THE ANALYSIS OF MATHEMATICAL TEXTS,

AND THEIR CALIBRATION IN TERMS OF INTRINSIC STRENGTH I

by

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This is the first of a series of reports on an ongoing mathematical program that will evidently take me several years to complete. This first report is intended to communicate the general features of the program, which have now become clear to me.

Perhaps the best way to describe this program is in autobiographical terms.

1. In 1969, I discovered that a certain subsystem of second order arithmetic based on a <u>mathematical</u> statement (that every perfect tree which does not have at most countably many paths, has a perfect subtree), was provably equivalent to a <u>logical principle</u> (the weak Π^1_1 axiom of choice) modulo a weak base theory (comprehension for arithmetic formulae).

2. Certain persons were claiming that certain formalisms were adequate to "formalize" various bodies of mathematics. I did not agree that they did, at least directly, formalize the bodies in question, since they were laden with <u>coding</u> devices and other features <u>alien</u> to the mathematics.

3. In 1974, I made a systematic study of the phenomenon in 1 above, and found that <u>that</u> situation, in the setting of second order arithmetic, was not the exception, but actually the <u>rule</u>. Provable equivalences between mathematical statements and logical principles abound in second order arithmetic. I reported these findings to the

ICM meeting at Van Couver.

4. I realize the limitations of second order arithmetic in dealing with mathematical statements (say, about functions on the real numbers, Banach spaces, etc.), and did not want to subject myself to the same criticism I levelled in 2 above.

5. I saw that a precise analysis of what it means to "formalize" mathematical text could have far reaching consequences. Suppose one had this. Then if one purports to formalize a certain body of mathematical text this claim is subject to a precise test; and one can state in precise terms what has been accomplished when this is done. Or one may be able to prove, precisely, that no formalism F can be given for the text which meets certain conditions; e.g., that F have a consistency proof in Peano arithmetic.

6. I also saw that there could be a kind of precise analysis of what it means to "formalize" which would be even more far reaching. That is, suppose one could associate to each body of mathematical text, a specific finite combinatorial structure which clearly constitutes a formalism; furthermore, this formalism is chosen in a uniform manner throughout the whole of mathematics, independently of the nature of any particular text. Let us also assume that this formalism F includes "nothing more and nothing less" than the original text. So F is completely free of coding !! We would then require of a formalization G of the text, that there be given a "translation" of F into G .

In particular, it will be clear that F will be the minimum formalization of the text, and so any formalism G of the text must have the property that there is a finitist reduction of the consistency problem for F to that of G .

Normally, the text includes basic arithmetic, and a formalization G will also of course. Also the translation of F in G will normally preserve all arithmetic (or at least Π^0_2 or Π^0_1) sentences. Hence any formalization G will have to at least prove all the arithmetic (of Π^0_2 or Π^0_1) sentences that F does .

In addition, various logicians have calibrated various formal systems in terms of their "logical strength". (For more about this, see Appendix 1.) If we calibrate the formal systems F arising from the association above, then <u>we have really calibrated the original texts in terms of their intrinsic logical strength</u>.

7. I found the needed association discussed in 6, in the Fall of 1974. I discovered that mathematics, as informally given in texts, is far closer to being formal than people realize, and that one can uniformly and faithfully translate all mathematical text in a formal calculus U, in such a way that the deductive relations between the informal mathematical statements are preserved in the translation, in both directions. U is an extension of the usual first order predicate calculus without identity , with certain extra apparatus.

8. I now envisioned a combining of 3 with 7. That is, for various texts, I would pass to the associated "<u>raw</u> formal system" of 7, and then prove that the latter has the same arithmetic (Π^0_2, Π^0_1) consequences as some standard logical system. I fully realized,however, that I could not possibly make full use of the axioms of the raw formal system, since the number of them could easily be over 10^3, depending on the length of the

text under consideration.

9. I further envisioned a two stage process. Let F be one of these raw formal systems. One first gets a "foundational" idea, so that one writes down a system G which is manageable; and as a consequence of the idea, one sees that any arithmetic (Π^0_2, Π^0_1) statement provable in F is also provable in G. That is, G, as well as F, is a formalization of the original text. But G is manageable, (which is somehow related to its having a "foundational" idea in it). Then a standard logical system S is shown to have as consequences, all the arithmetic (Π^0_2, Π^0_1) consequences of G. So S gives an upper bound for F, and hence the text.

Secondly, one extracts a manageable system H, all of whose axioms are theorems of F. One forms a system B based on a "foundational" idea, and shows that any arithmetic (Π^0_2, Π^0_1) consequence of B is also a consequence of H. Then a standard logical system T is shown to have the property that all the arithmetic (Π^0_2, Π^0_1) consequences of T are consequences of B. So T gives a lower bound for F, and hence the text.

Of course, S and T are compared. Often S is just T, and so we have trapped the text from below and above, thus calibrating its strength as S.

NOTE THAT THE BASE THEORIES IN 1 AND 3 DISAPPEAR

10. It was hoped that there would be a small handful of distinct strengths of important texts; say $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7,$ in increasing order. I further hoped that the texts corresponding to the α_n could be distinguished in general terms, I.e., say, seven different brands of mathematics emerge, which can be distinguished in general

informal mathematical (not only logical) terms. Furthermore, that these distinctions already are vaguely understood by mathematicians now.

This of course necessitates the discovery of very carefully drawn formalisms $A_1, A_2, A_3, A_4, A_5, A_6, A_7,$ of respective strengths α_n, which provide a formalism for the union (fusion) of all texts of its strength. That is, A_n must provide a formalization of all the texts of intrinsic strength α_n.

11. This brings us to the present. My work to date has followed the lines of 8-10. The whole program has so far been successful, and it draws heavily from a large group of areas of modern mathematical logic. I am now working out the program for the (classical) theory of analysis on Euclidean space, and somewhat delicate analytic constructions and arguments also play a role.

Let me end this informally distributed report with some rather grandiose remarks, which I hope the reader will forgive. It is my way of expressing my utter ecstasy with the way this program has proceeded to date.

There seems to be emerging an intimate and delicate interaction of ideas in analysis and contemporary mathematical logic, on a scale that I had not, until recently, even envisioned. A subject that has gone through great difficities may well need an environment of proper technical tools and machinery, in addition to the necessary conceptual insights. The technical tools and machinery of modern mathematical logic are now impressive. It appears that they are now sufficient (and to a reasonable extent necessary!) for us to attack the subject of metamathematics, in radically new ways.

APPENDIX 1

ON STRENGTHS OF FORMAL SYSTEMS

Let PRA be primitive recursive arithmetic. Consider the relation $\leq$ among formal systems with finitely many axioms: $T_1 \leq T_2$ iff there is a primitive recursive function symbol F such that PRA $\vdash$ "if x is the index of a proof of a contradiction in T_1 then $F(x)$ is the index of a proof of a contradiction in T_2". Let $T_1 \sim T_2$ be $(T_1 \leq T_2$ and $T_2 \leq T_1)$. Then $\sim$ is an equivalence relation. Clearly $(T_1 \leq T_2 \;\&\; T_2 \leq T_3) \rightarrow T_1 \leq T_3$. Also $(T_1 \sim T_2 \;\&\; T_3 \sim T_4) \rightarrow (T_1 \leq T_3 \leftrightarrow T_2 \leq T_4)$. Let $T_1 < T_2 \leftrightarrow (T_1 \leq T_2$ & not $T_1 \sim T_2)$.

From now on, let us suppose all our theories contain sufficiently much arithmetic, in addition to having only finitely many axioms.

Consider the relations i) $T_2 \vdash \mathrm{Con}(T_1)$ ii) $\Pi^0_1(T_1) = \Pi^0_1(T_2)$ ii) $\Pi^0_1(T_1) \subset \Pi^0_1(T_2)$ iv) $\Pi^0_1(T_1) \subsetneqq \Pi^0_1(T_2)$. Here $\Pi^0_1(T)$ is the Π^0_1 consequences of T.

EXPERIMENTAL FACTS. $T_1 \sim T_2$ iff $\Pi^0_1(T) = \Pi^0_1(T_2)$. $T_1 \leq T_2$ iff $\Pi^0_1(T_1) \subset \Pi^0_1(T_2)$. $T_1 < T_2$ iff $T_2 \vdash \mathrm{con}(T_1)$ iff $\Pi^0_1(T_1) \subsetneqq \Pi^0_1(T_2)$. In particular, we have the trichotomy laws $(T_1 \vdash \mathrm{con}(T_2)$ or $T_2 \vdash \mathrm{con}(T_1)$ or $T_1 \sim T_2)$, and $(T_1 \leq T_2$ or $T_2 \leq T_1)$.

The above holds for all the formal systems we know of, which are presented in a Hilbert style predicate calculus setting, and which have even a remote connection with mathematical practice.

Even more is true. Replace Π^0_1 in the above by Π^0_∞.

The "logical strength" of a _formal_ system (of the above kind) is identified with its equivalence class; or, alternatively, with its set of Π^0_1 consequences.

APPENDIX 2

John Corcoran pointed out to me the following quotation from N. Bourbaki, "Foundations of Mathematics for the Working Mathematician", JSL, 14(1949), 1-8, Corcoran used in his paper "Gaps between Logical Theory and Mathematical Practice", M. Bunge, The Methodological Unity of Science, 23-50, 1973, D. Reidel Publishing Co., Holland. "By a proof, I understand a section of a mathematical text ... proofs, however, had to exist before the structure of a proof could be logically analyzed; and this analysis ... must have rested ... on a large body of mathematical writings. in other words, logic , so far as we mathematicians are concerned, is no more and no less than the grammar of the language which we use, a language which had to exist before the grammar could be constructed .... The primary task of the logician is thus the analysis of the body of mathematical texts." Bourbaki immediately goes on to say "particularly of those [texts] which by common agreement are regarded as the most correct ones, or, as one formerly used to say, the most"rigorous". In this, he will do well to be guided more by what the mathematician does than by what he thinks, or, as it would be more accurate to say, by what he thinks he thinks; for the mental images which occur to the working mathematician are of psychological rather than logical interest. Also, if logic (as grammar) is to acquire a normative value, it must, with proper caution, allow the mathematician to say what he really wants to say, and not try to make him conform to some elaborate and useless ritual. [The logician] may then set himself further objectives ... some of these deal with various aspects of the problem of non-contradiction; this is a question which, since the earliest times, has played a prominent part in the relations between logic and mathematics .... ."