THE LOGICAL STRENGTH OF MATHEMATICAL STATEMENTS I

by

Harvey Friedman*
SUNYAB
August, 1976
PRELIMINARY REPORT

The motivation for the program presented here is the so called consistency problem for mathematical analysis, or more generally, for bodies of mathematics.** We were led to more careful formulations of these problems than are customary, including a new methodology for the construction of negative solutions.

The legitimacy of the formulations and methodology rests on a syntactic and semantic analysis of mathematics (mathematical text). This theoretical analysis of text is in turn based on a logical calculus U which is an augmentation of the ordinary first order predicate calculus. The analysis serves to legitimize the notions of provability, consistency, and satisfiability in the context of genuine mathematical statements - rather than merely in the usual framework of formalized mathematics.

In particular, the analysis shows how any finite set S of mathematical statements can be construed as a finitely axiomatized formal system - the <u>raw formal system</u> $|S|$ - whose axioms are just the statements themselves (presented as sentences in U).

Let S_1, S_2 be any two formal theories in U whose axioms consist of

---

*This research was partially supported by NSF MPS 75-05856.

**In fact, we regard this work as the proper formulation of, and answer to, Hilbert's second problem (about the consistency of analysis). Or, "everything metamathematics was meant to be but never was".

a finite number of statements together with a finite number of schemes. (Thus, Peano arithmetic is allowed). We say that S_1 and S_2 are equiconsistent if it can be proved in primitive recursive arithmetic (PRA) that S_1 is consistent if and only if S_2 is. In this case we write $S_1 \sim S_2$. We also write $S_1 \leq S_2$ to indicate that it can be proved in PRA that if S_2 is consistent, S_1 is consistent.

We say that two finite sets S_1, S_2 of mathematical statements have the same logical strength of $|S_1| \sim |S_2|$. We say that the logical strength of S_2 is at least that of S_1 if $|S_1| \leq |S_2|$.

By analysis on Euclidean space, or more loosely speaking, real variables, we will mean the study of pointsets in E^n and functions from pointsets in E^n into E^m, and more generally, of pointsets in spaces of power $\leq c$, and functions from such pointsets into such spaces.

We have made an extensive study of the raw formal systems arising out of real variables and have uncovered the following remarkable empirical phenomenon. Once a set of statements of real variables includes a handful of certain basic ones (which we call RV), the raw formal system will closely correspond to one of the principal metamathematically oriented formal systems studied in mathematical logic.

One aspect of this correspondence between mathematical and metamathematical statements is equiconsistency. The raw formal systems arising out of real variables (including $|RV|$ as a base) all seem to be equiconsistent with metamathematical systems. Thus for finite sets S of sentences of real variables (including RV as a base), we can "calculate" the logical strength of S by presenting a standard metamathematical system T for which $T \sim |S|$. Other aspects of this correspondence relate to provable ordinals and conservative extension results.

The calculus U is presented in a preliminary report "A universal calculus for the logic of mathematics", August, 1976.

Finally, we emphasize that there is another approach to logical strength which is not in any way dependent on a system of axioms and rules of logic. Indeed, one may question whether such axioms and rules are a faithful modelling of mathematical practice. Perhaps some restriction on them - which now may be transcendentally difficult for us to discover - form just as inclusive a modelling of mathematical practice, and lead to a very altered situation regarding consistency; perhaps even the demise of the second incompleteness theorem for the "new consistency".

The approach that avoids the issue of the modelling of mathematical practice is the semantic approach. Semantic consistency just means the existence of a system of objects in which all the statements hold. If there are only finitely many statements in the class - and this is always the case here - then semantic consistency is straightforwardly a single statement. No extrapolation or modelling of mathematical practice is involved.

In the treatment of $\sim$ and $\leq$, we cannot of course use PRA. One approach is to expand PRA to arithmetic comprehension (ACA_0 of my abstract in JSL of June, 1976). But then we know, of course, by the Gödel completeness theorem that the approach in terms of proofs is equivalent (if PA is used instead of PRA ; in practice, this makes no difference either).

1. Arithmetic 3rd order comprehension - the logician's RV.

RV and |RV| are presented in the next section, and are quite lengthy because they have real live genuinely mathematical primitives and principles which are finite in number and explicitly used in actual text. It is quite impossible to think efficiently in RV , |RV| for obtaining metamathematical results, just as it is impossible to think efficiently in arithmetic 3rd order comprehension (ACA^3) for obtaining mathematical results.

We have variables k_n over natural numbers, x_n over sets of natural numbers, and F_n^m over m-ary total functions from sets of natural numbers to sets of natural numbers. We have constants 0,1 , and function symbols $+,\cdot$. We have $=,\in$.

The numerical terms are given by 1) k_n,0,1 are 2) if s,t are then $(s + t)$, $(s\cdot t)$ are. The set terms are given by 1) x_n are 2) if $s_1,\ldots,s_m$ are, so is $F_n^m(s_1,\ldots,s_m)$. The function terms are just the F_n^m .

The atomic formulae are $s = t$, $\alpha \in \beta$, where s,t are either both numerical terms or both set terms, and α is a numerical term, β a set term.

A formula is arithmetic if the only quantifiers are numerical. The axioms and rules of logic are obvious. The proper axioms are as follows.

1. $n + (m + 1) = (n + m) + 1$, $n\cdot(m + 1) = (n\cdot m) + n$, $n + 1 \neq 0$, $n + 1 = m + 1 \rightarrow n = m$.
2. $(0 \in x \;\&\; (\forall n)(n \in x \rightarrow n + 1 \in x)) \rightarrow n \in x$.
3. $(\exists x)(\forall n)(n \in x \leftrightarrow A)$, where A is arithmetic, x not free in A .
4. $(\exists F)(\forall x_1)\ldots(\forall x_m)(\forall k)(k \in F(x_1,\ldots,x_m) \leftrightarrow A)$, where A is arithmetic and F is not free in A .
5. $x = y \leftrightarrow (\forall n)(n \in x \leftrightarrow n \in y)$.

ACA^3 is a conservative extension of Peano arithmetic, or even the ACA_0 of my abstract in the JSL of June, 1976.

For metamathematical purposes, one always thinks of ACA^3 instead of RV, $|RV|$.

2. The statements of RV and the formal system |RV| .

The primitives of the raw formal system |RV| are as follows.

a. Variables α_j over finite nonempty sequences of real numbers; variables x_j over 1-tuples of real numbers, which are identified with real numbers; variables n_j over natural numbers (1,2,3...) , which are certain real numbers.

b. Variables A_j over subsets of Euclidean spaces.

c. Variables f_j over partial functions from one Euclidean space into another.

d. Variables F_j over partial binary functions from some $E^p \times E^q$ into some E^r .

e. The sets E^n , as a function symbol from natural numbers into sets, and the set constant N .

f. The function symbols $+, \cdot$ on both the natural numbers and the real numbers; the function symbols $-, 1/, |\ |$ on the real numbers; the function symbol ℓth from finite sequences (of real numbers) into natural numbers; the real number constant 0; the natural number constant 1.

g. The relation symbols $<$ on real numbers, and $=$ among natural numbers, or real numbers, or finite sequences, or sets; the relation symbol $\subset$ among sets; the 3-ary relation symbol $f_j: A_p \to A_q$ among unary functions and two sets; the 4-ary relation symbol $F_j: A_p \times A_q \to A_r$ among binary functions and three sets.

h. The variable binding operators $\sum_{k=1}^{n}$, $\prod_{k=1}^{n}$ on real numbers, or on natural numbers; the variable binding operator $\lim_{n_j \to \infty}$ on real numbers.

Thus in $|RV|$ there are six sorts of variables - natural number variables, real variables, Euclidean variables, set variables, unary variables, and binary variables. The unary terms, binary terms are respectively the unary variables, binary variables.

The natural number terms are given by i) $1, n_j$ are natural number terms ii) if s,t are natural number terms, so are $s + t$, $s \cdot t$ iii) if s is a Euclidean term then $\ell\mathrm{th}\ (s)$ is a natural number term iv) if s,t are natural number terms, then $\sum_{n_j=1}^{s} t$, $\prod_{n_j=1}^{s} t$ are natural number terms.

The real terms are given by i) all natural number terms are real terms ii) $0, x_j$ are real terms iii) if s,t are real terms, so are $s + t$, $s \cdot t$, $s - t$, $1/s$, $|s|$ iv) if s is a Euclidean term, t a natural number terms then $s(t)$ is a real term v) if s is a natural number term, t a real term, then $\sum_{n_j=1}^{s} t$, $\prod_{n_j=1}^{s} t$, $\lim_{n_j \to \infty} t$ are real terms.

The Euclidean terms are given by i) all real terms are Euclidean terms ii) α_j is a Euclidean term iii) if s,t are Euclidean terms, so are $f_j(s)$, $F_j(s,t)$.

The set terms are given by i) N, A_j are set terms ii) if s is a natural number term then E^s is a set term.

The atomic formulas of $|RV|$ consist of i) $s = t$, $s \simeq t$, where s,t are either both natural number terms, both real terms, both Euclidean terms, or both set terms ii) $s < t$, where s,t are real terms iii) $s \in t$, where s is a Euclidean term, t is a set term iv) $s \subset t$, where s,t are set terms v) $f_j\colon s \to t$, $F_j\colon s \times t \to r$, where s, t, r are set terms vi) $D(s)$ for any term s.

1. <u>Equality axioms</u>. Two sets are equal if and only if they have the same elements. Two finite sequences of real numbers are equal if and only if they agree everywhere.

2. <u>Miscellaneous axioms</u>. N is the set of all natural numbers. Every natural number is a real number. The real numbers are the finite sequences of length 1. The length of all finite sequences is a natural number. $+,\cdot,-,|\ |$ are defined on exactly the real numbers. Reciprocal is defined exactly for all nonzero real numbers.

3. <u>Axioms of explicit definition</u>. The unary and binary functions together are closed under substitution. $+,\cdot,-,1/,|\ |$ define functions. All constant functions exist. For each $n \geq 1$ there is an F such that $F(\alpha,x) = y$ if and only if $\alpha(x) = y$, for all $\alpha \in E^n$. Every set is the domain of some function. Every real valued function can be extended to a total function by making it 0 off the domain. The restriction of any function to a set exists. For all $F\colon \{1,\ldots,n\} \times A \to E^1$ there is a $g\colon A \to E^n$ such that for $\alpha \in A$, $1 \leq k \leq n$, $g(\alpha)(k) = F(k,\alpha)$.

4. <u>Normal Archimedean ordered field axioms</u>. The reals are an Archimedean ordered field under $+,\cdot,-,1/,<$. $0 \leq x \to |x| = x$, $x < 0 \to |x| = 0 - x$, $(0 < x \ \&\ 0 < y) \to (\exists n)(x < n\cdot y)$.

5. <u>ΣΠ axioms</u>. $\sum_{k=1}^{n} f(k) \simeq \prod_{k=1}^{n} f(k) \simeq f(1)$, $\sum_{k=1}^{n+1} f(k) \simeq f(n+1) + \sum_{k=1}^{n} f(k)$, $\prod_{k=1}^{n+1} f(k) \simeq f(n+1) \cdot \prod_{k=1} f(k)$.

6. <u>Sequential induction axioms.</u> $0 \notin N$, $1 \in N$, $n + m \in N$. If $f\colon N \to E^1$ and $f(1) = 0$, $(\forall n)(f(n) = 0 \to f(n+1) = 0)$, then $(\forall n)(f(n) = 0)$.

7. Cauchy completeness axioms. $\lim_{n\to\infty} f(n) = x$ if and only if $(\forall p)(\exists q)(\forall r)(q < r \to |x - f(r)| < 1/p)$. $(\exists x)(\lim_{n\to\infty} f(n) = x)$ if and only if $(\forall p)(\exists q)(\forall r)(\forall s)((q < r \;\&\; q < s) \to |f(r) - f(s)| < 1/p)$.

8. Pointwise limit axiom. If $F\colon N \times A \to E^1$ and $\lim_{n\to\infty} F(n,\alpha)$ exists for all $\alpha \in A$, then there is a $g\colon A \to E^1$ such that $g(\alpha) = \lim_{n\to\infty} F(n,\alpha)$, for all $\alpha \in A$.

We now present the raw formal system $|RV|$. The purely logical axioms and rules of inference are as follows.

A. All propositional tautologies. B. $((\forall\lambda)(\varphi) \;\&\; (t)) \to \varphi[\lambda/t]$. C. $(\varphi[\lambda/t] \;\&\; D(t)) \to (\exists\lambda)(\varphi)$. D. $D(\lambda)$. E. $D(s) \to D(t)$, for regular subterms t of s. F. $\varphi \to D(s)$, for all terms s appearing in the atomic formula φ. G. $s \simeq t \leftrightarrow ((D(s) \vee D(t)) \to s = t)$. H. $\lambda = \lambda$, $\lambda = \kappa \to K = \lambda$, $(\lambda = \kappa \;\&\; \kappa = \mu) \to \lambda = \mu$. I. $\lambda = s \to (\varphi \to \varphi[\lambda/s])$. J. $(\forall k)(s \simeq t) \to (\sum_{k=1}^{n} s \simeq \sum_{k=1}^{n} t \;\&\; \prod_{k=1}^{n} s \simeq \prod_{k=1}^{n} t \;\&\; \lim_{k\to\infty} s \simeq \lim_{k\to\infty} t)$. From $\varphi, \varphi \to \psi$ derive ψ. L. From $\varphi \to \psi$ derive $\varphi \to (\forall\lambda)(\psi)$. M. From $\psi \to \varphi$ derive $(\exists\lambda)(\psi) \to \varphi$.

Here, λ, k, μ are variables and s,t are terms so that the above are all formulas; and m free occurence of λ in φ lies within the scope of a quantifier $(\forall v)$ or $(\exists v)$, where v is free in t; λ is not free in ψ.

The proper axioms of $|RV|$ are as follows.

I. Equality axioms. $A = B \leftrightarrow (\forall\alpha)(\alpha \in A \leftrightarrow \alpha \in B)$ $\alpha = \beta \leftrightarrow (\forall n)(\alpha(n) \simeq \beta(n))$.

II. Miscellaneous axioms. $\alpha \in N \leftrightarrow (\exists n)(n = \alpha)$. $(\forall n)(\exists x)(n = x)$. $(\forall\alpha)(\ell th(\alpha) = 1 \leftrightarrow (\exists x)(x = \alpha))$. $\ell th(\alpha) \in N$. $(\forall\alpha)(\alpha \in E^n \leftrightarrow \ell th(\alpha) = n)$. $A \subset B \leftrightarrow (\forall\alpha)(\alpha \in A \to \alpha \in B)$. $(\exists n)(A \subset E^n)$. $(f\colon A \to B) \leftrightarrow (\forall\alpha)((D(f(\alpha)) \leftrightarrow \alpha \in A) \;\&\; D(f(\alpha)) \to f(\alpha) \in B)$. $(F\colon A \times B \to C) \leftrightarrow (\forall\alpha)(\forall\beta)((D(F(\alpha,\beta)) \leftrightarrow (\alpha \in A \;\&\; \alpha \in B)) \;\&\; D(F(\alpha,\beta)) \to F(\alpha,\beta) \in C)$.

$D(|\alpha|) \leftrightarrow \alpha \in E^1$. $D(1/\alpha) \leftrightarrow (\alpha \in E^1 \ \& \ \alpha \neq 0)$.

III. Axioms of explicit definition. $(\exists J)(\forall\alpha)(\forall\beta)(J(\alpha,\beta) \simeq F(G(\alpha,\beta),H(\alpha,\beta)))$, $(\exists J)(\forall\alpha)(\forall\beta)(J(\alpha,\beta) \simeq f(G(\alpha,\beta)))$, $(\exists h)(\forall\alpha)(h(\alpha) \simeq J(f(\alpha),g(\alpha)))$, $(\exists F)(\exists G)(\forall\alpha)(\forall\beta)((\alpha \in E^n \ \& \ \beta \in E^m) \rightarrow (F(\alpha,\beta) = \alpha \ \& \ G(\alpha,\beta) = \beta))$, $(\exists J)(\forall\alpha)(\forall\beta)((\alpha \in E^n \ \& \ \beta \in E^m) \rightarrow J(\alpha,\beta) = \gamma)$. $(\exists F)(\exists G)(\exists H)(\forall\alpha)(\forall\beta)(F(\alpha,\beta) \simeq \alpha + \beta \ \& \ G(\alpha,\beta) \simeq \alpha \cdot \beta \ \& \ H(\alpha,\beta) \simeq \alpha - \beta)$, $(\exists f)(\exists g)(\forall\alpha)(f(\alpha) \simeq 1/\alpha \ \& \ g(\alpha) \simeq |\alpha|)$. $(\exists F)(\forall\alpha)(\forall\beta)(\alpha \in E^n \rightarrow F(\alpha,\beta) \simeq \alpha(\beta))$. $(\forall A)(\forall B)(\exists F)(F\colon A \times B \rightarrow E^1)$. $(\forall\alpha)(\forall\beta)(D(F(\alpha,\beta)) \rightarrow (\alpha \in E^n \ \& \ \beta \in E^m \ \& \ F(\alpha,\beta) \in E^1)) \rightarrow (\exists G)(\forall\alpha)(\forall\beta)((D(F(\alpha,\beta)) \rightarrow G(\alpha,\beta) = F(\alpha,\beta)) \ \& \ (\alpha \in E^n \ \& \ \beta \in E^m \ \& \sim D(F(\alpha,\beta))) \rightarrow G(\alpha,\beta) = 0))$. $(\exists A)(\forall\alpha)(\alpha \in A \leftrightarrow D(f(\alpha)))$. $(\exists g)(\forall\alpha)(\forall\beta)(g(\alpha) = \beta \leftrightarrow (f(\alpha) = \beta \ \& \ \alpha \in A))$. $(F\colon B \times A \rightarrow E^1 \ \& \ (\forall x)(x \in B \leftrightarrow (x \in N \ \& \ 1 \leq x \leq n))) \rightarrow (\exists g)(g\colon A \rightarrow E^n \ \& \ (\forall\alpha)(\forall k)(g(\alpha)(k) \simeq F(k,\alpha)))$, $(\exists G)((\forall n)(\forall\alpha)(\forall\beta)(G(n,\alpha) \simeq \sum_{k=1}^{n} F(k,\alpha)) \ \& \ D(G(\beta,\alpha)) \rightarrow \beta \in N)$, $(\exists G)((\forall n)(\forall\alpha)(\forall\beta)(G(n,\alpha) \simeq \prod_{k=1}^{n} F(k,\alpha) \ \& \ D(G(\beta,\alpha)) \rightarrow \beta \in N)$.

IV. Normed Archimedean ordered field axioms. $x + y = y + x$, $x + (y + z) = (x + y) + z$, $0 + x = x$, $x + (0 - x) = 0$, $x - y = x + (0 - y)$, $x \cdot y = y \cdot x$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $1 \cdot x = x$, $x \neq 0 \rightarrow x \cdot (1/x) = 1$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, $0 \leq x \rightarrow |x| = x$, $x < 0 \rightarrow |x| = 0 - x$, $\sim(x < x)$, $(x < y \ \& \ y < z) \rightarrow x < z$, $x < y \vee y < x \vee x = y$, $x < y \rightarrow x + z < y + z$, $(x < y \ \& \ 0 < z) \rightarrow x \cdot z < y \cdot z$, $(0 < x \ \& \ 0 < y) \rightarrow (\exists n)(x < n \cdot y)$.

V. $\Sigma\Pi$ axioms. $f(1) \in E^1 \rightarrow (\sum_{k=1}^{1} f(k) = \prod_{k=1}^{1} f(k) = f(1))$, $D(\sum_{k=1}^{n} f(k)) \leftrightarrow D(\prod_{k=1}^{n} f(k)) \leftrightarrow f(1) \in E^1$, $\sum_{k=1}^{n+1} f(k) \simeq f(n + 1) + \sum_{k=1}^{n} f(k)$, $\prod_{k=1}^{n+1} f(k) \simeq f(n + 1) \cdot \prod_{k=1}^{n} f(k)$.

VI. Sequential induction axioms. $0 \notin N$, $1 \in N$, $n + m \in N$, $(f\colon N \rightarrow E^1 \ \& \ f(1) = 0 \ \& \ (\forall n)(f(n) = 0 \rightarrow f(n + 1) = 0) \rightarrow (\forall n)(f(n) = 0)$.

VII. Cauchy completeness axioms. $\lim_{n\to\infty} f(n) = \alpha \leftrightarrow (\forall p)(\exists q)(\forall r)(q < r \rightarrow |\alpha - f(r)| < 1/p)$. $(\exists x)(\lim_{n\to\infty} f(n) = x) \leftrightarrow (\forall p)(\exists q)(\forall r)(\forall s)(q < r \ \& \ q < s) \rightarrow |f(r) - f(s)| < 1/p)$.

VIII. Pointwise limit axiom. $(F\colon N \times A \rightarrow E^1 \ \& \ (\forall \alpha)(\alpha \in A \rightarrow D(\lim_{n\to\infty} F(n,\alpha)))) \rightarrow (\exists g)(g\colon A \rightarrow E^1 \ \& \ (\forall \alpha)(\alpha \in A \rightarrow g(\alpha) = \lim_{n\to\infty} F(n,\alpha)))$.

3. Some calculations of logical strengths.

A very substantial amount of analysis on Euclidean space is already provable in RV . This is discussed in detail in "The analysis of mathematical texts, and their calibration in terms of intrinsic strength IV", pp. 3-24, August, 1975. Also, a number of strengths were given there, which we append at the end.

As a guide to the results, we have come across four strengths. Class I is measured by Peano arithmetic, or $TI(<\epsilon_0)$. By the latter, we mean the quantifier theory of α-induction and α-recursion, based on standard notations for $\alpha<\epsilon_0$. Class II is measured by the theory $ATR(<\omega^\omega)$, which is the subsystem of second order arithmetic based on the closure of sets under the α-jump (and recursivity, join), for each standard notation for $\alpha<\omega^\omega$: Or also $TI(<K^\omega(0))$, where standard notations are used for all $\alpha<K^\omega(0)$. Here $K^0(\beta)=\omega^\beta$, $K^{\gamma+1}(\beta)$ is the βth fixed point of K^γ, $K^\lambda(\beta)$ is the βth simultaneous fixed point for all K^γ, $\gamma<\lambda$. Another characterization of the strength is the subsystem of second order arithmetic HDC with arithmetic induction only, which we call HDC_0 in the June 1976 JSL abstract. Class III is measured by the theory $ATR(<\Gamma_0)$, or $TI(<\Gamma_0)$. Here Γ_0 is the least α with $K^\alpha(0)=\alpha$. Also class III is measured by the sybsystem of analysis ATR with arithmetic induction only, which we call ATR_0 in the June 1976 JSL abstract. The ATR stands for "arithmetic transfinite recursion". Class IV is measured by the system $ID(<\omega)$ of finitely iterated inductive definitions. See Feferman's paper in the North Holland volume Intuitionism and Proof Theory, 1968, for detailed axioms. It is the theory of $\Theta, \Theta^{\Theta}, \Theta^{(\Theta^{\Theta})}$, etc. Or the subsystem of second order arithmetic given by Π^1_1 - CA with arithmetic induction only, which we call Π^1_1 - CA_0 in the JSL abstract.

Class I can also be measured by HAC_0 . Class IV can also be measured by Σ^1_2 - AC_0 . (The latter was independently observed by S. Feferman, observing its being omitted in the JSL abstract).

At this point, we only intend to state some results obtained since the "Text paper IV". A thorough discussion of the proper organization of these results, general conclusions, and further results, will be in "The logical strength of mathematical statements II".

We know that all of the below which are designated (I); i.e., of class I when added to RV , are in fact not provable in RV .

A few remarks about the below.

A. Item 19 is of special interest. By bounded variation, we mean that there is an upper bound to the variations. This is different than the definition adopted in "Text paper IV". All other notions are taken from there.

B. Item 14 corrects an error in "Text IV".

C. In item 15, w.m. in [0,1] means that for all $\epsilon > 0$ there are two sequences of balls in [0,1] whose total lengths are respectively α, β , the first of which covers the set, the second of which covers the complement of the set, and where $\alpha + \beta < 1 + \epsilon$.

D. In item 16, we mean that $f: [0,1] \to [0,1]$ has the following property. There is a function which assigns to each $0 \leq x \leq 1$ and $\epsilon > 0$, two sequences of balls in [0,1] with the property in 3 above for $f^{-1}[[0,X)]$.

This is all you seem to need for Lebesgue theory.

1. Continuous functions on bounded closed sets are unif. cont. (I).

2. Infinite sets have arb. large finite subsets. (II).

3. Infinite sets have denumerable subsets. (II).

4. Uncountable sets have limit points. (III).

5. Uncountable sets have perfect subsets. (III).

6. Limit pts are approachable. (I).

7. The least upper bound principle. (IV).

8. The inverse of a one-one function exists. (III).

9. $(\forall\alpha)(\exists\beta)(F(\alpha,\beta) = 0) \rightarrow (\exists G)(\forall\alpha)(F(\alpha,G(\alpha)) = 0)$. (III).

10. The sup of a function exists. (IV).

11. Bded Infinite sets have limit points. (II).

12. Every open set is the union of a sequence of open intervals. (IV).

13. The least upper bound principle for closed sets. (IV).

14. The intersection of a sequence of dense open sets is dense. (I).

15. There is an open set which is not weakly measurable in $[0,1]$. (I).

16. Every function on $[0,1]$ is strongly measurable. (I).

17. If $f: [0,1] \rightarrow F$ is bounded and is cont. a.e., then f is Riemann integrable. (I).

18. If $f: [0,1] \rightarrow E$ is Riemann integrable then f is cont. a.e. (I).

19. Every function of bounded variation is the difference between two monotone increasing functions. (III).

20. Every subset of E^n closed under addition and scalar multiplication, is spanned by a finite independent set. (II).

21. The set of points of continuity of functions on the reals exists. (IV).

22. The conjunction of 1, 6, 14, 16, 17, 18. (I).

23. The conjunction of 22, 2, 3, 11, 20. (II).

24. The conjunction of 23, 4, 5, 8, 9, 19. (III).

25. The conjunction of 1-21. (IV).

Taken from "The analysis of mathematical texts, and their calibration in terms of intrinsic strength IV"

Theorems of Strength PA

The theorems below are provable in RV_1 + sequential choice, which reads $(\forall n)(\exists\alpha)(f(n,\alpha) = 0) \rightarrow (\exists g)(\forall n)(f(n,g(n)) = 0)$, and is equiconsistent with PA .

THEOREM 1. Every limit point x of A is the limit of a sequence of elements from A other than x .

THEOREM 2. If a set contains arbitrarily large finite subsets, then it contains a denumerable subset.

THEOREM 3. Every nhbd of a limit point of A contains a denumerable subset from A .

THEOREM 4. Every sequence of open sets covering a compact set has an initial segment covering it also.

THEOREM 5. A continuous function on a compact set is uniformly continuous.

THEOREM 6. If $f_n \geq f_{n+1}$ on a compact set E , $f_n \rightarrow f$ on E , and f_n, f are continuous on E , then $f_n \rightarrow f$ uniformly on E .

THEOREM 7. Let K be a compact set. If $\{f_n\}$ is a uniformly convergent sequence of continuous functions on K , then $\{f_n\}$ is equicontinuous.

THEOREM 8. If $f: I^k \rightarrow \mathbb{R}$, f is continuous a.e., then f is Riemann integrable.

THEOREM 9. If $A \subset B$, A compact, B open, then there is a finite sequence of open rectangles from B which cover A .

In Theorem 8, cont. a. e. means that for each $\epsilon > 0$ there is a sequence of open rectangles, the sum of whose volumes is $< \epsilon$, such that the function is continuous off of these rectangles.

Theorems of Strength $ATR(<\omega^{\omega})$

The theorems below are equiconsistent with $ATR(<\omega^{\omega})$ when added to RV_1, and are provable in RV_1 + dependent choice, which is

$(\forall\alpha)(\exists\beta)(f(\alpha,\beta) = 0) \rightarrow (\forall\alpha)(\exists g)(g(0) = \alpha \;\&\; (\forall n)(f(g(n),g(n+1)) = 0))$,

and is also equiconsistent with $ATR(<\omega^{\omega})$.

THEOREM 1. A set is infinite if and only if it contains a denumerable subset.

THEOREM 2. A set is infinite if and only if it contains arbitrarily large finite subsets.

THEOREM 3. If $A \subset \mathbb{R}^n$ is closed under addition and scalar multiplication, then A is a vector space.

THEOREM 4. The finite union of finite sets is finite.

Theorems of Strength $ATR(<\Gamma_0)$, ATR_0

The following theorems are equiconsistent with $ATR(<\Gamma_0)$ and ATR_0, when added to RV_1.

THEOREM 1. Every one-one function has an inverse.

THEOREM 2. Every graph determines a function.

THEOREM 3. Countable union of countable sets is countable.

THEOREM 4. The countable intersection of dense open subsets of $\mathbb{R}^n$ is dense.

Theorems of Strength $ID(<\omega)$

The theorems below are equiconsistent with $ID(<\omega)$ when added to RV_1.

THEOREM 1. Every nonempty bounded set of reals has a least upper bound.

THEOREM 2. Every open set is sequentially open.

THEOREM 3. Every bounded closed set of reals has a least upper bound.

THEOREM 4. Every indexed family of open sets covering the unit interval contains a finite subcover.

THEOREM 5. The set of interior points of every set exists.