INTERPRETING SET THEORY IN ORDINARY THINKING: CONCEPT CALCULUS

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Lecture 3
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1. Interpretation hierarchy (lecture #1).
2. The general nature of concept calculus.
3. Better than, much better than.
4. Single varying quantity.
5. Single varying bit.
6. Persistently varying bit.
7. Some models.

1. INTERPRETATION HIERARCHY (Lecture #1).

The first of these Tarski lectures was devoted to a discussion of Tarski interpretability. This bears repeating.

STARTLING OBSERVATION. *Any two natural theories S,T, known to interpret EFA, are known (with small numbers of exceptions) to have: S is interpretable in T or T is interpretable in S. The exceptions are believed to also have comparability.*

Exceptions to linearity are known, which have clearly identifiable elements of artificiality. It would be interesting to explore just how natural one can (artificially!) make the incomparability.

Because of this observation, there has emerged a rather large linearly ordered table of “interpretation powers” represented by natural formal systems. Generally, several natural formal systems may occupy the same position.

We call this growing table, the *Interpretation Hierarchy*.

2. THE GENERAL NATURE OF CONCEPT CALCULUS.

Concept Calculus is a new mathematical/philosophical program of wide scope. The development of Concept Calculus began in Summer, 2006. There is a report in [1].

Concept Calculus promises to connect mathematics, philosophy, and commonsense thinking in a radically new way.

Advances in Concept Calculus are made through rigorous mathematical findings, and promise to be of immediate and growing interest to philosophers.

Developments in Concept Calculus generally consist of the following.

a. An identification of a few related concepts from informal thinking. In the various developments, the choice of these concepts will vary greatly. In fact, all concepts from ordinary language are prime targets.

b. Formulation of a variety of fundamental principles involving these concepts. These various principles may have various degrees of plausibility, and may even be incompatible with each other. There may be no agreement among philosophers as to just which principles to accept. Concept Calculus is concerned only with logical structure.

c. Formulation of a variety of systems of such fundamental principles in b. These systems generally combine several such fundamental principles in some attractive way.

d. An identification of the location of the theory in the interpretation hierarchy.

e. In particular, for each of the resulting systems, a determination of whether they interpret mathematics - as formalized by ZFC.

A system T having interpretation power at least that of mathematics (ZFC) has special significance. This means that

if T is without contradiction then mathematics (ZFC) is without contradiction.

I.e., we have a consistency proof for mathematics (ZFC) relative to that of T.

Furthermore, relative consistency proofs arising this way are generally very finitary.

AMBITION:

Newton/Leibniz calculus:
Science and engineering.

=

Concept Calculus:
Everything else.

3. BETTER THAN, MUCH BETTER THAN.

We begin with the notions: better than (>), and much better than (>>). These are binary relations. The passage of > to >> is an example of what we call concept amplification.

One can also view > and >> mereologically, as

x > y iff y is a “proper part of x”.

x >> y iff y is a “small proper part of x”.

Define z is minimal if and only if (w)(z > w).

Define z is minimally > y   (w)(z > w   (y > w   y = w)).

The nonlogical axioms of T(>,>>,=)

BASIC. x > x, x > y y > z   x > z, x >> y   x > y. x >> y y > z   x >> z. x > y y >> z   x >> z. (x)(x >> y,z). x >> y   (z)(x >> z z is minimally > y).

MINIMAL (Political Axiom). There is nothing that is better than all minimal things.

EXISTENCE (Exact Bound, Plenitude). Let x be a thing better than a given range of things. There is something that is better than the given range of things and the things that they are better than, and nothing else. Here we use L(>,>>,=) to present the range of things.

Existence is like fusion. Here the “range of things” is given by a first order formula in L(>,>>,=) with parameters allowed.

AMPLIFICATION (Special Indiscernibility). Let y > x be given, as well as a true statement about x, using the binary relations >,= and the unary relation >> x. The corresponding statement about x, using >,= and >> y, is also true.

THEOREM 3.1. Basic + Minimal + Existence + Amplification is mutually interpretable with ZFC. This is provable in EFA.

AMPLIFIED LIMIT (Star). There is something that is better than something, and also much better than everything it is better than.

BINARY AMPLIFICATION. Let y > x be given, as well as a true statement about x, using the binary relations >,= and the binary relation z >> w >> x. The corresponding statement about x, using >,= and z >> w >> y, is also true.

Leads to much higher places in Interpretation Hierarchy than ZFC:

THEOREM 3.2. Basic + Minimal + Existence + Amplification + Amplified Limit interprets ZFC + “there is an almost ineffable cardinal” and is interpretable in ZFC + “there exists an ineffable cardinal”.

THEOREM 3.3. Basic + Minimal + Existence + Binary Amplification interprets ZFC + “there exists a Ramsey cardinal” and is interpretable in ZFC + “there exists a measurable cardinal”.

THEOREM 3.4. Basic + Minimal + Existence + Binary Amplification + Amplified Limit interprets ZFC + “there exists a measurable cardinal with arbitrarily large lesser measurable cardinals” and is interpretable in ZFC + “there exists a measurable cardinal with a normal measure 1 set of lesser measurable cardinals”.

There are also some results involving elementary embedding axioms.

4. SINGLE VARYING QUANTITY.

We now consider a single varying quantity - where the time and quantity scale are the same, and are linearly ordered.

This is common in ordinary physical science, where the time scale and the quantity scale may both be modeled as nonnegative real numbers.

The language has >,>>,=,F, where >,>> are binary relations, and F is a one place function.

F(x) is the value of the varying quantity at time x.

When thinking of time, >,>> is later than and much later than. When thinking of quantity, >,>> is greater than and much greater than.

BASIC. > is a linear ordering. x >> y □ x > y. x >> y > z □ x >> z. x > y >> z □ x >> z. (□x)(x >> y,z). x >> y □ (□z)(x >> z □ z is minimally > y).

ARBITRARY BOUNDED RANGES. Every bounded range of values is the range of values over some bounded interval. Here we use L(>,>>,=,F) to present the bounded range of values.

AMPLIFICATION. Let y > x be given, as well as a true statement about x, using F, binary >,= and unary >> x. The corresponding statement about x, using F, binary >,= and unary >> y is also true.

This also lands at ZFC in the Interpretation Hierarchy. We can strengthen as before:

AMPLIFIED LIMIT. There is something that is greater than something, and also much greater than everything it is greater than.

BINARY AMPLIFICATION. Let y > x be given, as well as a true statement about x, using F, binary >,= and binary z >> w >> x. The corresponding statement about y, using F, binary >,= and binary z >> w >> y, is also true.

As before, these latter two principles push the interpretation power well into the large cardinal hierarchy.

There are versions where we do not assume that the time scale is the same as the quantity scale. Some of these versions use two varying quantities, and there are three separate scales (time, first quantity, second quantity).

5. SINGLE VARYING BIT.

We now use a bit varying over time. Physically, this is like a flashing light. Mathematically, it corresponds to having a time scale with a unary predicate.

In order to get logical power out of this particularly elemental situation, we need to use forward translations of time. We think of b+c so that the amount of time from b to b+c is the same as the amount of time before c.

We use >,>>,=,+,P, where P(t) means that the varying bit at time t is 1.

Instead of a time scale, we can think of one dimensional space with a direction. P(t) means that there is a pointmass at position t.

In the earlier contexts, we did not support continuity. Here we simultaneous support discreteness and continuity.

BASIC. > is a linear ordering. x >> y □ x > y. x >> y > z □ x >> z. x > y >> z □ x >> z. (□x)(x >> y,z). x >> y □ (□z)(x >> z □ z > y).

BOUNDED TIME TRANSLATION. For every given range of times before a given time b, there exists a translation time c such that a time before b lies in the range of times if and only the bit at time b+c is 1. Here we use L(>,>>,=,+,P) to present the range of times.

The idea is our usual one: the behavior of P over bounded intervals is arbitrary, up to translation.

AMPLIFICATION. Let y > x be given, as well as a true statement about x, using P, binary >,= and unary >> x. The corresponding statement about x, using P, binary >,= and unary >> y is also true.

AMPLIFIED LIMIT. There is something that is greater than something, and also much greater than everything it is greater than.

BINARY AMPLIFICATION. Let y > x be given, as well as a true statement about x, using P, binary >,= and binary z >> w >> x. The corresponding statement about x, using P, binary >,= and binary z >> w >> y is also true.

The analogous results hold.

6. PERSISTENTLY VARYING BIT.

The objection can be raised that a varying bit realistically has to have persistence. It cannot be varying "densely". Specifically, if the bit is 1 then it remains 1 for a while, and if the bit is 0 then it remains 0 for a while.

Define a persistent range of times in the obvious way.

PERSISTENT TIME TRANSLATION. For any time b and persistent range of times before b, there exists a translation time c such that any time before b lies in the range of times if and only if the bit at time b+c is 1. Here we use L(>,>>,=,P,+) to present the range of times.

We need to have two additional time principles.

ADDITION. $y < z$ $\square$ $x+y < x+z$.

ORDER COMPLETENESS. Every nonempty range of times with an upper bound has a least upper bound. Here we use L(>,>>,=,P,+) to present the nonempty range of times.

We also have our usual AMPLIFICATION, AMPLIFIED LIMIT, BINARY AMPLIFICATION.

We get the analogous results.

7. SOME MODELS.

We first form an underlying structure (D,>,=).

We define pairs $(D_{\square},>_{\square})$, for all ordinals $\square$. Define $(D_0,>_0) = (\emptyset,\emptyset)$. Suppose $(D_{\square},>_{\square})$ has been defined, and is transitive and irreflexive. Define $(D_{\square+1},>_{\square+1})$ to extend $(D_{\square},>_{\square})$ by adding an exact strict upper bound for every subset of $D_{\square}$ - even if it already has an exact strict upper bound. For limit ordinals $\square$, define $D_{\square} = \square_{\square<\square}D_{\square}$, $>_{\square} = \square_{\square<\square}>_{\square}$. Let $D = \square_{\square}D_{\square}$, $> = \square_{\square}>_{\square}$.

The new elements introduced at each stage are incomparable.

The (eventual) predecessors of x are introduced earlier than x. Each exact upper bound introduced remain valid later.

LEMMA 7.1. (D,>) is irreflexive and transitive, satisfies Minimality, and also Existence in second order form. The same claims are true for ($D_{\square}$,$>_{\square}$), where □ is a limit ordinal.

Proof: Let A □ D, with a strict upper bound x □ $D_{\square}$. Then A □ $D_{\square}$. Hence an exact upper bound for A is introduced in $D_{\square+1}$, which remains an exact upper bound in D. The same argument works for $D_{\square}$.

For Minimality, note that at every stage we introduce an exact upper bound for ∅, and these exact upper bounds are incomparable. Thus (D,>) has a proper class of minimal elements. But all x □ D have a set of strict predecessors.

For $D_{\square}$, note that ($D_{\square}$,$>_{\square}$) has minimal elements introduced at every stage □ < □. Let x □ $D_{\square}$, □ < □. Then any minimal element introduced at any stage ≥ □ will not be < x. QED

Note also that for every x □ D, we introduce some y minimally > x at every stage after x is introduced, and y remains minimally > x at every later stage. This also applies to ($D_{\square}$,$>_{\square}$), for limit ordinals □.

Now fix S to be a nonempty set of limit ordinals, with no greatest element, whose union is □. We define M[S] to be ($D_{\square}$,$>_{\square}$,$>>_S$), where $D_{\square}$,$>_{\square}$ is as above. We define x $>>_S$ y if and only if

x,y □ $D_{\square}$ □ (□□,□ □ S)(□ < □ □ y □ $D_{\square}$ □ (□w □ $D_{\square}$)(x > w)).

LEMMA 7.2. M[S] satisfies Basic.

Proof: x >> y □ x > y. Let x,y □ $D_{\square}$. Let □,□ □ S, □ < □, y □ $D_{\square}$, (□w □ $D_{\square}$)(x $>_{\square}$ w)). Since y □ $D_{\square}$, x $>_{\square}$ y.

x >> y □ y > z □ x >> z. Let x,y,z □ $D_{\square}$. Let □,□ □ S, □ < □, y □ $D_{\square}$, (□w □ $D_{\square}$)(x $>_{\square}$ w), y $>_{\square}$ z. Then z □ $D_{\square}$, and so x $>>_S$ z.

x > y □ y >> z □ x >> z. Let x,y,z □ $D_{\square}$. Let □,□ □ S, □ < □, z □ $D_{\square}$, (□w □ $D_{\square}$)(y $>_{\square}$ w), x $>_{\square}$ y. Then (□w □ $D_{\square}$)(x $>_{\square}$ w), and so x $>>_S$ z.

(□x)(x >> y,z). Let □,□ □ S, □ < □, y,z □ $D_□$. Let x □ $D_{□+1}$ be the exact upper bound of $D_□$ introduced in $D_{□+1}$. Then x $>>_S$ y,z.

x >> y □ (□z)(x >> z □ z is minimally > y). Let □,□ □ S, □ < □, y □ $D_□$, (□w □ $D_□$)(x $>_□$ w)). Let y □ $D_□$, □ < □. Let z □ $D_{□+1}$ be minimally $>_□$ y. Note that x $>>_S$ z. QED

LEMMA 7.3. MK proves that there exists a set S of limit ordinals, with no greatest element, with the following indiscernibility property. For all □ < □ < □ from S, □,□ have the same first order properties over V, relative to any parameters from V(□).

LEMMA 7.4. Let n < □. ZF proves that there exists a set S of limit ordinals, with no greatest element, with the following indiscernibility property. For all □ < □ < □ from S, □,□ have the same first order properties over V, with at most n quantifiers, relative to any parameters from V(□).

LEMMA 7.5. Let S to be a nonempty set of limit ordinals, with no greatest element. In M[S], x >> y if and only if (□w □ $D_□$)(x > w), where □ is the least ordinal in S after □, and □ is the least ordinal in S such that y □ $D_□$.

LEMMA 7.6. Suppose S has the indiscernibility property in Lemma 7.3. Then M[S] satisfies Basic + Minimal + Existence + Amplification.

Proof: By Lemmas 7.1 and 7.2, we need only verify Amplification. Let y $>_□$ x, and □(x) true in M[S], using $>_□$,=, and $>>_S$ x. Let □ be the least element of S such that x □ $D_□$. Let □ be the least element of S after □. Let □’ be the least element of S such that y □ $D_□$’. Let □’ be the least element of S after □’.

The statement M[S] |= □(x,>,>> x)) can be viewed as a statement in V(□S) about x,□. Likewise, the statement M[S] |= □(x,>,>> y) can be viewed as a statement in V(□S) about x,□’. By property ii), the two statements must have the same truth value. QED

THEOREM 7.7. Basic + Minimal + Existence + Amplification has the same Tarski degree as ZF. I.e., they are mutually interpretable.

For Basic + Minimal + Existence + Binary Amplification, we need S of type □ satisfying a more powerful form of indiscernibility.

LEMMA 7.8. Suppose there is a countable transitive model of ZC + "there exists a measurable cardinal". There is a countable transitive model A of ZF, and an unbounded S □ A consisting of limit ordinals, of order type □, with the following indiscernibility property. Let □ < □ < □ be from S. Then S\□ and S\□ have the same first order properties over A, relative to any parameters from the V(□) of A.

Proof: Inside the given model, shoot a Prikry sequence of limit ordinals through the measurable cardinal, □. Take A to be the V(□) of the given model, and S to be the range of the Prikry sequence. QED

Let A be a transitive model of ZF, and S be an unbounded subset of A consisting of limit ordinals. Define M[A,S] as follows. The domain and the > is defined internally in M, as proper classes of M, as before. We write these as DM, $>_M$. The >>, which is a binary relation on D_M, is then defined as before. We write this as $>>_S$.

LEMMA 7.9. M[A,S] satisfies Basic + Minimality + Existence.

LEMMA 7.10. Let A,S be as in the conclusion of Lemma 7.8. Then M[A,S] satisfies Binary Amplification.

Proof: Let y $>_M$ x, and □(x) true in M[A,S], using $>_M$,=, and the binary relation y $>>_S$ z $>>_S$ x. Let □ be the least element of S such that x □ $D_{M□}$. Let □ be the least element of S after □. Let □' be the least element of S such that y □ $D_{M□'}$. Let □' be the least element of S after □'.

Let □(x) be a statement in M[A,S] about x □ D_M, using $>_M$, and the binary relation z $>>_S$ w $>>_S$ x. The statement M[A,S] |= □(x,>,binary >> x,=)) can be viewed as a statement in (M,□,S\□) about x. This is because the binary relation z $>>_S$ w $>>_S$ x can be defined from M and S\□. Likewise, the statement M[A,S] |= □(x,>,binary >> y,=) can be viewed as the corresponding statement in (M,□,S\□') about x. By the indiscernibility property, the two statements must have the same truth value. QED

THEOREM 7.11. Basic + Minimality + Existence + Binary Amplification interprets ZFC + "there exists a Ramsey cardinal” and is interpretable in ZFC + “there exists a measurable cardinal”.

PRINCIPLE OF PLENITUDE

From Wikipedia, Plenitude Principle.

The principle of plenitude asserts that everything that can happen will happen.

The historian of ideas Arthur Lovejoy was the first to discuss this philosophically important Principle explicitly, tracing it back to Aristotle, who said that no possibilities which remain eternally possible will go unrealized, then forward to Kant, via the following sequence of adherents:

Augustine of Hippo brought the Principle from Neo-Platonic thought into early Christian Theology.

St Anselm 's ontological arguments for God's existence used the Principle's implication that nature will become as complete as it possibly can be, to argue that existence is a 'perfection' in the sense of a completeness or fullness.

Thomas Aquinas's belief in God's plenitude conflicted with his belief that God had the power not to create everything that could be created. He chose to constrain and ultimately reject the Principle.

Giordano Bruno's insistence on an infinity of worlds was not based on the theories of Copernicus, or on observation, but on the Principle applied to God. His death may then be attributed to his conviction of its truth.

Leibniz believed that the best of all possible worlds would actualize every genuine possibility, and argued in *Théodicée* that this best of all possible worlds will contain all possibilities, with our finite experience of eternity giving no reason to dispute nature's perfection.

Kant believed in the Principle but not in its empirical verification, even in principle.

The Infinite monkey theorem and Kolmogorov's zero-one law of contemporary mathematics echo the Principle. It can also be seen as receiving belated support from certain radical directions in contemporary physics, specifically the many-worlds interpretation of quantum mechanics and the cornucopian speculations of Frank Tipler on the ultimate fate of the universe.

REFERENCE

[1] H. Friedman, Concept Calculus, Preprints, #53, http://www.math.ohio-state.edu/%7Efriedman/manuscripts.html

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