1 are not present.

Proof: In RCA_0. Let $R = Q(0,1)^{2<} \times Q(0,1)^{2<}$. Let E = {(1/6,1/4),(1/7,1/3),(0,1/5),(1/2,1)}. The idea behind E is that no coordinate of an element of E is a coordinate of any other element of E, and the pairs of elements are in general position relative to that restriction. Then S is an emulator of $E \subseteq Q[0,1]^2$ if and only if $S \subseteq Q[0,1]^{2<}$ and no coordinate of an element of S is a coordinate of any other element of S.

Let S be a maximal emulator of $E \subseteq Q[0,1]^2$. Suppose p < q are both not present in S. Then S ∪ {(p,q)} is an emulator of $E \subseteq Q[0,1]^2$, contradicting maximality. Hence all numbers in Q[0,1] are present in S except for possibly one number. In particular, S has at least three elements, and so S has an element (p,q), where 0 < p < q < 1. If S contains its R image, then (p,q') ∈ S for any p < q < q' < 1, contradicting that E has no has no repeated numbers. Hence S does not contain its R image. QED

MAXIMAL EMULATION LARGE USE/3. MELU/3. $Q[1/2,1)^{2<} \times Q[1/2,1)^{2<}$ is ME usable.

Proof: In RCA_0. Let $E \subseteq Q[0,1]^2$ be finite. We find a maximal emulator S of E which contains or is disjoint from $Q[1/2,1)^{2<}$, which suffices according to Theorem 3.3.2. The first case that applies is operational.

case 1. E has no (p,q), $p < q$. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$. Then S is disjoint from $Q[1/2,1)^{2<}$.

case 2. E has no (p,q),(p',q'), $p < q < p' < q'$. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$ containing (0,1/3). Then S is disjoint from $Q[1/2,1)^{2<}$.

case 3. E has no (p,q),(p',q'), $p' < p < q < q'$. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$ containing (1/3,1). Then S is disjoint from $Q[1/2,1)^{2<}$.

case 4. E has no (p,q),(p',q), $p < p' < q$. Let $C = \{(p,1-2p): 0 \leq p \leq 1/4\}$. Since case 3 does not apply, C is an emulator of $E \subseteq Q[0,1]^2$. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$ containing C. Then every number in [1/2,1] is the right endpoint of a unique element of C, and that unique element of C has left endpoint $\leq 1/4$. Therefore S is disjoint from $Q[1/2,1)^{2<}$.

case 5. E has no (p,q),(p,q'), $p < q < q'$. Clearly $Q[0,1) \times \{1\}$ is an emulator of $E \subseteq Q[0,1]^2$ since case 4 does not apply. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$ containing $Q[0,1] \times \{1\}$. Then S is disjoint from $Q[1/2,1)^{2<}$.

case 6. E has no (p,q),(p',q'), $p < p' < q < q'$. Let $B = \{(1/3,q): 1/3 < q \leq 1$. Since case 5 does not apply, B is an emulator of $E \subseteq Q[0,1]^2$. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$ containing B. Because $(1/3,1) \in B$, clearly S is disjoint from $Q[1/2,1)^{2<}$.

case 7. Otherwise. Then every element of $Q[0,1]^{2<} \times Q[0,1]^{2<}$ is order equivalent to an element of E^2. Hence $Q[0,1]^{2<}$ is an emulator of $E \subseteq Q[0,1]^2$, and so some maximal emulator S of E contains $Q[0,1]^{2<}$.

QED

The situation changes considerably with dimension $k \geq 3$.

MAXIMAL EMULATION LARGE USE/4. MELU/4. For $k \geq 3$,

$Q[1/3,1/2]^{k<} \times Q[1/3,1/2]^{k<}$ is not ME usable.

Proof: In RCA_0. Fix $k \geq 3$. Let $x_1,y_1,\ldots,x_n,y_n \in Q[0,1]^{k<}$ be such that
i. For each i, it is not the case that x_i,y_i have the same first and last coordinates.
ii. Each x_{i+1},y_{i+1} has all 2k combined coordinates greater than all 2k combined coordinates in x_i,y_i.
iii. Every $(x,y) \in Q[0,1]^{k<} \times Q[0,1]^{k<}$, where it is not the case that x,y have the same first and last coordinates, is order equivalent (as a 2k-tuple) to some (x_i,y_i).

Let $E = \{x_1,\ldots,x_n,y_1,\ldots,y_n\}$. We claim that S is an emulator of E if and only if $S \subseteq Q[0,1]^{k<}$, and for all distinct $x,y \in S$, it is not the case that x,y have the same first and last coordinates. Suppose S is an emulator of E, and let $x,y \in S$ be distinct, where x,y have the same first and last coordinates. Then (x,y) cannot be order equivalent to any (x_i,y_i), and also (x,y) cannot be order equivalent to any (x_i,y_j), $i \neq j$. So S is not an emulator of E. Conversely, suppose $S \subseteq Q[0,1]^{k<}$, where for all distinct $x,y \in S$, it is not the case that x,y have same first and last coordinate. Then every $(x,y) \in S^2$ is order equivalent to some $(x_i,y_i) \in E$ by iii.

Let S be a maximal emulator of E. We claim that for all $p < q$ there exists $p < b_1 < \ldots < b_{k-2} < q$ such that $(p,b_1,\ldots,b_{k-2},q) \in S$. Suppose this is false for $p < q$. Choose $p < c_1 < \ldots < c_{k-2} < q$. Then $S \cup \{(p,c_1,\ldots,c_{k-2},q)\}$ is an emulator of E, and so $(p,c_1,\ldots,c_{k-2},q) \in S$, which is a contradiction. Now let $(1/3,b_1,\ldots,b_{k-2},1/2) \in S$. S cannot contain its R image since for $1/3 < b_1',b_2,\ldots,b_{k-2} < 1/2$, $1/3 < b_1' < b_2$, we have $(1/3,b_1',b_2,\ldots,b_{k-2},1/2) \in S$. QED

We are still very far from obtaining any kind of complete determination of the order theoretic usable $R \subseteq Q[0,1]^k \times Q[0,1]^k$, even in dimension $k = 2$. MELU/2,3,4 indicate that, roughly speaking, large order theoretic R are not usable in dimension $k \geq 3$ but somewhat usable in dimension $k = 2$. Just as one example of the many issues remaining here with large order theoretic R, consider the following possibility.

MAXIMAL EMULATION LARGE USE/5. MELU/5. All order preserving order theoretic $R \subseteq Q[1/3,1/2]^2 \times Q[1/3,1/2]^2$ are usable.

We do not know the status of MELU/5, and its stronger forms with $Q[1/2,1)^2 \times Q[1/2,1)^2$ and even $Q[1/2,1]^2 \times Q[1/2,1]^2$. But note that MELU/5 with dimension $k \geq 3$ is strongly refuted by MELU/4.

3.4. SMALL RELATIONS

Here we first confront independence from ZFC. The most immediately transparent examples presented here are MESU/1 and MED/1.

We start with the most obvious parameterization of finite R, obtained by simply adding a new dummy variable.

PARAMETERIZATION DEFINITION. Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$. The parameterization of R is the relation $R' \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ given by $R'(x,y) \leftrightarrow R((x_1,\ldots,x_k),(y_1,\ldots,y_k)) \wedge x_{k+1} = y_{k+1}$.

Parameterizations are too strong to be used for ME usability. This is because parameterizations are generally not even order preserving - a necessary condition for ME usability - see MENU in section 3.2.

THEOREM 3.4.1. (RCA_0) Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$. The following are equivalent.
i. The parameterization of R is ME usable.
ii. The parameterization of R is order preserving.
iii. R alters no numbers.

Proof: Let R be as given. i $\rightarrow$ ii, and iii $\rightarrow$ i are immediate. Assume ii and let R' be the parameterization of R. Suppose R alters some number, and let $R(x,y)$ where $x_i \neq y_i$. Clearly $R'((x,x_i),(y,x_i))$, and so $(x,x_i),(y,x_i)$ are order equivalent. Therefore $x_i = y_i$, which is a contradiction. QED

The obvious way to fix parameterizations is to use the weaker notion of lower parameterization, which does behave well with order preservation.

LOWER PARAMETERIZATION DEFINITION. Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$. The lower parameterization of R is the relation $R' \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ given by $R'(x,y) \leftrightarrow$
$R((x_1,\ldots,x_k),(y_1,\ldots,y_k)) \wedge x_{k+1} = y_{k+1} < x_1,\ldots,x_k,y_1,\ldots,y_k$.

THEOREM 3.4.2. (RCA_0) The lower parameterization of any $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is order preserving if and only if $R|>0$ is order preserving.

Proof: Let $R' \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ be the lower parameterization of $R \subseteq Q[0,1]^k \times Q[0,1]^k$. Suppose R' is order preserving. Let $R|>0(x,y)$. Then $R'((x,0),(y,0))$, and so $(x,0),(y,0)$ are order equivalent. Hence x,y are order equivalent. Conversely, suppose $R|>0$ is order preserving, and let $R'(x,y)$. Then $R((x_1,\ldots,x_k),(y_1,\ldots,y_k)) \wedge x_{k+1} = y_{k+1} < x_1,\ldots,x_k,y_1,\ldots,y_k$. It is clear that x,y are order equivalent. QED

MAXIMAL EMULATION SMALL USE/1. MESU/1. The lower parameterization of any order preserving finite $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable.

We shall see that MESU/1 is neither provable nor refutable from ZFC, assuming SRP is consistent. We first discuss some weaker and stronger forms of MESU/1. On the weaker side, we use what we call drop equivalence. This allows us to conveniently state a very specific special case of MESU/1 without using the ME use terminology, while still having independence from ZFC.

DROP EQUIVALENCE DEFINITION. Let $S \subseteq Q[0,1]^k$. S is drop equivalent at x,y if and only if $x,y \in Q[0,1]^k \wedge x_k = y_k \wedge (\forall p \in Q[0,x_k))(S(x_1,\ldots,x_{k-1},p) \leftrightarrow S(y_1,\ldots,y_{k-1},p))$.

We can think of x,y as raindrops in the space $Q[0,1]^k$, at the same height $x_k = y_k$ over the ground. As they fall to the ground in tandem, they generally go in and out of a given set $S \subseteq Q[0,1]^k$. Drop equivalence asserts that as they fall in tandem, one is in S if and only if the other is in S. I.e., x,y have the same pattern of membership in S as they fall in tandem.

MAXIMAL EMULATION DROP/1. MED/1. For finite subsets of $Q[0,1]^k$, some maximal emulator is drop equivalent at $(1,1/2,\ldots,1/k),(1/2,\ldots,1/k,1/k)$.

Thus we have presented the most immediately transparent statements independent of ZFC in this paper - MESU/1 and MED/1.

THEOREM 3.4.3. (RCA_0) MESU/1 → MED/1. For $k \geq 1$, MESU/1 for dimension k implies MED/1 for dimension k+1.

Proof: It is clear that MED/1 for k+1 asserts the ME usability of $R \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$, where $R(x,y) \leftrightarrow R'(x,y) \vee R'(y,x)$, and $R'(x,y) \leftrightarrow (x_1,\ldots,x_k) = (1/1/2,\ldots,1/k) \wedge (y_1,\ldots,y_k) = (1/2,\ldots,1/k+1) \wedge x_{k+1} = y_{k+1} < 1/(k+1)$. Note that R is the lower parameterization of $\{((1,\ldots,1/k),(1/2,\ldots,1/(k+1))),((1/2,\ldots,1/(k+1)),(1,\ldots,1/k))\}$, which is clearly order preserving and finite, ready for use in MESU/1 for k. QED

We now state a far reaching extension of MESU/1 using certain critical equivalence relations.

DEFINITION 3.4.1. Let $A \subseteq Q[0,1]$. The relation $R_k(A) \subseteq Q[0,1]^k \times Q[0,1]^k$ is given by $R_k(A)(x,y)$ if and only if
i. x,y are order equivalent.
ii. If $x_i \neq y_i$ then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A.

LEMMA 3.4.4. (RCA_0) $R_k(A)$ is order preserving. Let $R_k(A)(x,y)$. $(\forall i,j)(x_i \neq y_i \rightarrow x_i,y_i \in A)$. $(\forall i)(x_i \in A \leftrightarrow y_i \in A)$.

Proof: Fix $A \subseteq Q[0,1]$. Obviously $R_k(A)$ is order preserving. Let $R_k(A)(x,y)$. If $x_i \neq y_i$ then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A, and in particular, $x_i,y_i \in A$. If $x_i = y_i$ then we are done. If $x_i \neq y_i$ then $x_i,y_i \in A$ and we are done. QED

THEOREM 3.4.5. (RCA_0) Each $R_k(A)$ is an order preserving equivalence relation on $Q[0,1]^k$. If $|A| \leq 1$ then $R_k(A)$ alters no numbers. If $|A| \geq 2$ then $R_k(A)$ alters exactly the elements of A. If A is finite then $R_k(A)$ alters finitely many numbers and $R_k(A)$ is order theoretic.

Proof: Obviously $R_k(A)$ is reflexive and symmetric. For transitivity, let $R_k(A)(x,y)$, $R_k(A)(y,z)$. We want $R_k(A)(x,z)$. Clearly x,y,z are order equivalent. Let $x_i \neq z_i$. Clearly $x_i \neq y_i \vee y_i \neq z_i$.

case 1. $x_i \neq y_i$. Then every $x_j \geq x_i$ lies in A. Let $z_j \geq z_i$. By order equivalence, $y_j \geq y_i \wedge y_j \in A$. By Lemma 3.4.4, $z_j \in A$.

case 2. $y_i \neq z_i$. Then every $z_j \geq z_i$ lies in A. Let $x_j \geq x_i$. By order equivalence, $y_j \geq y_i \wedge y_j \in A$. By Lemma 3.4.4, $x_j \in A$.

This establishes the first claim.

Suppose $|A| \leq 1$. If $R_k(A)$ alters p,q then by Lemma 3.4.4, $p,q \in A$ and $p = q$. Hence $R_k(A)$ alters no numbers. Suppose $|A| \geq 2$. Let $p \in A$. Then $R_k(A)((p,\ldots,p)(q,\ldots,q))$ where $p \neq q \in A$. Conversely, if $R_k(A)$ alters p then by Lemma 3.4.4, $p \in A$.

Let $A \subseteq Q[0,1]$ be finite. Then obviously $R_k(A)$ is order theoretic with parameters from A. By the previous claims, $R_k(A)$ alters at most the elements of A, which is finite. QED

MAXIMAL EMULATION SMALL USE/2. MESU/2. For finite $A \subseteq Q(0,1]$, $R_k(A) \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable.

We now give a finiteness condition on $R \subseteq Q[0,1]^k \times Q[0,1]^k$.

FINITENESS CONDITION. $R \subseteq Q[0,1]^k \times Q[0,1]^k$ has the Finiteness Condition if and only if there are finitely many p appearing in some element of R that alters some $q \leq p$.

THEOREM 3.4.6. (RCA_0) An order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ has the Finiteness Condition if and only if it is contained in some $R_k(A)$, A finite. An order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ has the Finiteness Condition and does not alter 0 if and only if it is contained in some $R_k(A)$, $0 \notin A$, A finite.

Proof: For the first claim, let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ be as given. Let A be the set of all p such for some $(x,y) \in R$, p is both present in (x,y) and at least as large as some number altered by (x,y). Then A is finite. Let R(x,y). Then x,y are order equivalent. Suppose $x_i \neq y_i$. Then obviously all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A. Hence $R_k(A)(x,y)$. Conversely, suppose $R \subseteq R_k(A)$, A finite. We claim that all p appearing in some element of $R_k(A)$ that alters some $q \leq p$ lie in A. To see this, let $R_k(A)(x,y) \wedge x_i \neq y_i \wedge q \in \{x_i,y_i\} \wedge q \leq p = x_j$. Then $p \in A$. Therefore $R_k(A)$ has the Finiteness Condition. Since $R \subseteq R_k(A)$, clearly R has the Finiteness Condition.

For the second claim, let order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ have the Finiteness Condition and not alter 0. Repeat the proof of the first claim, noting that the A constructed there must not contain 0. Conversely, note that $R_k(A)$, $0 \notin A$, A finite, has the Finiteness Condition and does not alter 0. Therefore R has the Finiteness Condition and does not alter 0. QED

MAXIMAL EMULATION SMALL USE/3. MESU/3. Every order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ with the Finiteness Condition, not altering 0, is ME usable.

THEOREM 3.4.7 (RCA_0) MESU/2 ↔ MESU/3. For $k \geq 1$, MESU/2 for dimension k ↔ MESU/3 for dimension k.

Proof: The first claim follows from the second claim. Assume MESU/2 for k, and let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ be order preserving with the Finiteness Condition, not altering 0. By Theorem 3.4.6, let $R \subseteq R_k(A)$, A finite, $0 \notin A$. By MESU/2 for k, $R_k(A)$ is ME usable. Hence by Theorem 2.6, R is ME usable. Assume MESU/3 for k, and let $A \subseteq Q(0,1]$ be finite. By Theorem 3.4.6, $R_k(A)$ is order preserving and has the Finiteness Condition and does not alter 0. Hence $R_k(A)$ is ME usable. QED

In MED/1 above, it is natural to ask which numbers can be used.

DROPPABLE TUPLES DEFINITION. $x,y \in Q[0,1]^k$ are droppable if and only if
i. $x_k = y_k$.
ii. $(x_1,\ldots,x_{k-1}),(y_1,\ldots,y_{k-1})$ are order equivalent.
iii. For all $1 \leq i \leq k$, $(x_i < x_k \vee y_i < y_k) \rightarrow x_i = y_i$.

MAXIMAL EMULATION DROP/2. MED/2. Let $x,y \in Q[0,1]^k$. The following are equivalent.
i. For finite subsets of $Q[0,1]^k$, some maximal emulator is drop equivalent at x,y.
ii. x,y are droppable or $x_k = y_k = 0$.

In fact, we have the following multiple form.

MAXIMAL EMULATION DROP/3. MED/3. Let $x_1,y_1,\ldots,x_n,y_n \in Q[0,1]^k$. The following are equivalent.
i. For finite subsets of $Q[0,1]^k$, some maximal emulator is

drop equivalent at every x_i,y_i.
ii. For finite subsets of $Q[0,1]^k$ and $1 \leq i \leq k$, some maximal emulator is drop equivalent at (x_i,y_i).
iii. For all i, x_i,y_i is droppable or $(x_i)_k = (y_i)_k = 0$.

THEOREM 3.4.8. (RCA_0) Each of MESU/1,2,3, MED/1,2,3 follows from MESU/2 and implies MED/1. For $k \geq 1$, MESU/1 for dimension k, MESU/2,3, MED/1,2,3 for dimension k+1 each follow from MESU/2 for dimension k+1 and each imply MED/1 for dimension k+1. Each of MESU/2,3, MED/1,2,3 hold for dimension k = 1.

Proof: Fix $k \geq 1$. The first claim follows from the second claim. Assume MESU/2 for k+1. We have MESU/3 for k+1 by Theorem 3.4.7. For MESU/1 for k, let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ be finite and order preserving. Let A be the set of numbers appearing in R. We claim that the lower parameterization R' of R is contained in $R_{k+1}(A)$. To see this, let $R'(x,y)$. Then $R(x_1,\ldots,x_k) \wedge R(y_1,\ldots,y_k) \wedge x_{k+1} = y_{k+1} < x_1,\ldots,x_k,y_1,\ldots,y_k \wedge x_1,\ldots,x_k,y_1,\ldots,y_k \in A$. If $x_i \neq y_i$ then $1 \leq i \leq k$ and obviously all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A. Hence $R_{k+1}(A)(x,y)$, and so by MESU/2 for k+1, R' is ME usable.

For MED/3 for k+1, let $x_1,y_1,\ldots,x_n,y_n \in Q[0,1]^{k+1}$. We show i $\rightarrow$ ii $\rightarrow$ iii $\rightarrow$ i. i $\rightarrow$ ii is immediate, and we now show ii $\rightarrow$ iii (for k+1), even without using MESU/2 for k+1. Assume ii for k+1. By the drop equivalence in ii, we have $(x_i)_{k+1} = (y_i)_{k+1}$. Assume $(x_i)_{k+1} = (y_i)_{k+1} = 0$ is false. Then $(x_i)_{k+1} = (y_i)_{k+1} > 0$. To show that x_i,y_i is droppable, we first restate a consequence of drop equivalence in terms of ME usability. Let $R<x_i,y_i> \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ given by $R(z,w) \leftrightarrow (z_1,\ldots,z_k) = ((x_i)_1,\ldots,(x_i)_k) \wedge (w_1,\ldots,w_k) = ((y_i)_1,\ldots,(y_i)_k) \wedge z_{k+1} = w_{k+1} < (x_i)_k = (y_i)_k$. By ii, $R<x_i,y_i>$ is ME usable, and so by MENU, $R<x_i,y_i>$ is order preserving. Hence $(x_1,\ldots,x_k,0),(y_1,\ldots,y_k,0)$ are order equivalent, and therefore $(x_1,\ldots,x_k),(y_1,\ldots,y_k)$ are order equivalent. Now suppose $(x_i)_j < (x_i)_{k+1} \vee (y_i)_j < (x_i)_{k+1}$. Then $((x_i)_1,\ldots,(x_i)_k,(x_i)_j),((y_i)_1,\ldots,(y_i)_k,(x_i)_j)$ are $R<x_i,y_i>$ related and order equivalent, and therefore $(y_i)_j = (x_i)_j$. This verifies that x_i,y_i is droppable.

Finally, assume iii in MED/3 for k+1. We show i in MED/3 for k+1 using MESU/2. We first claim that each $R<x_i,y_i> \subseteq R_{k+1}(A)$, where $A = \{(x_i)_1,\ldots,(x_i)_k,(y_i)_1,\ldots,(y_i)_k\}\backslash Q[0,x_{k+1})$. If $(x_i)_{k+1} = (y_i)_{k+1} = 0$ then we have i vacuously. Assume x_i,y_i

is droppable. The relation $R<x_i,y_i>$ defined above, associated with x_i,y_i, is order preserving, using that x_i,y_i is droppable. We claim that each $R<x_i,y_i> \subseteq R_k(A)$, where $A = \{(x_i)_1,\ldots,(x_i)_k,(y_i)_1,\ldots,(y_i)_k\}\backslash Q[0,x_{k+1})$. We need to examine $((x_i)_1,\ldots,(x_i)_k,p),((y_i)_1,\ldots,(y_i)_k,p)$, $0 \leq p < (x_i)_{k+1} = (y_i)_{k+1}$. Let $(x_i)_j \neq (y_i)_j$. By droppability, $(x_i)_j,(y_i)_j \geq x_{k+1} = y_{k+1}$. Also the $x_n \geq (x_i)_j$ and $y_n \geq (y_i)_j$ lie in A. The claim is established.

It is now clear that $\cup_i R<x_i,y_i> \subseteq R_{k+1}(A)$, and so $\cup_i R<x_i,y_i>$ is ME invariantly usable. It is now clear that i of MED/3 holds.

To obtain MED/1,2 for k+1, obviously MED/3 for k+1 → MED/2 for k+1 → MED/1 for k+1.

So see that MESU/1 for k, MESU/2,3, MED/1,2,3 for k+1 each imply MED/1 for k+1, it suffices to show that MESU/1 for k and MESU/3 for k+1 imply MED/1 for k+1. Note that MED/1 for k+1 asserts that the following $R \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is ME invariantly usable. $R(x,y) \leftrightarrow (x_1,\ldots,x_k) = (1,1/2,\ldots,1/k) \wedge (y_1,\ldots,y_k) = (1/2,\ldots,1(k+1) \wedge x_{k+1} = y_{k+1} < 1/(k+1)$. Since R is contained in the lower parameterization of a finite order preserving $R' \subseteq Q[0,1]^k \times Q[0,1]^k$, and also is order preserving with the Finiteness Condition and not altering 0, we can simply apply MESU/1 for k and MESU/3 for k+1.

For the final claim, MED/1,2,3 are vacuous for k = 1. For MESU/2 for k = 1, note that for finite $A \subseteq Q[0,1]$, $R_1(A)$ is ME usable by, e.g., MELU/1 of section 3.3. For MESU/3, use Theorem 3.4.7. QED

We now obtain dimension reduction as in Theorem 3.4.10 below.

DEFINITION 3.4.2. Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ and $1 \leq i < k$. $\gamma(R,i) \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is given by $\gamma(R,i)(x,y) \leftrightarrow R((x_1,\ldots,x_i,x_{i+2},\ldots,x_k),(y_1,\ldots,y_i,y_{i+2},\ldots,y_k)) \wedge x_i = x_{i+1} \wedge y_i = y_{i+1}$.

LEMMA 3.4.9. (RCA_0) Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ and $1 \leq i < k$. R is ME usable if and only if $\gamma(R,i)$ is ME usable.

Proof: Left to the reader. QED

THEOREM 3.4.10. For all $k \geq 1$, any of MESU/1,2,3, MED/1,2,3 for dimension k+1 implies that same statement for dimension k.

Proof: Assume MESU/1 for dimension k+1. Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ be finite and order preserving. We want to show that the lower parameterization $LP(R) \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is ME usable. By Lemma 3.4.9, it suffices to show that $\gamma(LP(R),k)$ is ME usable. But $\gamma(LP(R),k) = LP(\gamma(R,k-1))$. Since $\gamma(R,k-1) \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is finite and order preserving, LP((R,k-1)) is ME usable by MESU/1 for dimension k+1.

Assume MESU/2 for dimension k+1. Let $A \subseteq Q(0,1]$ be finite. We want to show that $R_k(A) \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable. Now $\gamma(R_k(A),k) \subseteq R_{k+1}(A)$, and so $\gamma(R_k(A),k) \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is ME usable. By Lemma 3.4.9, $R_k(A)$ is ME usable.

Assume MESU/3 for dimension k+1. Apply Theorem 3.4.7 to the previous claim.

Assume MED/2 for dimension k+1. Let $x,y \in Q[0,1]^k$. We have seen that in MED/2 for dimension k, i $\rightarrow$ ii is provable in RCA_0 in the proof of Theorem 3.4.8 (without assuming MED/2 for dimension k+1). Assume ii in MED/2. If $x_k = y_k = 0$ then i in MED/2. Now assume x,y are droppable. Let $x' = (x_1,\ldots,x_{k-1},x_{k-1},x_k)$, $y' = (y_1,\ldots,y_{k-1},y_{k-1},y_k)$. Then $x',y' \in Q[0,1]^{k+1}$ are droppable. Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ be the symmetric relation associated with drop equivalence at x,y. Then $\gamma(R,k-1) \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is the symmetric relation associated with drop equivalence at x',y'. By MED/2 for dimension k+1, $\gamma(R,k-1)$ is ME usable. Hence by Lemma 3.4.9, R is ME usable. Therefore i in MED/2 holds.

Assume MED/3 for dimension k+1. Let $x_1,y_1,\ldots,x_n,y_n \in Q[0,1]^k$. We have seen that in MED/3 for dimension k, i $\rightarrow$ ii $\rightarrow$ iii is provable in RCA_0 in the proof of Theorem 3.4.8. (without assuming MED/3 for dimension k+1). Assume iii in MED/3. For $1 \leq i \leq k$, let x_i',y_i' be defined as in the previous paragraph. Let $R \subseteq Q[0,1]^k \times Q[0,1]^k$ be the union of the symmetric relations associated with drop equivalence at the x_i,y_i that do not have $(x_i)_k = (y_i)_k = 0$. Then $\gamma(R,k-1) \subseteq Q[0,1]^{k+1} \times Q[0,1]^{k+1}$ is the symmetric relation associated with drop equivalence in the multiple form for these x_i',y_i'. By MED/3 for dimension k+1, $\gamma(R,k-1)$ is ME usable. Hence by Lemma 3.4.9, R is ME usable. Therefore i in MED/3 holds.

The claim for MED/1 will be addressed elsewhere in [Fr18]. QED

The following result shows that MESU/1,2,3, MED/1,2,3 are far beyond the reach of ZFC.

THEOREM 3.4.11. RCA_0 proves MED/1 → Con(SRP). The following is provable in EFA. $(\forall n)(\exists k)$(RCA_0 proves (MED/1 for dimension k) → Con(SRP[n])).

Proof: This reversal will appear in [Fr18]. QED

3.5. EXOTIC PROOF

This entire section is devoted to the proof of MESU/2 in dimension k. As a Corollary, we obtain all six of MESU/1,2,3, MED/1,2,3 through Theorems 3.4.8 and 3.4.10. We have already dispensed with the trivial case k = 1, and thus focus on k ≥ 2. The key technique for proving MESU/2 is to use a certain transfinite extension of our space Q[0,1] = (Q[0,1],<). The length of the transfinite constructions that we will use will be roughly some strategically chosen limit ordinal κ. At some point in the proof, we will need that κ is an uncountable regular cardinal, the smallest of which is ω_1. Then soon later, difficulties arise, and the proof starts to work only for dimension k = 2. At that point in the proof, we will continue to assume that κ is an uncountable regular cardinal (such as ω_1) for dimension k = 2. However, to carry on the proof for dimensions k ≥ 3, we will assume that κ is a (k-2)-subtle cardinal. The rest of the proof of MESU/2 proceeds normally and uniformly in k.

Thus the proof of MESU/2 for dimension k = 2 goes through naturally using $\kappa = \omega_1$, and therefore well within ZFC, and, with standard modifications, in Z or even in Z_3. For dimension k ≥ 3 we use ZFC + $(\exists\kappa)$(κ is (k-2)-subtle), which is a weak fragment of SRP[k-1]. See Appendix A. We conjecture that MESU/2 is provable in RCA_0 for dimension k = 2, and that such a proof will be far more complicated but of a totally different character than the proof given here using ω_1, involving a painstaking combinatorial analysis of maximal emulation in two dimensions. However, we conjecture that the natural extension of MESU/2 for dimension k = 2, discussed in section 3.6, cannot be proved in ZFC\P or Z_2.

Towards the proof of MESU/2, fix $k \geq 2$, finite $E \subseteq Q[0,1]^k$, and finite $A \subseteq Q(0,1]$. We construct a maximal emulator S of $E \subseteq Q[0,1]^k$ that is $R_k(A)$ invariant. We can assume without loss of generality that $A = \{q_1 < ... < q_n = 1\}$, where $n \geq 1$ and $q_1 > 0$. We also assume, without loss of generality, that $E \subseteq Q(0,1)^k$. Let κ be a limit ordinal. At certain points in the proof we will be making further requirements on κ. Ordinals are treated in the usual set theoretic way as the set of their predecessors. This is the epsilon connected transitive set definition. < is used to compare ordinals, but also to compare rationals. For other comparisons, we adorn the < symbol.

DEFINITION 3.5.1. $T[\kappa] = \kappa+1 \times Q[0,1)$. For $x \in T[\kappa]$, $\mathrm{ord}(x) = x_1$. $<_{T[\kappa]}$ is the linear ordering on $T[\kappa]$ given by $x <_{T[\kappa]} y$ if and only if $\mathrm{ord}(x) < \mathrm{ord}(y) \vee (\mathrm{ord}(x) = \mathrm{ord}(y) \wedge x_2 < y_2)$. $x \leq_T y \leftrightarrow (x <_T y \vee x = y)$. S is a $T[\kappa]$-emulator of E if and only if $S \subseteq T[\kappa]^k$, and every $(x,y) \in S^2$ is order equivalent to some $(z,w) \in E^2$ (using $<_{T[\kappa]}$ and numerical <). S is a maximal $T[\kappa]$-emulator of E if and only if S is a $T[\kappa]$-emulator of E which is not a proper subset of any $T[\kappa]$-emulator of E. For $x \in T[\kappa]^k$, $\mathrm{ord}(x) = \max(\mathrm{ord}(x_1),...,\mathrm{ord}(x_k))$. λ always denotes a limit ordinal.

Note that in the above definition, we use a notion of emulator ($T[\kappa]$-emulator) in which the emulators live in a different space than the sets being emulated. We have avoided this earlier in the paper, but here it is very convenient. And here, and later, since there are so many linear orderings being defined, we parenthetically mention which linear orderings are being presently used.

We view $\kappa+1 \times \{0\}$ as the preferred closed subset of $T[\kappa]$ under $<_{T[\kappa]}$.

DEFINITION 3.5.2. Fix an effective enumeration $0 = p_0,p_1,...$ of Q[0,1), without repetition. <' is the linear ordering ordering on Q[0,1) of type ω given by the p's. <* is the linear ordering on $T[\kappa]$ given by x <* y if and only if ord(x) < ord(y) ∨ (ord(x) = ord(y) ∧ x_2 <' y2. <** is the linear ordering on $T[\kappa]^k$ given by x <** y if and only if max(x) <* max(y) ∨ (max(x) = max(y) ∧ x is lexicographically earlier than y using <* on the k coordinates), where the four max's here are with respect to <*. x ≤* y ⇔ x <* y ∨ x = y. x ≤** y ⇔ x <** y ∨ x = y. The greedy T[κ]-emulator of E, GE(E,T[κ]) ⊆ $T[\kappa]^k$, is defined by transfinite recursion on <** given by the equation GE(E,T[κ]) = {x ∈ $T[\kappa]^k$: GE(E,T[κ])|<**x ∪ {x} is a T[κ]-emulator of E}.

Although we write $<_{T[\kappa]}$ to distinguish it from < (numerical comparison of rationals), we decided to write just <* and <** for readability.

Note that Definition 3.5.2 relies on the easily established fact that <* and <** are well orderings. Note also that we are using two orderings on T[κ], the dense $<_{T[\kappa]}$ and the well ordering <*. Here the T[κ]-emulator is constructed along the particular extension <** of <* to $T[\kappa]^k$.

LEMMA 3.5.1. The following hold. Below, α < κ.
i. GE(E,T[κ]) is uniquely defined, by the equation in Definition 3.4.4.
ii. GE(E,T[κ]) is a maximal T[κ]-emulator of E.
iii. For x,y ∈ T[κ], x ≤* y → ord(x) ≤ ord(y).
iv. For x,y ∈ $T[\kappa]^k$, x ≤** y → ord(x) ≤ ord(y).
v. For x ∈ T[κ], x $<_{T[\kappa]}$ (α,0) ⇔ x <* (α,0) ⇔ ord(x) < α.
vi. For x ∈ $T[\kappa]^k$, (∀i)(x_i $<_{T[\kappa]}$ (α,0)) ⇔ ord(x) < α ⇔ x <** ((α,0),(0,0),...,(0,0)).

Proof: i is left to the reader. For ii, GE(E,T[κ]) = $\cup_x$GE(E,T[κ])|<**x, and so is the union of T[κ]-emulators of E. Hence GE(E,T[κ]) is a T[κ]-emulator of E. For maximality, suppose GE(E,T[κ]) ∪ {x} is a T[κ]-emulator of E. Then GE(E,T[κ])|<**x ∪ {x} is a T[κ]-emulator of E, putting x in GE(E,T[κ]).

iii is immediate from the definition of <*. For iv, let x ≤** y. Then max(x) ≤* max(y), where the max's use ≤*. Now apply iii.

For v, let x $<_{T[\kappa]}$ (α,0). ord(x) = α is impossible since x is lexicographically earlier than $(\alpha,0)$. Hence $\text{ord}(x) < \alpha$.

For vi, we show $(\forall i)(x_i <_{T[\kappa]} (\alpha,0)) \rightarrow \text{ord}(x) < \alpha \rightarrow x <^{**} ((\alpha,0),(0,0),\ldots,(0,0)) \rightarrow (\forall i)(x_i <_{T[\kappa]} (\alpha,0))$. The first implication is by vii. The second implication is clear by comparing the max's of both sides, max with respect to $<^*$. For the third implication, suppose $x <^{**} ((\alpha,0),(0,0),\ldots,(0,0))$. If $\max(x) = (\alpha,0)$ then x is lexicographically earlier than $((\alpha,0),(0,0),\ldots,(0,0))$, which is impossible. If $\max(x) <^* (\alpha,0)$, max with respect to $<^*$, then $\text{ord}(x) < \alpha$ and so $(\forall i)(x_i <_{T[\kappa]} (\alpha,0))$. QED

DEFINITION 3.5.3. Let $x \in T[\kappa]^k$. $x\backslash\kappa$ is the tuple of length $\leq k$ obtained by deleting all coordinates of x whose first term is κ, from x. For this purpose only, we allow the 0-tuple, whose ord is taken to be 0. $x(\kappa|\alpha)$ and $x(\alpha|\kappa)$ are the results of replacing κ by α and α by κ, respectively, throughout x, where $\alpha \leq \kappa$. These ordinal replacements are done at the first terms of the coordinates of x only.

LEMMA 3.5.2. Let κ be an uncountable regular cardinal. There is a closed unbounded $C \subseteq \kappa$ of limit ordinals such that the following holds. Let $x \in T[\kappa]^k$. If $\text{ord}(x\backslash\kappa) < \lambda \in C$, then $x \in GE(E,T[\kappa]) \leftrightarrow x(\kappa|\lambda) \in GE(E,T[\kappa])$. If $\text{ord}(x) \leq \lambda \in C$, then $x \in GE(E,T[\kappa]) \leftrightarrow x(\lambda|\kappa) \in GE(E,T[\kappa])$.

Proof: Let κ be as given. We form the structure M = $(\kappa+1,Q[0,1),E,<_{\kappa+1},<_{Q[0,1)},<',GE(E,T[\kappa]))$, where
i. The two domains are $\kappa+1$ and $Q[0,1)$.
ii. E is the current E as a k-ary predicate on $Q[0,1)$.
iii. $<_{\kappa+1}$ is the usual ordering on $\kappa+1$.
iv. $<_{Q[0,1]}$ is the usual numerical ordering on $Q[0,1)$.
v. $<'$ is the ordering of $Q[0,1)$ given by the p's from Definition 3.5.2.
vi. $GE(E,T[\kappa])$ is used as a 2k-ary predicate whose 2k arguments are of sorts $\kappa+1,Q[0,1),\kappa+1,Q[0,1),\ldots,\kappa+1,Q[0,1)$.

There is a crucial sentence φ that holds in M. φ asserts that $GE(E,T[\kappa])$ is a greedy T-emulator of E (which we know is unique), and is formulated as follows. A given 2k-tuple x lies in (this form of) $GE(E,T[\kappa])$ if and only if for all $y \leq^{**} x$ from $GE(E,T[\kappa])$, (x,y) is order equivalent to some $(z,w) \in E^2$ (using $<_{T[\kappa]}$ and $<_{Q[0,1)}$). The $\leq^{**}$ here is our usual

$\leq^{**}$, and is formulated in the obvious way using $<_{\kappa+1}$ and $<'$. The $<_{T[\kappa]}$ here is formulated in the obvious way using $<_{\kappa+1}$ and $<_{Q[0,1)}$.

We will ignore these differences in presentation, and effectively regard the present 2k-ary predicate GE(E,T[κ]) as the same as GE(E,T[κ]).

By standard techniques from elementary model theory, we form a transfinite sequence of elementary substructures M_α, $\alpha < \kappa$, of M, whose first domains have the following property: the set of first coordinates of its elements is of the form $\gamma_\alpha \cup \{\kappa\}$. Furthermore, the set of these γ_α, $\alpha < \kappa$, forms a closed unbounded set of limit ordinals $< \kappa$. I.e, each $\gamma_\alpha < \gamma_{\alpha+1}$ and each γ_λ, $\lambda < \kappa$, is the sup of the γ_α, $\alpha < \lambda$. This construction crucially relies on κ being an uncountable regular cardinal. Clearly κ appears in all M_α, and the second domains are all Q[0,1).

To clarify this picture for the reader, it is automatic that κ appears in the first domain of an elementary substructure of M, and the second domain is just Q[0,1), but generally there could be lots of gaps in the ordinals appearing in the first domain, whereby many ordinals less than many ordinals appearing in the first domain are missing from that first domain. In the above construction, we have arranged that all such gaps have been filled in (of course, there must be one gigantic gap below κ itself). Also the use of elementary substructures is very convenient but clearly overkill, as we need only weak forms of elementarity.

Write $M_\alpha = (\{\kappa\} \cup \gamma_\alpha, Q[0,1), E, <_{\kappa+1}|, <_{Q[0,1)}, <', GE(E,T[\kappa])|)$, where | is used to restrict to the first domain $\{\kappa\} \cup \gamma_\alpha$. Let $j:\{\kappa\} \cup \gamma_\alpha \cup Q[0,1) \to \gamma_\alpha+1 \cup Q[0,1)$ be the identity on $\gamma_\alpha \cup Q[0,1)$ and send κ to γ_α. Let M_α^* be the unique structure such that j is an isomorphism from M_α onto M_α^*. Write $M_\alpha^* = (\gamma_\alpha+1, Q[0,1), E, \leq_{\kappa+1}|, <_{Q[0,1)}, <', X_\alpha)$, where $<_{\kappa+1}|$ is the usual ordering on $\gamma_\alpha+1$. Then M_α^* is no longer an elementary substructure of M, but it is elementarily equivalent to M. Therefore M_α^* satisfies φ.

Now read the description of φ above as a sentence about M_α^*. It is now clear that $X_\alpha = GE(E,T[\kappa]) \cap (\gamma_\alpha+1)^k$.

Now set $C = \{\gamma_\alpha: \alpha < \kappa\}$. Let $x \in T[\kappa]^k$. Suppose $ord(x\backslash\kappa) < \gamma_\alpha$. Clearly $j(x) = x(\kappa|\gamma_\alpha)$ since j is the identity below γ_α and sends κ to γ_α. Since j is an isomorphism, we have $x \in GE(E,T[\kappa]) \leftrightarrow j(x) \in X_\alpha \leftrightarrow j(x) \in GE(E,T[\kappa])$, j acting coordinatewise. For the second claim, let $ord(x) \leq \lambda \in C$. Then $ord(x(\lambda|\kappa)\backslash\kappa) < \lambda$, and hence $x(\lambda|\kappa) \in GE(E,T[\kappa]) \leftrightarrow x(\lambda|\kappa)(\kappa|\lambda) = x \in GE(E,T[\kappa])$. QED

We fix the closed unbounded $C \subseteq \kappa$ given by Lemma 3.5.2.

DEFINITION 3.5.4. Let $A \subseteq \kappa$, $x \in T[\kappa]^k$. The max coordinates in x are the x_i such that every $x_j \leq_{T[\kappa]} x_i$. The top coordinates/A of x are the max coordinates that lie in $A \times \{0\}$. The high coordinates/A of x are the x_i for which all $x_j \geq_{T[\kappa]} x$ lie in $A \times \{0\}$, and x_i is not max in A. The low coordinates/A of x are the coordinates that are neither top/A nor high/A. It may be that all x_i are low coordinates/A of x. We often write x_i is max in x, x_i is top/A, x_i is high/A, x_i is low/A.

Note that all max coordinates in x have the same first coordinate.

LEMMA 3.5.3. Let $A \subseteq \kappa$, $x \in T[\kappa]^k$.
i. All low coordinates/A of x are less than all high coordinates/A of x are less than all top coordinates/A of x, using $<_{T[\kappa]}$.
ii. If there is a high coordinate/A of x then the top coordinates/A of x are exactly the max coordinates of x.
iii. $x_i \geq_{T[\kappa]} x_j \wedge x_j$ is high/A $\rightarrow$ (x_i is high/A $\vee$ x_i is top/A).

Proof; For i, let x_i be low/A, x_j be high/A, $x_i \geq_{T[\kappa]} x_j$. Then all $x_n \geq_{T[\kappa]} x_i$ are in $A \times \{0\}$. Since x_i is not high/A, x_i is max in x. But then x_i is top/A, which is a contradiction. Now let x_i be high/A, x_j be top/A, $x_i \geq_{T[\kappa]} x_j$. Since x_j is max in x, clearly x_i is max in x, which is a contradiction.

For ii, let x_i be high/A. Then all $x_j \geq x_i$ lie in $A \times \{0\}$. If x_j is max in x then $x_j \geq_{T[\kappa]} x_i$, $x_j \in A \times \{0\}$, and so x_j is top/A. If x_j is top/A then x_j is max in x by definition.

For iii, let $x_i \geq_{T[\kappa]} x_j$ and x_j is high/A. Then all $x_n \geq_{T[\kappa]} x_i$

lie in $A \times \{0\}$, which makes x_i high/A or max in x. In the latter case, x_i is top/A. QED

DEFINITION 3.5.5. As in Definition 3.4.1, Let $A \subseteq \kappa$. $R_k(A,T[\kappa]) \subseteq T[\kappa]^k \times T[\kappa]^k$ is given by $R_k(A,T[\kappa])(x,y)$ if and only if

i. x,y are order equivalent (using $<_{T[\kappa]}$).

ii. If $x_i \neq y_i$ then all $x_j \geq_{T[\kappa]} x_i$ and $y_j \geq_{T[\kappa]} y_i$ lie in $A \times \{0\}$.

LEMMA 3.5.4. Let $A \subseteq \kappa$ and $R_k(A,T[\kappa])(x,y)$. The following hold.

i. x,y are order equivalent (using $<_{T[\kappa]}$).

ii. $x_i \neq y_i \rightarrow x_i,y_i \in A \times \{0\}$.

iii. $x_i \in A \times \{0\} \leftrightarrow y_i \in A \times \{0\}$.

iv. $\mathrm{ord}(x_i) \in A \leftrightarrow \mathrm{ord}(y_i) \in A$.

v. x_i is top/A $\leftrightarrow$ y_i is top/A.

vi. x_i is high/A $\leftrightarrow$ y_i is high/A.

vii. x_i is low /A $\leftrightarrow$ y_i is low/A.

Proof: i,ii are immediate from the definition. For iii, let $x_i \in A \times \{0\}$. If $x_i = y_i$ then $y_i \in A \times \{0\}$. If $x_i \neq y_i$ then $x_i,y_i \in A \times \{0\}$. The converse is proved in the same way. For iv, let $\mathrm{ord}(x_i) \in A$. If $x_i = y_i$ then $\mathrm{ord}(y_i) \in A$. If $x_i \neq y_i$ then $x_i,y_i \in A \times \{0\}$, and so $\mathrm{ord}(x_i),\mathrm{ord}(y_i) \in A$. The converse is proved in the same way.

For v, let x_i be top/A. Then $x_i \in A \times \{0\}$ is max in x. By order equivalence (using $<_{T[\kappa]}$) and iii, $y_i \in A \times \{0\}$ is max in y. The converse is proved the same way. For vi, let all $x_j \geq x_i$ lie in $A \times \{0\}$, where x_i is not max in x. By order equivalence (using $<_{T[\kappa]}$) and iii, all $y_j \geq y_i$ lie in $A \times \{0\}$ and y_j is not max in x. The converse is proved the same way. For vii, let y_i be not high/A and not top/A. Then y_i is not high/A and not top/A by v,vi. The converse is proved the same way. QED

Those readers who wish to continue to stay within ZFC are going to have to now assume $k = 2$. For more adventurous readers, assume $k \geq 3$. In the proof of the following Lemma, we will separate the case $k = 2$ from the case $k \geq 3$. But after Lemma 3.5.5, there will be no difference.

LEMMA 3.5.5. Let $k = 2$, or let $k \geq 3$ and our existing κ be a

$(k-2)$-subtle cardinal. There exists infinite $C' \subseteq C$ of order type ω such that $GE(E,T[\kappa])$ is invariant under the relation $R_k(C',T[\kappa]) \subseteq T[\kappa]^k \times T[\kappa]^k$.

Proof: We first assume $k = 2$. Let $R_2(C,T[\kappa])(x,y)$. We can almost get $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$ without even shrinking C to C'. We will show this under the assumption $x,y \notin (C \times \{0\})^2$, and worry about the general case later. We can obviously assume $x \neq y$. Thus we have $x_1 \neq y_1 \vee x_2 \neq y_2$.

case 1. $x_1 \neq y_1$. Then $x_1,y_1 \in C \times \{0\}$, and therefore $x_2,y_2 \notin C \times \{0\}$. Hence $x_2 = y_2 <_{T[\kappa]} x_1,y_1$. Given this relationship between x,y, we can now apply Lemma 3.5.2, claim 2. Thus raising x_1,y_1 both to $(\kappa,0)$ does not change the status of membership in $GE(E,T[\kappa])$, and x,y become identical. Therefore $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$.

case 2. $x_2 \neq y_2$. Argue as in case 1 with subscripts 1,2 switched.

In order to handle the general case (still with $k = 2$), we must shrink C. Let $C' \subseteq C$ be of order type ω such that membership of $((a,0),(b,0))$ in $GE(E,T[\kappa])$ depends only on the order type of $(a,b) \in C'^2$, using the usual infinite Ramsey theorem from [Ra30]. Suppose $x,y \notin (C \times \{0\})^2$ is false. By Lemma 3.5.4,iii, $x,y \in (C \times \{0\})^2$. Let $R_k(C',T[\kappa])(x,y)$. We claim $x \in GE(E,T) \leftrightarrow y \in GE(E,T)$. This is clear since x,y are order equivalent (using $<_{T[\kappa]}$). We have established that $GE(E,T)$ is invariant under $R_2(C',T[\kappa])$.

Now let $k \geq 3$ and assume κ is a $(k-2)$-subtle cardinal. At this point, we refer the reader to Appendix A for the relevant definitions. The condition satisfied by κ is as follows (for any closed unbounded $C \subseteq \kappa$, not just the C we are using at this point in the present proof).

1) Let $f:S_{k-2}(C) \rightarrow S(\kappa)$ be regressive. There is an f-homogenous $C' \subseteq C$ of cardinality $k-1$.

This is not quite strong enough for our purposes. Fortunately, in [Fr01], Lemma 1.6, we prove the following.

2) Let $f:S_{k-2}(C) \to S(\kappa)$ be regressive. There are f-homogenous $C' \subseteq C$ of every cardinality $< \kappa$.

We can obviously assume that our closed unbounded $C \subseteq \kappa$ consists entirely of uncountable cardinals by shrinking if necessary. We will use 2) only to obtain f-homogenous $C^* \subseteq C$ of order type ω.

Let $D \in S_{k-2}(C)$ and $x \in T[\kappa]^k$. We say that x is D-controlled/C if and only if
i. x has a top/C coordinate.
ii. Every high/C coordinate of x lies in $D \times \{0\}$.
iii. Every low/C coordinate of x lies in $\min(D) \times Q[0,1)$.

Suppose x is D-controlled/C. Define $\gamma(D,x) \in T[\kappa]^k$ to be the result of replacing each high/C coordinate $x_i = (u,0) \in D \times \{0\}$ of x, with $(m,0)$, where u is the m-th element of D, counting from 1, and each top coordinate of x with $(0,0)$. If x is not D-controlled, set $\gamma(D,x) = (0,0)$.

We claim that every $\gamma(D,x) \in (\min(D) \times Q[0,1))^k$. This is evident by inspection for D-controlled/C x. For other x, $\gamma(D,x) = (0,0) \in (\min(D) \times Q[0,1))^k$.

We use standard tuple coding whereby finite sequences from $T[\kappa] \cup \omega$ are coded by ordinals in a standard one-one way. Thus the standard tuple code of every $\gamma(D,x)$ is $< \min(D)$.

We define regressive $f:S_{k-2}(C) \to S(\kappa)$ as follows. $f(D) = \{\beta: (\exists x \in GE(E,T[\kappa]))(x$ is D-controlled/C $\wedge$ β is the standard code of $\gamma(D,x))\}$.

We now use 2) to obtain an f-homogeneous set $C^* \subseteq C$ of order type ω which is f-homogenous. By the infinite Ramsey theorem from [Ra30], we fix $C' \subseteq C^*$ of order type ω such that for all order equivalent $x,y \in C'^k$ (using $<_{T[\kappa]}$), $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$.

We now fix x,y such that $R_k(C',T[\kappa])(x,y)$. We want to prove that $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$. Clearly x,y are order equivalent (using $<_{T[\kappa]}$). For the easy case, suppose $x \in (C' \times \{0\})^k \vee y \in (C' \times \{0\})^k$. It is clear that $x,y \in (C' \times \{0\})^k$ using Lemma 3.5.4,iii. Since $x,y \in (C' \times \{0\})^k$ and x,y are order equivalent (using $<_{T[\kappa]}$), we have $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$ by the construction of C'.

We can now assume $x,y \notin (C' \times \{0\})^k$, and show $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$. By Lemmas 3.5.3,3.5.4, if there are no top/C' coordinates in x or there are no top/C' coordinates in y, then there are no high/C' coordinates in x,y, and so all coordinates of x,y are low/C'. In this case, x = y, and we are done.

So we assume that x,y both have top/C' coordinates.

Let W be the set of all high/C' coordinates in x and W' be the set of all high/C' coordinates in y. Note that x,y both must have the same one or more low/C' coordinates, because x,y both have coordinates not in $C' \times \{0\}$. Thus W,W' both omit at least two coordinates, and hence $|W|,|W'| \leq k-2$.

In fact, since the high/C' coordinates of x,y appear at the same positions (Lemma 3.5.4,vi), we have $|W| = |W'| \leq k-2$.

Now extend W,W' with the same new ordinals from C' that are greater than $\max(W \cup W')$,ord(x),ord(y), and all low/C' coordinates of x,y, arriving at $D,D' \subseteq C'$, each with exactly k-2 elements.

We claim that x is D-controlled/C' and y is D'-controlled/C'. x,y both have top/C' coordinates. Every high/C' coordinate of x lies in $W \times \{0\} \subseteq D \times \{0\}$, and every high/C' coordinate of y lies in $W' \times \{0\} \subseteq D' \times \{0\}$. Also given the construction of W,W',D,D', we see that every low/C' coordinate of x,y lies in $\min(D) \times Q[0,1)$.

We claim that every D-controlled/C' u is also D-controlled/C. To see this, assume u is D-controlled/C'. Obviously every top/C' coordinate of u is a top/C coordinate of u. Suppose u_i is high/C'. Then every $u_j \geq_{T[\kappa]} u_i$ lies in $C' \times \{0\}$ and u_i is not max in u. Then every $u_j \geq_{T[\kappa]} u_i$ lies in $C \times \{0\}$, and so u_i is high/C.

We claim that $\gamma(D,x) = \gamma(D',y) \in \min(D \cup D')^k$. To see this, $\gamma(D,x)$ results from x by replacing each high/C' coordinate $x_i = (u,0) \in D \times \{0\}$ of x, with the integer m, where u is the m-th element of D, and each top/C' coordinate of x with

0. Also $\gamma(D',y)$ results from y by replacing each high/C' coordinate $y_i = (u,0) \in D' \times \{0\}$ of y, with the integer m, where u is the m-th element of D', and each top/C' coordinate of y with 0. By Lemma 3.5.4, the top/C' and high/C' coordinates of x,y lie at the same positions, and x,y have the same low/C' coordinates in the same positions, evidently $\gamma(D,x) = \gamma(D',y)$. The non integer coordinates are the necessarily common low/C' coordinates in x,y which we have seen are < min(D),min(D'), and hence < $\min(D \cup D')$.

We claim that if u,v are D-controlled/C and $\gamma(D,u) = \gamma(D,v)$, then $u \in GE(E,T[\kappa]) \leftrightarrow v \in GE(E,T[\kappa])$. Let u,v be D-controlled/C and $\gamma(D,u) = \gamma(D,v)$. Given this relationship between u,v, we can now apply Lemma 3.5.2, claim 2. Thus raising the top/C coordinates of u,v to $\kappa \times \{0\}$ does not change the status of membership in GE(E,T[κ]), and after this raising, u,v become identical. Therefore $x \in GE(E,T[\kappa]) \leftrightarrow y \in GE(E,T[\kappa])$.

By the definition of f, the standard code β for $\gamma(D,x)$ lies in f(x) if and only if there exists $x^* \in GE(E,T[\kappa])$ such that x* is D-controlled/C ∧ β is the standard code of $\gamma(D,x^*)$. We also have that $\beta \in f(y)$ if and only if there exists $y^* \in GE(E,T[\kappa])$ such that y* is D'-controlled/C ∧ β is the standard code of $\gamma(D',y^*)$.

Suppose $x \in GE(E,T[\kappa])$. Since x is D-controlled/C', x is D-controlled/C. Hence the standard tuple code of $\gamma(D,x)$ lies in f(x). Hence the standard tuple code of $\gamma(D',y)$ lies in f(y). Let $y^* \in GE(E,T[\kappa])$, y* is D'-controlled/C, and the standard tuple code of $\gamma(D',y)$ equals the standard tuple code of $\gamma(D',y^*)$. Then $\gamma(D',y) = \gamma(D',y^*)$. By the previous claim, $y \in GE(E,T[\kappa]) \leftrightarrow y^* \in GE(E,T[\kappa])$. Hence $y \in GE(E,T[\kappa])$.

Conversely, suppose $y \in GE(E,T[\kappa])$. Argue as in the previous paragraph, that $x \in GE(E,T[\kappa])$, by switching x with y. QED

We fix $C' \subseteq C$ given by Lemma 3.5.5, and write $C' = \{\lambda_1 < \lambda_2 < \ldots\}$.

We now capture the needed properties of our T[κ], GE(E,T[κ]), and C', into a single structure with basic internal properties. In so doing, we will be removing all mention of κ and C' ⊆ C ⊆ κ. Recall the data that we fixed at the outset of this exotic proof. Dimension $k \geq 2$, finite $E \subseteq Q(0,1)^k$, and $A = \{q_1 < ... < q_n = 1\}$, $q_1 > 0$.

We use λ_n for the n-th element of $C \subseteq \lambda$, counting from 1.

LEMMA 3.5.6. There is a structure $M = (D,<_D,P,0,d_1,...,d_n)$, such that the following holds.
i. $<_D$ is a dense linear ordering on D with the left endpoint 0 and the right endpoint d_n.
ii. $0 <_D d_1 <_D ... <_D d_n$.
iii. P is a k-ary relation on D.
iv. P is an emulator of E in the following sense. Every pair from P (as a 2k-tuple) is order equivalent to a pair from E (using $<_D$).
v. P is point maximal in the following sense. If any k-tuple is added to P, then we no longer have an emulator of E.
vi. P is invariant under the binary relation $R_k(\{d_1,...,d_n\})$ on k-tuples.
In iv, we do not use E directly, but only a finite list of order types of its elements (k-tuples).
In vi, $R_k(\{d_1,...,d_n\})((x_1,...,x_k),(y_1,...,y_k)) \leftrightarrow$ $(x_1,...,x_k),(y_1,...,y_k)$ are order equivalent (using $<_D$) ∧ if $x_i \neq y_i$ then all $x_j \geq_D x_i$ and all $y_j \geq_D y_i$ are among $d_1,...,d_n$.

Proof: Take $D = \{x: x \leq_{T[k]} (\lambda_n,0)\}$. Take $<_D = <_{T[\kappa]} \cap D^2$. Take $P = GE(E,T[\kappa]) \cap D^k$. Take $0 = (0,0)$. Take each $d_i = (\lambda_i,0)$. Note that D is also $\{x: x <^* (\lambda_n,0)\}$. In addition, $D^k = T[\kappa]^k \cap \{x: x \leq^{**} (d_n,...,d_n)\}$ because $<^{**}$ orders first according to max, using $<^*$, and then lexicographically using $<^*$. So since P is the initial segment of the greedy T[κ]-emulator of E, GE(E,T[κ]), up through D^k, in the sense of $<^{**}$, it is clear that P is a maximal emulator of E in the sense required by iii,iv,v. Also the relation $R_k(\{d_1,...,d_n\})$ defined here is $R_k(C',T[\kappa]) \cap D^{2k}$. By Lemma 3.5.5, GE(E,T[κ]) is invariant under the relation $R_k(C',T[\kappa]) \subseteq T[\kappa]^k \times T[\kappa]^k$. Hence P is invariant under the relation $R_k(\{d_1,...,d_n\})$. QED

In Lemma 3.5.6, we have not yet reached countability, as λ_n is generally uncountable.

LEMMA 3.5.7. The structure M given by Lemma 3.5.6 can be taken to be countable.

Proof: By an obvious sequential construction. The need for a sequential argument arises from density in i, and point maximality in v. The rest of the conditions take care of themselves. Construct finite sets $B_1,B_2,\ldots \subseteq D$ as follows. Take $B_1 = \{0,d_1,\ldots,d_n\}$. Suppose finite B_i has been defined. For each pair of distinct elements of B_i, put an intermediate element of D in B_{i+1}. For each $x \in B_i^k$ such that $P \cup \{x\}$ is not an emulator of E in the sense of iv, choose $y \in P \cup \{x\}$ such that (x,y) is not order equivalent to an element of E^2 (using $<_D$ and numerical <). Then take M restricted, in the obvious sense, to the countable set $\cup_i B_i$. QED

We now fix countable $M = (D,<_D,P,0,d_1,\ldots,d_n)$ as given by Lemma 3.5.7.

LEMMA 3.5.8. M is isomorphic to a system $(Q[0,1],<,S,0,q_1,\ldots,q_{n-1},1)$, where S is a maximal emulator of $E \subseteq Q[0,1]^k$ in the usual sense used in MESU/2, and S is invariant under the equivalence relation $R_k(\{q_1,\ldots,q_n\})$ on $Q[0,1]^k$.

Proof: Let $h:D \to Q[0,1]$ be any isomorphism from $(D,<_D,0,d_1,\ldots,d_n)$ onto $(Q[0,1],<,0,q_1,\ldots,q_{n-1},1)$. This is clear from well known facts about countable dense linear orderings with endpoints, $0,d_n$ are the left/right endpoints of the first system and 0,1 are the left/right endpoints of the second system. Then h is an isomorphism from $M = (D,<_D,P,0,d_1,\ldots,d_n)$ onto $(Q[0,1],<,h[P],0,q_1,\ldots,q_{n-1},1)$, where h is the image of P acting coordinatewise. It is easy to see that the properties i-vi in Lemma 3.5.8 are preserved under the isomorphism h. Hence h[P] is a maximal emulator of $E \subseteq Q[0,1]^k$ which is $R_k(\{q_1,\ldots,q_{n-1},1\}) = R_k(\{q_1,\ldots,q_n\})$ invariant. QED

THEOREM 3.5.9. MESU/2 for dimension k = 2 is provable in ZFC. In fact, Z and even Z_3 suffices.

Proof: The entire proof through Lemma 3.5.9 uses only that κ is an uncountable regular cardinal, except for the second part of the proof of Lemma 3.5.5. But if k = 2 then the second part of the proof of Lemma 3.5.5 is not needed (it uses $k \geq 3$). For $k = 2$, we can set $\kappa = \omega_1$, as the actual transfinite ordinal ω_1 is available in ZF, and the countable axiom of choice can be used to prove that ω_1 is regular (regularity of κ was used in the proof).

However, we can avoid any use of the axiom of choice, and even stay within Z. We can achieve this using a truncated version of Gödel's constructible hierarchy known to be available in Z. We can build an initial segment of the constructible hierarchy in a well known coded fashion, cut off so that there are exactly three internal infinite cardinalities. This well known kind of construction does not use the axiom of choice or replacement, and the second internal infinite cardinal can serve as the $\kappa = \omega_1$ for the argument. With some additional care, we can in fact stay within Z_3. QED

THEOREM 3.5.10. The following hold.
i. MESU/1,2,3, MED/1,2,3 are provable in SRP^+ but not in SRP (assuming SRP is consistent).
ii. MESU/1,2.3, MED/1,2,3 are provably equivalent to Con(SRP) over WKL_0.
iii. MESU/1,2,3, MED/1,2,3 are not provable in ZFC (or SRP), assuming ZFC (or SRP) is consistent.
iv. MESU/1,2,3, MED/1,2,3 are independent of ZFC (and even SRP), assuming SRP is 1-consistent.

Proof: We proved Lemma 3.5.10 by using a (k-2)-subtle cardinal for all $k \geq 3$ (only ZFC for $k = 2$ and RCA_0 for $k = 1$). So it is clear that we have given a proof of MESU/2 in SRP^+. Now apply Theorem 3.4.8 to see that we have proved all six statements in SRP^+. Now suppose that any of the six are provable in SRP. Then by Theorem 3.4.8, MED/1 is provable in SRP. By Theorem 3.4.11, SRP then proves Con(SRP), and so by Gödel's Second Incompleteness Theorem, SRP is inconsistent.

For ii, we first argue in WKL_0 + Con(SRP). By Corollary 3.1.7, let $\varphi(k)$ be a Π^0_1 formula such that WKL_0 proves $(\forall k)((\text{MESU/2 for dimension } k) \leftrightarrow \varphi(k))$ is provable in WKL_0, and display such a proof. According to the way Lemma 3.5.8 was proved, we have that for all $k \geq 1$, ZFC + "there exists a k-subtle cardinal" proves MESU/2 for dimension k (actually much better than this). Hence for all $k \geq 1$, ZFC + "there exists a k-subtle cardinal" proves $\varphi(k^*)$, where k^*

is the usual closed term for k. Hence, using Con(SRP), we have (∀k)(φ(k)). From the displayed proof, we derive (∀k)(MESU/2 for dimension k), which is MESU/2. Thus we have derived MESU/2 in WKL_0 + Con(SRP). Now apply Theorem 3.4.8.

For the other direction of ii, we have RCA_0 + MED/1 proves Con(SRP) by Theorem 3.4.11. Now apply Lemma 3.4.8.

For iii, if any of the six is provable in ZFC (or SRP), then Con(SRP) is provable in ZFC (or SRP), and so by Gödel's Second Incompleteness Theorem, ZFC (or SRP) is inconsistent.

For iv, suppose SRP refutes one of the six statements. Then SRP refutes Con(SRP), and so SRP is not 1-consistent. QED

THEOREM 3.5.13. The following are provable in EFA.
i. MESU/1,2,3, MED/1,2,3 for any fixed dimension k is provable in SRP.
ii. ZFC proves that for all k ≥ 1, if there is a max(k-1,0)-subtle cardinal then MESU/1 holds for dimension k and MESU/2,3, MED/1,2,3 holds for dimension k+1.
iii. For all sufficiently large k ≥ 1, MESU/1,2,3, MED/1,2,3 for dimension k are not provable in ZFC, assuming ZFC is consistent.
iv. For all sufficiently large k ≥ 1, MESU/1,2,3, MED/1,2,3 for dimension k are independent of ZFC, or even any SRP[n] fixed in advance, assuming SRP is consistent.

Proof: For i, let k ≥ 1. By the way Lemma 3.5.10 was proved, we proved the six statements in dimension k over ZFC using a k-subtle cardinal (and much better), and so we stayed within SRP.

For ii, let k ≥ 2. if there is a (k-1)-subtle cardinal then MESU/2 holds for dimension k+1 (since k+1 ≥ 3). Now apply Theorem 3.4.8. And for k = 1, we proved MESU/2 for dimension 2 in a weak fragment of ZFC (see Theorem 3.5.11). Again apply Theorem 3.4.8.

For iii, by Theorem 3.4.10 and the last claim of Theorem 3.4.11, MED/1 for sufficiently large dimension proves the consistency of ZFC over RCA_0. By Theorem 3.4.8, this is true for MESU/1,2,3, MED/1,2,3,. Hence we have unprovability in ZFC in sufficiently large dimension, assuming ZFC is consistent.

For iv, let n ≥ 1. by Theorem 3.4.10 and the last claim of Theorem 3.4.11, MED/1 for sufficiently large dimension proves the consistency of SRP[n] over RCA_0. By Theorem 3.4.8, this is true for MESU/1,2,3, MED/1,2,3. Hence we have unprovability in SRP[n], in sufficiently large dimension, if SRP is consistent. We also cannot have refutability in any given fixed SRP[n], since we have provability in SRP, assuming SRP is consistent. QED

3.6. r-EMULATION

Recall the general Emulation definitions, ME/DEF/1,2 of section 2. Also the specific versions in Definitions 3.1.2, 3.1.3 for M = (Q[0,1],<). Notice the exponent 2 in all of these Definitions, indicating pairs of k-tuples. Here we introduce the very natural parameter r ≥ 1, which we have spared the reader from thus far.

DEFINITION 3.6.1. S is an r-emulator of $E \subseteq M^k$ if and only if $S \subseteq M^k$ and every element of S^r is M equivalent to an element of E^r. S is a maximal r-emulator of $E \subseteq M^k$ if and only if S is an r-emulator of $E \subseteq M^k$ which is not a proper subset of an r-emulator of $E \subseteq M^k$.

DEFINITION 3.6.2. S is an r-emulator of $E \subseteq Q[0,1]^k$ if and only if $S \subseteq Q[0,1]^k$ and every element of S^r is order equivalent to an element of E^r. S is a maximal r-emulator of $E \subseteq Q[0,1]^k$ if and only if S is an r-emulator of $E \subseteq Q[0,1]^k$ which is not a proper subset of an r-emulator of $E \subseteq Q[0,1]^k$.

Note that the elements of S^r,E^r are rk-tuples. Also note that (maximal) emulators are just the (maximal) 2-emulators.

The r-emulators give rise to a strengthened notion of usability. General usability was defined in section 2 as MEU/DEF, MEIU/DEF. Usability for M = (Q[0,1],<) was defined in Definition 3.1.5.

DEFINITION 3.6.3. $R \subseteq M^k \times M^k$ is ME usable* if and only if for all subsets of M^k and r ≥ 1, some maximal r-emulator contains its R image. $R \subseteq M^k \times M^k$ is ME invariantly usable* if and only if for all subsets of M^k and r ≥ 1, some maximal r-emulator is R invariant.

DEFINITION 3.6.4. $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable* if and only if for finite subsets of $Q[0,1]^k$ and $r \geq 1$, some maximal r-emulator contains its R image. $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME invariantly usable* if and only if for finite subsets of $Q[0,1]^k$ and $r \geq 1$, some maximal r-emulator is R invariant.

Most of what we have said about emulators, maximal emulators, ME usable, ME invariantly usable, lifts straightforwardly to r-emulators, maximal r-emulators, ME usable*, and ME invariantly usable*. Here we restate the starred form of the previous statements, and indicate their resulting status.

FROM SECTION 2

MAXIMAL EMULATION*/1. ME*/1. Every $E \subseteq M^k$ has a maximal r-emulator.

MAXIMAL EMULATION*/2. ME*/2. Every $E \subseteq M^k$ has a maximal r-emulator containing any given r-emulator.

FINITE SUBSET EMULATION*. Assuming M has finitely many components, every $E \subseteq M^k$ is an r-emulator of some finite $E' \subseteq M^k$ with $E' \subseteq E$. This is provable in RCA_0 for countable M with finitely many components.

EMULATION TRANSITIVITY*. If S is an r-emulator of $E \subseteq M^k$ and E is an r-emulator of $E' \subseteq M^k$, then S is an r-emulator of $E' \subseteq M^k$. Let E be an r-emulator of $E' \subseteq M^k$ and E' be an r-emulator of $E \subseteq M^k$. The r-emulators of $E \subseteq M^k$ are the same as the r-emulators of $E' \subseteq M^k$. The maximal r-emulators of $E \subseteq M^k$ are the same as the maximal r-emulators of $E' \subseteq M^k$. This is provable in RCA_0 for countable M.

MAXIMAL EMULATION*/3. ME*/3. (RCA_0) Let M have countable domain and finitely many components. Every subset of M^k has a maximal r-emulator. The following are equivalent.
i. ACA_0.
ii. Every subset of M^k has a maximal r-emulator containing any given r-emulator.
iii. In every equivalence relation M on N, every finite subset of D^2 has a maximal r-emulator containing any given

r-emulator.

THEOREM 2.2*. (RCA_0) Let M have finitely many components. R $\subseteq M^k \times M^k$ is ME usable* if and only if for all finite subsets of M^k and $r \geq 1$, some maximal r-emulator contains its R image.

MAXIMAL EMULATION*/4. ME*/4. If $R \subseteq M^k \times M^k$ is ME usable* then R is M preserving in the sense that $(\forall x,y)(R(x,y) \rightarrow$ x,y are M equivalent).

THEOREM 2.4*. $R \subseteq M^k \times M^k$ is ME invariantly usable* if and only if $R \cup R^{-1}$ is ME usable*. If $R \subseteq M^k \times M^k$ is symmetric then R is ME invariantly usable* if and only if R is ME usable*.

THEOREM 2.5*. (RCA_0) Let $x,y \in M^k$. The following are equivalent.
i. For finite subsets of M^k, some maximal r-emulator is equivalent at $x,y \in M^k$.
ii. $\{(x,y)\}$ is ME invariantly usable*.
iii. $\{(x,y),(y,x)\}$ is ME usable*.

All of the proofs of the above in section 2 go through without modification with the exception of iii $\rightarrow$ i in ME*/3, which we have not thought through.

FROM SECTION 3.1

THEOREM 3.1.2*. (RCA_0) Every $E \subseteq Q[0,1]^k$ is an r-emulator of a finite subset. E has a recursive maximal r-emulator.

LEMMA 3.1.5*. Same as Lemma 3.1.5 with $k,r \geq 1$ fixed, and emulator replaced by r-emulator.

THEOREM 3.1.6*. (EFA) Consider the statement $\varphi(k,r,E,R)$ = "For finite $E \subseteq Q[0,1]^k$, some maximal r-emulator S of $E \subseteq Q[0,1]^k$ has $R[S] \subseteq S$".
i. If k,r,E,R are fixed in advance, where $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then $\varphi(k,r,E,R)$ is implicitly Π^0_1 over WKL_0.
ii. If k,r,R are fixed in advance, where $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then $(\forall E \subseteq Q[0,1]^k)(\varphi(k,r,E,R))$ is implicitly Π^0_1 over WKL_0.

iii. If k,R are fixed in advance, where $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then $(\forall E \subseteq Q[0,1]k)(\forall r)(\varphi(k,r,E,R))$ is implicitly Π^0_1 over WKL_0. Furthermore, the associated Π^0_1 forms and the equivalence proofs in WKL_0 can be constructed effectively from fixed parameters in a way that RCA_0 can verify.

COROLLARY 3.1.7*. (EFA) Consider the statement $\varphi(k,R)$ = "$R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME* usable". If k,R are fixed in advance, where $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is order theoretic, then $\varphi(k,R)$ is implicitly Π^0_1 over WKL_0

All of the proofs of the above in section 3.1 go through without modification.

FROM SECTION 3.2

MAXIMAL EMULATION* NECESSARY USE. MENU*. If $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable* then R is order preserving.

MAXIMAL EMULATION FINITE USE*/1. MEFU*/1. Any finite order preserving $R \subseteq Q(0,1)^k \times Q(0,1)^k$ is ME usable*.

MAXIMAL EMULATION FINITE USE*/2. MEFU*/2. Any finite order preserving $R \subseteq Q(0,1]^k \times Q(0,1]^k$ is ME usable*.

MAXIMAL EMULATION FINITE USE*/3. MEFU*/3. Any finite order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ not altering both of 0,1 is ME usable*.

MAXIMAL EMULATION SINGLETON USE*/1. MEOU*/1. Any order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ of cardinality 1 is ME usable*.

All of the proofs of the above in section 3.1 go through without modification.

MEOU/2 needs to be reconsidered for r-emulators.

FROM SECTION 3.3

LEMMA 3.3.1*. (RCA_0) The maximal r-emulators of $E \subseteq Q[0,1]$, $|E| < r$, are the $S \subseteq Q[0,1]$, $|S| = |E|$. The maximal r-emulator of $E \subseteq Q[0,1]$, $|E| \geq r$, is just $Q[0,1]$.

MAXIMAL EMULATION LARGE USE*/1. MELU*/1. $R \subseteq Q[0,1] \times Q[0,1]$ is ME usable* if and only if for all n there is an n element subset of Q[0,1] containing its R image.

Proof: In RCA_0. Let $R \subseteq Q[0,1] \times Q[0,1]$ be ME usable*. Let $|E| = n$ and S be a maximal (n+1)-emulator of E containing its R image. Then $|S| = n$ contains its R image. Conversely, suppose that for all n there is an n element subset of Q[0,1] containing its R image. Let $E \subseteq Q[0,1]$ be finite and $r \geq 1$. If $|E| \geq r$ then E has the maximal r-emulator Q[0,1] containing its R image. If $|E| < r$ then E has the maximal r-emulator S containing its R image, where $|S| = |E|$ and S contains its R image. QED

MAXIMAL EMULATION LARGE USE*/2. MELU*/2. $Q(0,1)^{2<} \times Q(0,1)^{2<}$ is not ME usable*. It is order preserving, order theoretic, and 0,1 are not present.

MAXIMAL EMULATION LARGE USE*/4. MELU*/4. For $k \geq 3$, $Q[1/3,1/2]^{k<} \times Q[1/3,1/2]^{k<}$ is not ME usable*.

All of the proofs of the above in section 3.3 go through without modification.

MELU/3 needs to be reconsidered for r-emulators.

FROM SECTIONS 3.4, 3.5

Most importantly, we now come to MESU/1,2,3 and MED/1,2,3.

THEOREM 3.4.1*. Same as Theorem 3.4.1 with usable replaced by usable*.

MAXIMAL EMULATION SMALL USE*/1. MESU*/1. The lower parameterization of any order preserving finite $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable*.

MAXIMAL EMULATION SMALL USE*/2. MESU*/2. For finite $A \subseteq Q(0,1]$, $R_k(A) \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable*.

MAXIMAL EMULATION SMALL USE*/3. MESU*/3. Every order preserving $R \subseteq Q[0,1]^k \times Q[0,1]^k$ with the Finiteness Condition, not altering 0, is ME usable*.

MAXIMAL EMULATION DROP*/1. MED*/1. For finite subsets of

$Q[0,1]^k$, some maximal r-emulator is drop equivalent at (1,1/2,...,1/k),(1/2,...,1/k,1/k).

MAXIMAL EMULATION DROP*/2. MED*/2. Let $x,y \in Q[0,1]^k$. The following are equivalent.
i. For finite subsets of $Q[0,1]^k$, some maximal r-emulator is drop equivalent at x,y.
ii. x,y are droppable or $x_k = y_k = 0$.

MAXIMAL EMULATION DROP*/3. MED*/3. Let $x_1,y_1,...,x_n,y_n \in Q[0,1]^k$. The following are equivalent.
i. For finite subsets of $Q[0,1]^k$, some maximal r-emulator is drop equivalent at every x_i,y_i.
ii. For finite subsets of $Q[0,1]^k$ and $1 \leq i \leq k$, some maximal r-emulator is drop equivalent at (x_i,y_i).
iii. For all i, x_i,y_i is droppable or $(x_i)_k = (y_i)_k = 0$.

All of the proofs of the results concerning the above statements in sections 3.4, 3.5 go through without modification. This includes the Exotic Proof of section 3.5.

We conjecture that MESU/2 for k = 2 is provable in RCA_0. However, we conjecture that none of MESU*/1,2,3, MED*/1,2,3 for dimension k = 2 is provable in ZFC\P or Z_2. We also conjecture that none of MESU*/1,2,3, MED*/1,2,3 for dimension k = 3 is provable in ZFC. In fact, we conjecture that for each $k \geq 3$, MESU*/1 for dimension k-1 and MESU*/2,3, MED*/1,2,3 for dimension k is provable equivalent to Con(ZFC + "there exists a (k-2)-subtle cardinal").

4. GENERAL CONJECTURES

Here we discuss some General Conjectures in Basic Emulation Theory on Q[0,1] which do not specifically pertain to the statements discussed in section 3.

GENERAL CONJECTURE 1. GC1. There is an algorithm for determining whether a given order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is usable. For inputs, use a standardly digitized form of quantifier free formulas over (Q[0,1],<) with parameters.

GC1 is trivial for fixed dimension k = 1 by MELU/1. We have not proved GC1 even for fixed dimension k = 2. In fact, we have no significant results about GC1.

The following sharper form of GC1 would not seem to be different than GC1 in any significant way.

GENERAL CONJECTURE 2. GC2. There is a Turing machine with at most $2^{2^{\wedge}1000}$ states/symbols each, that determines whether a given order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ is ME usable. For inputs, use a standardly digitized form of quantifier free formulas over (Q[0,1],<) with parameters.

Here $2^{2^{\wedge}1000}$ is merely a simply described ridiculously large number of states and symbols for any actual algorithm.

We will show below that GC2 is not provable in ZFC, or even in SRP, assuming SRP is consistent. Establishing this unprovability result for GC1 seems to require a new idea, and the unprovability may well be false.

Our result that GC2 is not provable in ZFC (assuming Con(SRP)) does tell us that, in a sense, ZFC (or even SRP) is not sufficient to analyze the ME usability of order theoretic R. There is another sense in which we know that ZFC (or SRP) is not sufficient to analyze this. What is different about the situation with GC2 is that a particular conjecture (GC2) concerning the nature of the ME usability of order theoretic relations is shown to be unprovable in ZFC (or SRP).

This other sense that we know ZFC (or SRP) is not sufficient to analyze the ME usability of order theoretic R is as follows.

LEMMA 4.1. (EFA) Let T be a recursively axiomatized first order system that interprets EFA. Let φ be a (interpreted) Σ^0_1 sentence such that T + φ proves Con(T). Then T refutes φ.

Proof: This is well known. Suppose T + φ proves Con(T). Since T + φ proves "T proves φ", we have that T + φ proves Con(T + φ). So by Gödel's Second Incompleteness Theorem, T + φ is inconsistent, and T refutes φ. QED

THEOREM 4.2. (EFA) There is no algorithm α such that ZFC (or even SRP) proves that α correctly decides whether or not a given order theoretic relation is ME usable -

assuming SRP is consistent. We do not need the hypothesis that α always returns with an answer.

Proof: Let α be such an algorithm, where α is proved to be correct in SRP. Let n be such that this correctness is proved in SRP[n]. By Theorem 3.4.11, let m be such that MED/1 for dimension m provably implies Con(SRP[n]) over RCA_0. Clearly SRP[n] proves "if α returns that MED/1 holds in dimension m then MED/1 holds in dimension m and so Con(SRP[n])". By Lemma 4.1, SRP[n] proves "it is not the case that α returns that MED/1 holds in dimension m". Hence SRP[n] proves "α returns that MED/1 fails in dimension m". Hence SRP[n] proves that MED/1 fails in dimension m. But SRP proves MED/1. Hence SRP is inconsistent.

Instead of using MED/1 here, we can use the sharper MESU/2. QED

GENERAL CONJECTURE 3. GC3. For every order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$, the statement "R is ME usable" is either provable in SRP or refutable in RCA_0.

THEOREM 4.3. (EFA)
i. GC2 → GC1.
ii. Assume Con(SRP). GC3 → GC2.
iii. GC3 is false for any SRP[n], assuming Con(SRP).

Proof: i is trivial. For ii, assume Con(SRP). Suppose GC3. We use the algorithm that searches for a proof in SRP or a refutation in RCA_0 of "R is ME usable". We only find one of these since Con(SRP). This algorithm can be given by a small enough Turing machine. Now assume Con(SRP) and GC3 holds for SRP[n]. By Theorem 3.4.11, let m be such that MED/1 (or MESU/2) for dimension m provably implies Con(SRP[n]) over RCA_0. Then SRP[n] proves or refutes Con(SRP[n]). The former case violates Con(SRP). The latter case also violates Con(SRP) since SRP proves Con(SRP[n]). QED

We may be very wrong about GC1,2,3, and obviously as the dimension k rises we feel less confident. Here is a weak form of Conjecture 3 that we have more confidence in than GC1.

GENERAL CONJECTURE 4. GC4. Let k be least such that there is an order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ for which the statement "R is ME usable" is independent of ZFC. Then for all order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$, the statement "R is ME usable" is provable or refutable in SRP.

What can we say about this least k? Nothing now except $k \geq 2$. But we will still venture a guess.

GENERAL CONJECTURE 5. GC5. Let k (k') be least such that there is an order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ for which the statement "R is ME usable" is independent of ZFC (ZFC\P). Then $2 < k' < k \leq 8$.

It is natural to modify the conjectures GC1-5 to GC1*-5*, where we replace usable by usable*. Here we greatly strengthen GC5.

GENERAL CONJECTURE 6. GC6. Let k (k') be least such that there is an order theoretic $R \subseteq Q[0,1]^k \times Q[0,1]^k$ for which the statement "R is ME usable*" is independent of ZFC (ZFC\P). Then k = 3 and k' = 2.

We now show that GC2 is not provable in ZFC, assuming Con(SRP), as promised. The same proof works for GC2*, the version with usable*. We first prove a very general non provability result.

LEMMA 4.4. Let $T + \{\varphi_1,\varphi_2,...\}$ be any consistent theory extending EFA which is not derivable from any $T + \{\varphi_1,...,\varphi_i\}$. Then for no $n \geq 1$ does T prove "there exists a Turing machine with at most n states/symbols each that determines whether any given sentence φ_i is true".

Proof: Let $T,\varphi_1,\varphi_2,...$ and n be as given. For each TM with at most n states/symbols each, let α(TM) be the least i for which α(TM) returns "false" (if i exists). Let r be the maximum of these numbers α(TM). Note that r depends on n. We claim that there exists s > r that the theory $T + \{\varphi_1,...,\varphi_s,\neg\varphi_{s+1}\}$ is consistent. For otherwise, $T + \{\varphi_1,...,\varphi_{r+1}\}$ derives $\varphi_{r+2},\varphi_{r+3},...$, contrary to the hypothesis. But obviously $T + \{\varphi_1,...,\varphi_s,\neg\varphi_{s+1}\}$ refutes "TM determines whether any given sentence φ_i is true" for any of these TM's, as they yield a smaller break point or no break point at all. So it is consistent with T that all of these TM's fail to do their task. QED

LEMMA 4.5. Assume Con(SRP). For each $k \geq 1$, let φ_k be MED/1 (or MESU/2) for dimension k. For all $n \geq 1$, SRP[n] + $\{\varphi_1,\varphi_2,...\}$ is a consistent theory which is not derivable from any ZFC + $\{\varphi_1,...,\varphi_i\}$.

Proof: These theories are fragments of SRP. Again use Theorem 3.4.11 and Gödel's Second Incompleteness Theorem. QED

THEOREM 4.6. Assume Con(SRP). GC2 is not provable in ZFC, or even in SRP. The same holds for GC2*.

Proof: Suppose SRP proves GC2. Let n be such that SRP[n] proves GC2. Then apply Lemmas 4.4 and 4.5, setting $n = 2^{2\wedge 1000}$, to obtain a contradiction. The same argument works for GC2*. QED

APPENDIX A
THE STATIONARY RAMSEY PROPERTY

reprinted from [Fr14]

All results in this section are taken from [Fr01]. All of these results, with the exception of Theorem 9.1.1, iv $\leftrightarrow$ v $\rightarrow$ vi, are credited in [Fr01] to James Baumgartner. Below, λ always denotes a limit ordinal.

DEFINITION A.1. We say that $C \subseteq \lambda$ is unbounded if and only for all $\alpha < \lambda$ there exists $\beta \in C$ such that $\beta \geq \alpha$.

DEFINITION A.2. We say that $C \subseteq \lambda$ is closed if and only if for all limit ordinals $x < \lambda$, if the sup of the elements of C below x is x, then $x \in C$.

DEFINITION A.3. We say that $A \subseteq \lambda$ is stationary if and only if it intersects every closed unbounded subset of λ.

DEFINITION A.4. For sets A, let S(A) be the set of all subsets of A. For integers $k \geq 1$, let $S_k(A)$ be the set of all k element subsets of A.

DEFINITION A.5. Let $k \geq 1$. We say that λ has the k-SRP if and only if for every $f:S_k(\lambda) \rightarrow \{0,1\}$, there exists a stationary $E \subseteq \lambda$ such that f is constant on $S_k(E)$. Here SRP stands for "stationary Ramsey property."

The k-SRP is a particularly simple large cardinal property. To put it in perspective, the existence of an ordinal with the 2-SRP is stronger than the existence of higher order indescribable cardinals, which is stronger than the existence of weakly compact cardinals, which is stronger than the existence of cardinals which are, for all k, strongly k-Mahlo (see Theorem A.1 below, and [Fr01], Lemma 1.11).

Our main results are stated in terms of the stationary Ramsey property. In particular, we use the following extensions of ZFC based on the SRP.

DEFINITION A.6. SRP^+ = ZFC + "for all k there exists an ordinal with the k-SRP". SRP = ZFC + {there exists an ordinal with the k-SRP}$_k$. We also use SRP[k] for the formal system ZFC + $(\exists\lambda)(\lambda$ has the k-SRP).

For technical reasons, we will need to consider some large cardinal properties that rely on regressive functions.

DEFINITION A.7. We say that $f:S_k(\lambda) \to \lambda$ is regressive if and only if for all $A \in S_k(\lambda)$,if $\min(A) > 0$ then $f(A) < \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, $f(B) = f(C)$.

DEFINITION A.8. We say that $f:S_k(\lambda) \to S(\lambda)$ is regressive if and only if for all $A \in S_k(\lambda)$, $f(A) \subseteq \min(A)$. (We take $\min(\emptyset) = 0$, and so $f(\emptyset) = \emptyset$). We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, we have $f(B) \cap \min(B \cup C) = f(C) \cap \min(B \cup C)$.

DEFINITION A.9. Let $k \geq 1$. We say that α is purely k-subtle if and only if
i) α is an ordinal;
ii) For all regressive $f:S_k(\alpha) \to \alpha$, there exists $A \in S_{k+1}(\alpha\backslash\{0,1\})$ such that f is constant on $S_k(A)$.

DEFINITION A.10. We say that λ is k-subtle if and only if for all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \in S_{k+1}(C)$.

DEFINITION A.11. We say that λ is k-almost ineffable if and only if for all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \subseteq \lambda$ of cardinality λ.

DEFINITION A.12. We say that λ is k-ineffable if and only if for all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous stationary $A \subseteq \lambda$.

THEOREM A.1. Let $k \geq 2$. Each of the following implies the next, over ZFC.
i. there exists an ordinal with the k-SRP.
ii. there exists a (k-1)-ineffable ordinal.
iii. there exists a (k-1)-almost ineffable ordinal.
iv. there exists a (k-1)-subtle ordinal.
v. there exists a purely k-subtle ordinal.
vi. there exists an ordinal with the (k-1)-SRP.
Furthermore, i,ii are equivalent, and iv,v are equivalent. There are no other equivalences. ZFC proves that the least ordinal with properties i - vi (whichever exist) form a decreasing (≥) sequence of uncountable cardinals, with equality between i,ii, equality between iv,v, and strict inequality for the remaining consecutive pairs.

Proof: i ↔ ii is from [Fr01], Theorem 1.28, iv ↔ v is from [Fr01], Corollary 2.17. The strict implications ii → iii → iv → vi are from [Fr01], Theorem 1.28. Same references apply for comparing the least ordinals. QED

DEFINITION A.13. We follow the convention that for integers $p \leq 0$, a p-subtle, p-almost ineffable, p-ineffable ordinal is a limit ordinal, and that the ordinals that are 0-subtle, 0-almost ineffable, 0-ineffable, or have the 0-SRP, are exactly the limit ordinals. An ordinal is called subtle, almost ineffable, ineffable, if and only if it is 1-subtle, 1-almost ineffable, 1-ineffable.

APPENDIX B
FORMAL SYSTEMS USED

PFA Polynomial function arithmetic. Based on 0, successor, addition, multiplication, and bounded induction. Same as $I\Sigma_0$, (HP93], p. 29, 405.

EFA Exponential function arithmetic. Based on 0, successor, addition, multiplication, exponentiation and bounded induction. Same as $I\Sigma_0(\exp)$, [HP93], p. 37, 405.

RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].

WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].

ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics. [Si99,09].

ACA' Arithmetic comprehension axiom prime. ACA_0 together with "for all $n < \omega$ and $x \subseteq \omega$, the n-th Turing jump of x exists".

Z_2 Second order arithmetic as a two sorted first order theory. [Si99,09].

Z_3 Third order arithmetic as a three sorted first order theory. Extends Z_2 with a new sort for sets of subsets of ω.

Z(C) Zermelo set theory (with the axiom of choice). This is the same as ZF(C) without the axiom scheme of replacement.

ZF(C)\P ZF(C) without the power set axiom. [Ka94]

ZF(C) Zermelo Frankel set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Ka94].

SRP[k] ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), for fixed k. Appendix A.

SRP ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. Appendix A.

SRP^+ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). Appendix A.

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