THIS IS AN AMPLIFIED VERSION OF THE INVITED LECTURE DELIVERED ON AUGUST 5, 2016 AT THE FIELDS INSITUTE IN TORONTO

EXTENDING FUNCTIONS

by
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August 5, 2016
Workshop on Model Theory: From Fields to Hardy Fields
Fields Institute
Toronto, Canada
slightly revised September 22, 2016

I wish to thank the organizers for letting me come here and vie a talk at this splendid event.

I am not sure when I first met Lou. It was probably around 1980 at a logic meeting at MIT (confirmed by Lou).

Lou is driven by a vision of great foundational significance. This is the tame/wild dichotomy, which everybody here is fully familiar with. The tame perspective provides deep links to core mathematics.

The tame perspective is of course deeply reflected in o-minimality. It is not yet clear just what tameness means more generally, as, for instance, mathematicians do like to work in the ring of integers and with the sine function on all of $\mathfrak{R}$.

Lou was an early pioneer in o-minimality, and has consistently remained one of the very few top researchers in the area. I am in no position to say anything serious about the book - but it obviously promises to provide new deep foundations for asymptotic analysis.

At least as impressive as this great vision and energy is his Ph.D. production. His Ph.D. list is amazing both in quantity and quality. He still has 6 Ph.D. students! And

these students go on to greatly advance o-minimality and various tame adventures, with great effectiveness.
Now lest you think I am merely a disciple of Lou's, let me say that I don't necessarily agree with everything that Loy says.

I remember being at a meeting with Lou, and several of us were at the hotel. Lou offered up the following opinion:

"the only interesting thing that Gödel ever did was his beta function"

(Lou immediately disavoed this statement).

If you happened to have forgotten what this is, this is Gödel's ternary function beta(n,m,i), definable in the ring of integers, where every finite sequence from N is of the form beta(n,m,0), beta(n,m,1), ..., beta(n,m,r).

Well, most people here know that I am not going to agree that this is the only interesting thing Gödel ever did. But in all fairness to Lou, I think that many model theorists at the time agreed with at least a weak form of this, and maybe this is true today. But they may not have the refreshing openness to express it.

In any case, Lou has since said that he has moderated his views on Goödel. It does seem that Lou has mellowed. (Lou added that it seems that I have mellowed also).

Now for the boring part of my talk.

ABSTRACT. Given a function or family of functions on a domain, can we extend it or all of them to a larger domain preserving certain properties? This general kind of problem seems to cut across a wide variety of subjects including model theory, recursion theory, set theory, algebra, geometry, and analysis. We discuss some contexts, including the extension of families of real functions to Conway's surreal number system.

INTRODUCTORY REMARKS

There appears to be a considerable subject surrounding the extension of functions on one domain to functions on a more inclusive domain, preserving various properties.

This is already highly nontrivial when we merely ask for relationships between an individual function and its extension. More generally, we ask for relationships between several functions and their extensions.

This topic seems to be in need of a systematic treatment that cuts across, at least, model theory, recursion theory, set theory, algebra, geometry, and analysis. I hate to disappoint you, but the possibility of a unified subject only consciously occurred to me in preparation of my talk. I don't have any really appropriate systematic development, but consider these basic observations.

i. In model theory, by ultrapower or compactness arguments, we can extend functions preserving first order properties. This can take familiar functions to necessarily wild functions.

ii. In algebra/geometry, we can extend functions from (quite general) o-minimal structures, preserving first order properties. This goes from tame functions to tame functions.

iii. In analysis, we can extend analytic functions via analytic continuation, preserving considerable properties. This goes from tame functions to tame functions, with qualifications.

iv. In set theory, we can extend big functions, preserving certain non first order properties. This requires such bigness that this is very wild.

v. In recent years, rather elaborate set theoretic constructions have emerged allowing us to go from any structure to a proper elementary extension (even with saturation properties), set theoretically defined from that structure. The usual construction via a single ultrapower is not explicitly definable, requiring the choice of a single nonprincipal ultrafilter. All of this is very wild.

vi. More speculatively, perhaps there is a relevant adaptation to functions of my recent Continuation Theory of finite sets, where finite functions get maximally continued within a countable space, with certain symmetries. (These provide clear finitary statements that can only be proved by using far more than the usual ZFC axioms for mathematics.)

How this all fits together is not so clear.

I was asked to talk about results concerning the extension of real functions to Conway's surreal numbers. Positive and negative results of this kind are found in

[CEF] O. Costin, P. Ehrlich, H. Friedman, A Conjecture of Conway, Kruskal and Norton, https://arxiv.org/abs/1505.02478.

In [CEF], very natural explicitly given extensions of Ecalle's level one transseriable functions are lifted into the surreals with associated integration operators. These functions include "most functions occurring naturally in analysis" such as:

semi-algebraic, restricted semi-analytic, analytic, and meromorphic functions, Borel summable functions and generic solutions to nonlinear systems of ODEs with a possible irregular singularity at infinity.

They also include the named classical special functions such as Airy, Bessel, error functions Ei, Erf, Erfi, and Gamma, Painleve, among many others. See

Écalle, Jean Six lectures on transseries, analysable functions and the construct- tive proof of Dulac's con- jecture. Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), 75–184, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 408, Kluwer Acad. Publ., Dordrecht, 1993.

for an account of transseriable and Borel summable functions.

In [CEF], it is also shown that other various classes are not explicitly extendable to surreals obeying very basic weak natural conditions. These negative results apply to very regular classes that were hoped to be extendable to surreals, such as the entire functions with decay at $\pm\infty$.

These negative results are viewed to have dashed some hopes of Conway, Kruskal, and Norton. Furthermore, these negative results are very general, and do not use hardly any properties of the surreals. They are, however, formulated using fundamental notions from descriptive and general set theory.

I will present a more digestible simplified framework for casting these negative results here than in [CEF].

First I talk about function extension in a more general context, going back to

[Fr] H. Friedman, Working with Nonstandard Models, in: Nonstandard Models of Arithmetic and Set Theory, American Mathematical Society, ed. Enayat and Kossak, 71-86, 2004. See 44. Working with Nonstandard Models, July 31, 2003, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/
This story is quite separate from the focus of this talk, but it has a common starting point. Given any infinite $f:A^k \to A$, there is a proper elementary extension $g:B^k \to B$.

There is a purely algebraic form of this that avoids use of predicate calculus.

THEOREM 1. Every infinite $f:A^k \to A$ has a proper extension $g:B^k \to B$ containing the same finite $h:C^k \to B$ ($h:C^k \to C$) up to isomorphism (not moving any element of A).

There are several places we can take this. Here's a delicious one.

THEOREM 2?. Every infinite $f:A^k \to A$ has a proper extension $g:B^k \to B$ containing the same countable $h:C^k \to B$ ($h:C^k \to C$) up to isomorphism (not moving any element of A). Variants with countable replaced by "finitely generated" or "1-generated".

Well, this is easily refuted with A = ω. But look here!

THEOREM 3. (ZFC) The following are equivalent.
i. There exists A for which any of the statements in Theorem 2? hold, even for k = 2.
ii. There exists a measurable cardinal.

THEOREM 4. (ZFC) The following are equivalent.
i. Any or all of the statements in Theorem 2? hold for A, even for just k = 2.
ii. There is a countably additive 0,1 valued measure on $\wp(A)$ with singletons of measure 0 and A of measure 1.

iii. The cardinality of A is at least some measurable cardinal.
We can try to take this to an extreme along certain lines. But we fail.

THEOREM 5. (Major component is Kunen). For every A there exists $f:A^2 \rightarrow A$ which is not isomorphic to any of its proper extensions or proper restrictions.

The above is proved in ZFC, but it may be consistent with ZF. Note the formulations with extensions and restrictions are equivalent.

The idea: let A is of least cardinality such that every $f:A^2 \rightarrow A$ is isomorphic to a proper restriction. Show |A| is a strongly inaccessible cardinal. So take A to be the complete diagram of $V(\theta)$, θ strongly inaccessible. Now apply Kunen's inconsistency (every proper elementary embedding from $V(\alpha)$ into $V(\alpha)$ has $cf(\alpha) = \omega$ or $cf(\alpha-1) = \omega$) to obtain a contradiction.

Now let's come back down to earth.

Even with

Every countably infinite $f:A^k \rightarrow A$ has a countably infinite proper extension $g:B^k \rightarrow B$ containing the same finite $h:C^k \rightarrow B$ ($h:C^k \rightarrow C$) up to isomorphism

there is the question of whether for familiar f we can always find such a familiar g.

For example, (Z,<,+) is an elementary substructure of (Z×Q,<,+), and Presburger functions extend tamely.

However $f:Z^2 \rightarrow Z$, $f(x,y) = x+y^2$, is quite different. No proper extension containing the same finite $h:C^k \rightarrow C$ up to isomorphism can be even recursively presented.

Do tame functions have tame proper elementary extensions?

It is clear that infinite o-minimal structures have 0-minimal proper elementary extensions. This is immediate from the fact that being 0-minimal is first order.

If the 0-minimal structure is an expansion of an ordered Abelian group then the construction of an elementary extension is rather explicit. Use the ultrapower with definable functions and the nonprincipal filter of intervals (x,∞). The o-minimality guarantees that this filter is large enough for this purpose.

The ordered Abelian group structure is used here for the existential quantifier step, proved by choice. Here we use choice for definable relations.

So there remains a question of explicitness which can be formulated as follows. The above argument shows that every o-minimal expansion of an ordered Abelian group, with a recursive complete diagram, has a proper elementary extension with a recursive complete diagram. Is this true if we drop the ordered Abelian group hypothesis?

In
[DE] L. van den Dries and P. Ehrlich, Fields of surreal numbers and exponentiation, Fund. Math. 167 (2001) 173-188; erratum, ibid. 168 (2001) 295-297.

the o-minimal real exponential field, with bounded real analytic functions, is elementarily extended to the field of surreals in a natural way.

In [CEF], rich classes of real functions are naturally extended to surreals, along with their definite integrals, with some key analytic properties preserved. However, preservation of first order properties has not been verified.

In particular, the functions with level one Ecalle-Borel summable transseries, and their definite integrals, are naturally extended to the surreals by a ZFC explicit construction - which up to any countable level of the surreals is, moreover, Borel. (Also the case in [DE] with exp and restricted analytic functions.)

The [CEF] construction is "genetic" in the semiformal sense championed by Conway, and requires very "tight" asymptotic inequalities, supported by theorems of Costin and Kruskal in hyperasymptotics.

It should also be noted that these [CEF] functions include sine on $\mathfrak{R}$, and thus the results do not stay within the realm of Hardy fields.

POSITIVITY SETS

We begin with a core negative result that doesn't mention surreals or in fact any ordered field whatsoever - and not even functions.

A positivity set is an $S \subseteq \{0,\pm 1\}^N$ such that the following holds. No binary sum from S has a negative last nonzero term. No binary sum from $\{0,\pm 1\}^N \setminus S$ has a positive last nonzero term.

(Here a binary sum from X is an a+b, a,b ∈ X).

ZFDC is ZF with the axiom of choice replaced by the axiom of dependent choice.

THEOREM 6. ZFC proves there is a positivity set. ZFDC proves the following. There is no Borel positivity set. No positivity set has the property of Baire. If there is a positivity set then there exists a set of reals without the property of Baire.

Argue in ZFC. Take an ultrapower of the ordered ring of integers by a nonprincipal ultrafilter on ω, obtaining a nonstandard model of signed arithmetic, and choose any nonstandard $\gamma > 0$. Given $f:N \to \{0,\pm 1\}$, multiply by $4^0,4^1,4^2,\ldots$, and take its image in the ultrapower, and add its first γ terms, in the sense of the ultrapower. Put f ∈ S iff this nonstandard length sum is positive. (We use N for the set of all nonnegative integers).

Now argue in ZFDC. Suppose S is a positivity set with the property of Baire. Let α be a finite sequence of length n ≥ 1 such that S is meager in X|α or comeager in X|α.

In the first case, choose f,g ∈ X\S starting with α, continuing with f(n+1) = g(n+1) = -1, f(m) = -g(m), m ≥ n+2. (Use a homeomorphism argument). Then f+g has negative last nonzero term. In the second case, choose f,g ∈ S starting with α, continuing with f(n+1) = g(n+1) = 1, and f(m) = -g(m), m ≥ n+2. (Use a homeomorphism argument). Then f+g has positive last nonzero term. QED

A set of reals without the property of Baire is quite a terrifying thing. We know from Solovay that these cannot be proved to exist in ZFDC.

But since people generally accept the axiom of choice, another aspect of this, also due to Solovay, is very important. There is no explicit way to get a set of reals without the property of Baire. We apply this to positivity sets to obtain the following.

THEOREM 7. ZFDC does not prove the existence of a positivity set. There is no definition which, provably in ZFC, defines a positivity set. This holds even with real parameters.

So in a precise sense, positivity sets are inaccessible to normal mathematics.

As we see below, this tells us that certain kinds of extensions of functions are also inaccessible to normal mathematics.

EXTENDING REAL FUNCTIONS

A real function is a function from $\Re$ to $\Re$. Let W be a set of real functions.

An additive extension of W consists of an ordered Abelian group $G \supseteq\neq \Re$, and operator $\varphi:W \to G^G$, where for all $f,g,h \in W$ and $x \in \Re$,
i. $\varphi(f)$ extends f.
ii. $f + g = h \to \varphi(f) + \varphi(g) = \varphi(h)$.
iii. $f > 0$ on $[x,\infty) \to \varphi(f) > 0$ on $[x,\infty)$ in G.

It is easy to see that if W contains $\equiv 0$ and closed under -, then $\varphi(\equiv 0) = \equiv 0$, $\varphi(-f) = -\varphi(f)$, and $f < 0$ on $[x,\infty) \to \varphi(f) < 0$ on $[x,\infty)$ in G.

An absolutely convergent series of real functions is a series

$\lambda = \lambda_0 + \lambda_1 + \ldots$

of real functions pointwise absolutely convergent. $T(\lambda)$ consists of derived series, obtained via coefficients 0, ±1,±2 on the λ's.

Using ZFC, there exists an additive extension of $T(\lambda)$ by an ultrapower. But this is not explicit. More about this later.

THEOREM 8. (ZFDC) Let λ be absolutely convergent, where $\forall n$ $\exists m$ $(\lambda_{n+1} > 4|\lambda_n|$ on $[m,\infty))$. Assume $T(\lambda)$ has an additive extension G,φ. Then $\exists$ a positivity set. Hence $\exists$ a set of real numbers without the property of Baire.

Proof: Let λ,G,φ be as given. Fix $\gamma \in G$, $\gamma > \Re$, and $S = \{f \in \{0,\pm1\}^N: \varphi(\sum f(n)\lambda_n)(\gamma) > 0\}$. We show S is a positivity set. The two parts are analogous.

Suppose $f,g \in S$, $f+g$ has last nonzero term < 0. Then $\sum(f+g)(n)\lambda_n \in T(f)$ is eventually < 0. Hence $\varphi(\sum(f+g)(n)\lambda_n)(\gamma) < 0$. This contradicts $\varphi(\sum f(n)\lambda_n)(\gamma)$, $\varphi(\sum g(n)\lambda_n)(\gamma) > 0$. QED

We say that λ is brisk iff λ is absolutely convergent, where for all n there exists m such that $\lambda_{n+1} > 4|\lambda_n|$ on $[m,\infty)$. In the next section, we will discuss brisk λ where the $T(\lambda)$ sharply decay at $\pm\infty$. This is particularly relevant to prior hopes for extending various real functions to the surreals.

Here are two immediate Corollaries of Theorem 8.

THEOREM 9. There is no Borel brisk λ with a Borel additive extension G,φ. It cannot be proved in ZFDC that there exists brisk λ for which $T(\lambda)$ has an additive extension G,φ.

THEOREM 10. There are no definitions which, provably in ZFC, define a brisk λ, an additive extension G,φ of $T(\lambda)$, and a distinguished $c > \Re$ in G. This holds even if we allow real parameters in the definitions.

Note that this c is used in the proof of Theorem 8. This cannot be simply removed, because of arbitrary ZFC verified definitions by Shelah and collaborators using iterated ultrapower constructions. See

V. Kanovei and S. Shelah. A definable nonstandard model of the reals. Journal of Symbolic Logic, 69(1):159–164, 2004.

V. Kanovei, Michael Reeken, S. Shelah, Fully saturated extensions of standard universe, http://shelah.logic.at/files/E39.ps

The nonstandard elementary extensions constructed in these two references necessarily cannot have explicitly defined nonstandard elements. This is in sharp contrast with, e.g., No, where the ω is an explicitly defined non Archimedean element.

I.e., Theorem 10 applies where G is No or even natural parts of No, as they contain many explicitly defined $c > \Re$.

Perhaps you have noticed that we never used that $\varphi(f)$ extends f. Thus what we are really talking about is more basic than extensions.

Let W be a set of real functions. An additive evaluation of W consists of an ordered Abelian group $G \supsetneq \Re$, and functional $\varphi:W \rightarrow G$, where for all $f,g,h \in W$ and $x \in \Re$,
i. $f + g = h \rightarrow \varphi(f) + \varphi(g) = \varphi(h)$.
ii. $f > 0$ on $[x,\infty) \rightarrow \varphi(f) > 0$.

THEOREM 8'. (ZFDC) Let λ be brisk, with an additive extension G,φ. There is a positivity set. There is a set of real numbers without the property of Baire.

THEOREM 9'. There is no Borel brisk λ with a Borel additive evaluation G,φ. It cannot be proved in ZFDC that there exists brisk λ for which $T(\lambda)$ has an additive evaluation G,φ.

THEOREM 10'. There are no definitions which, provably in ZFC, define a brisk λ and an additive evaluation G,φ of $T(\lambda)$. This holds even if we allow real parameters in the definitions.

BRISK DECAY

We use some particular brisk $\Gamma = (h_n)_{n\geq 0}$.

$h_n(x) = 0$ for $x \leq n$;
$(x-n)/8$ if $n < x < n+1$;
8^{n-x} if $x \geq n+1$.

Clearly $h_{n+1} > 4|h_n|$ on $[n+1,\infty)$. Fix $0 \leq m \leq x < m+1$. Then $\sum h_n(x) = 8^{-x} + 8^{-x+1} + \ldots + 8^{-x+m-1} + (x-m)/8 \leq 8^{-m} + \ldots + 8^{-1} + 1/8 < 1$. So $\sum h_n(x)$ is pointwise absolutely convergent, $\Gamma = (h_n)_{n\geq 0}$ is brisk, and $T(\Gamma)$ lies strictly between -1 and 1. Fix $0 \leq m \leq y < z \leq m+2$. $h_n(y)-h_n(z)$ is 0 if $n \geq m+2$; is $8^{n-y}-8^{n-z}$ if $n \leq m-1$; is in the difference set of $\{(y-m)/8,(z-m)/8,0,8^{m-y},8^{m-z}\}$ if $n \in \{m,m+1\}$. This gives us a constant upper bound for $\sum|h_n(y)-h_n(z)|$ as a function of m, and not of y,z. This suffices to establish that all elements of $T(\Gamma)$ are continuous.

Let $f:\Re \to (0,\infty)$ be continuous. A function trapped between -f and f corresponds to enforcing a decay condition. We say that λ is continuously f-brisk if and only if

1. λ is a pointwise absolutely convergent series of real functions.
2. $\forall n\ \exists m$ such that $\lambda_{n+1} > 4|\lambda_n|$ on $[m,\infty)$.
3. The functions in $T(\lambda)$ are continuous and lie strictly between -f and f.

LEMMA 11. For all continuous $f:\Re \to (0,\infty)$, there is a continuously f-brisk λ.

Proof: Multiply $\Gamma = (h_n)_{n\geq 0}$ by the function $\min(f(x),e^{-|x|})$. QED

THEOREM 12. (ZFDC) If there exists continuous $f:\Re \to (0,\infty)$ with an additive extension G,φ, of the continuous $-f < g < f$, then there is a set of reals without the property of Baire. The same holds for additive evaluations G,φ.

THEOREM 13. There is no continuous $f:\Re \to (0,\infty)$ with a Borel additive extension G,φ of the continuous $-f < g < f$. It cannot be proved in ZFDC that there exists a continuous $f:\Re \to (0,\infty)$ with an additive extension G,φ of the $-f < g < f$. The same holds for additive evaluations G,φ.

THEOREM 14. There is no definition which, provably in ZFC, defines continuous $f:\Re \to (0,\infty)$, an additive extension G,φ of the continuous $-f < g < f$, and $c > \Re$ in G. The same for additive evaluations G,φ (without the $c > \Re$).

Real entire functions are entirely better than continuous functions.

THEOREM 15. (O. Costin) For all continuous $f:\Re \to (0,\infty)$, there exists an entirely f-brisk λ. I.e., where the elements of $T(\lambda)$ are entire.

THEOREM 16. Theorems 12-14 hold with continuous f-brisk replaced by entirely f-brisk.