THE ANALYSIS OF MATHEMATICAL TEXTS, AND THEIR CALIBRATION IN TERMS OF INTRINSIC STRENGTH III

by

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This is the third in a series of informally distributed reports on an ongoing program.

> It's not what the mathematician would accept if confronted with; but what the mathematician has accepted and been confronted with.

The universal calculus U is presented in detail in Part A, with both a proof theory and a model theory. The completeness theorem tells us that the approach to consistency in terms of proofs (no proof of a contradiction) and the more mathematical approach in terms of models (having a model) are equivalent. Also, in Part A we give examples of how to translate sections of mathematical text into U . We can relate this in a precise way to the translation of bodies of mathematics as follows.

By a (fully rigorous) body of mathematics, we mean a series of mathematical statements, every one of which is labeled as an assumption or a theorem, such that the theorems follow purely deductively from the preceding assumptions and theorems. Thus we are taking a body of mathematics so as not to include proofs or lemmas, and where definitions are not distinguished

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from assumptions.

Given a body $\mathcal{B}$ of mathematics, we can, uniquely up to a change of letters, associate a sequence $|\mathcal{B}|$ of sentences in U . We can obtain $|\mathcal{B}|$ most simply by setting A to be the conjunction $(A_1 \& A_2 \& \ldots \& A_n)$ of all assumptions in $\mathcal{B}$, letting $\Phi = (\varphi_1 \& \varphi_2 \& \ldots \& \varphi_n)$ be a representation of A in U , and setting $|\mathcal{B}|$ to be the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_n)$.

We have thus associated a combinatorial structure $|\mathcal{B}|$ to every body of mathematics, which we call the <u>raw formal system</u> for $\mathcal{B}$.

Part B formulates some basic postulational axioms (and a few basic theorems) that are found in various bodies of analysis on Euclidean space. Actually, we confine ourselves to Advanced Calculus for this and several reports, before entering the Lebesgue theory on Euclidean space. In Part C we calculate the intrinsic strength of systems based on these axioms. It is here that we must be careful to avoid adding things into the mathematics that are not already there. Thus these parts relate to the providing of lower bounds on the intrinsic strengths of bodies of mathematics.

Parts D, E relate to the providing of upper bounds on the intrinsic strengths of bodies of mathematics. Again, we concentrate on axioms of a postulational nature, rather than theorems.

PLAN OF THE WORK

<u>Report III</u>. Basic results concerning postulational axioms in Advanced Calculus.

<u>Report IV</u>. Theorems of Advanced Calculus (uniform continuity theorem, Heine-Borel theorem, Bolzano-Weierstrass theorem, local maxima theorem,

mean value theorem, Riemann integrability theorem, equivalence of notions of closedness and compactness, equivalence of notions of continuity, Taylor's theorem, intermediate value theorem, inverse function theorem, convergence tests for series, Fourier convergence theorems, etc.) Their intrinsic strength. The proper formal systems in which they are naturally proved.

Report V. Theorems of Lebesgue theory. Their intrinsic strength. The proper formal systems in which they are naturally proved.

By way of background, ATR $(<\lambda)$ stands for the subsystem of second order arithmetic with full induction, and arithmetic comprehension, and the existence of an H-set based on any specified initial segment of λ, relative to any set. (The latter is therefore a scheme.) ATR stands for "arithmetic transfinite recursion". Γ_0 is the Feferman-Schutte ordinal for predicativity. ID $(<\omega)$ is the theory of $\mathcal{O}$, $\mathcal{O}^{\mathcal{O}}$, etc., presented as a first order theory of inductive definitions iterated arbitrarily finitely often. For discussions of these and other systems, and their strengths, see my article in the 1974 ICM proceedings, and the abstracts "Subsystems of second order arithmetic with restricted induction I, II", to appear in the JSL.

The following pairs of theories can be used to calculate the intrinsic strength of certain bodies of mathematics.

1 of Part C and 5 of Part E.

4 of Part C and 12 of Part E.

4 of Part C and 13 of Part E.

5 of Part C and 19 of Part E.

6 of Part C and 22 of Part E.

6 of Part C and 23 of Part E.

7 of Part C and 26 of Part E.

We have concentrated on providing equiconsistency or "strength" results, rather than the additional information that provable equivalences and conservative extension results provide. These will be taken up in a later report.

PART A

THE CALCULUS U

U will possess infinitely many constant symbols c_n, infinitely many m-ary relation symbols R_n^m, infinitely many m-ary partial function symbols F_n^m for $1 \leq m$, and the special unary relation symbol D, for "being defined". In addition U will have infinitely many parameters a_n, infinitely many variables x_n, the connectives $\sim, \&, \vee, \rightarrow, \leftrightarrow$, and the quantifiers $\forall, \exists$.

The terms of U are defined by the following clauses. Each constant is a term. Each parameter is a term. If F is an n-ary partial function symbol and $t_1,\ldots,t_n$ are terms, then $F(t_1,\ldots,t_n)$ is a term. If $s,t_1,\ldots,t_n$ are terms then $s(t_1,\ldots,t_n)$ is a term.

The atomic formulae of U are written $R(t_1,\ldots,t_n)$ and $D(t)$, where $t,t_1,\ldots,t_n$ are terms, and R is an n-ary relation symbol.

Before defining the formulae of U, it is necessary for us to make a technical definition. If φ is a string of symbols, $\underline{a}$ is any symbol, and α is any string of symbols, we let φ_α^a be the string of symbols resulting from φ by replacing each occurrence of $\underline{a}$ in φ by α.

The formulae of U are given by the following clauses. Every atomic formula of U is a formula. If φ, ψ are formulae, so are $(\sim\varphi)$, $(\varphi \vee \psi)$, $(\varphi \& \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$. If φ is a formula, $\underline{a}$ is a parameter, x is a variable, x does not occur in φ, then $(\forall x)(\varphi_x^a)$, $(\exists x)(\varphi_x^a)$ are formulae.

A U-structure $(D,d_m,r_m^n,f_m^n,g_p) = M$ consists of a nonempty set D, elements $d_m \in D$, everywhere defined n-ary relations r_m^n on D, n-ary partial functions f^n_m on D, and p-ary partial functions g_p on D. Here $n \geq 1$, $p \geq 2$.

We define $Val(M,s,\alpha)$, where s is a term and $\alpha \in D^\omega$, by i) $Val(M,a_m,\alpha) = \alpha(m)$ ii) $Val(M,c_m,\alpha) = d_m$ iii) $Val(M,F^n_m(s_1,\ldots,s_n),\alpha) \simeq f^n_m(Val(M,s_1,\alpha),\ldots,Val(M,s_n,\alpha))$ iv) $Val(M,t(s_1,\ldots,s_{p-1}),\alpha) \simeq g_p(Val(M,t,\alpha), Val(M,s_1,\alpha),\ldots,Val(M,s_{p-1},\alpha))$.

We define $M \models \varphi[\alpha]$ by induction on φ as follows. $M \models R^n_m(s_1,\ldots,s_n)[\alpha]$ iff each $Val(M,s_i,\alpha)$ is defined, and $r^n_m(Val(M,s_1,\alpha),\ldots,Val(M,s_n,\alpha))$. $M \models D(s)[\alpha]$ iff $Val(M,s,\alpha)$ is defined. $M \models \sim\varphi[\alpha]$ iff not $M \models \varphi[\alpha]$. $M \models (\varphi \,\&\, \psi)[\alpha]$ iff $M \models \varphi[\alpha]$ and $M \models \psi[\alpha]$. $M \models (\varphi \vee \psi)[\alpha]$ iff $M \models \varphi[\alpha]$ or $M \models \psi[\alpha]$. $M \models (\varphi \rightarrow \psi)[\alpha]$ iff (if $M \models \varphi[\alpha]$ then $M \models \psi[\alpha]$). $M \models (\varphi \leftrightarrow \psi)[\alpha]$ iff ($M \models \varphi[\alpha]$ iff $M \models \psi[\alpha]$). $M \models (\forall x_m)(\varphi^{a_n}_{x_m})[\alpha]$ iff for all $d \in D$, $M \models \varphi[\alpha^n_d]$. $M \models (\exists x_m)(\varphi^{a_n}_{x_m})[\alpha]$ iff for some $d \in D$, $M \models \varphi[\alpha^n_d]$.

We say $M \models \varphi$ just in case $M \models \varphi[\alpha]$, for all α.

The relation " T proves φ " is given by the following clauses. T proves every element of T. If T proves every element of the set X of formulae, and X tautologically implies the formula φ, then T proves φ. T proves every formula of the form $(D(t) \,\&\, (\forall x)(\varphi)) \rightarrow \varphi^x_t$. T proves every formula of the form $(\varphi^x_t \,\&\, D(t)) \rightarrow (\exists x)(\varphi)$. If T proves $(\varphi \rightarrow \psi)$ then T proves $(\varphi \rightarrow (\forall x)(\psi^a_x))$, provided the parameter a does not occur in φ, and the latter is a formula. If T proves $(\varphi \rightarrow \psi)$ then T proves $((\exists x)(\varphi^a_x) \rightarrow \psi)$, provided the parameter a does not occur in ψ, and the latter is a formula. T proves $D(a)$, for any parameter a. T proves $(D(F(t_1,\ldots,t_n)) \rightarrow D(t_j))$, provided the latter is a formula, $1 \leq j \leq n$. T proves $(D(s(t_1,\ldots,t_n)) \rightarrow D(t_j))$, provided $s,t_1,\ldots,t_n$ are terms,

$1 \leq j \leq n$. T proves $R(t_1,\ldots,t_n) \rightarrow D(t_j)$, where R is an n-ary relation symbol, $1 \leq j \leq n$. T proves $(D(s(t_1,\ldots,t_n)) \rightarrow D(s))$.

" T constructively proves φ " is given by precisely the same clauses, except that in clause 2, "tautologically implies" is replaced by "intuitionistically tautologically implies".

COMPLETENESS THEOREM. T proves φ just in case $M \models \varphi$ whenever $M \models T$.

We now describe how to represent mathematical assertions (including mathematical definitions) in U . The representations are unique up to changes in letters. We describe the representations by examples.

1. " f is an harmonic function." The representations are formulae $R(a)$.

2. " f is an harmonic function and f(0) is positive." The representations are $(R(a) \& S(a(c)))$, where R , S are distinct.

3. " f is defined at a ." The representations are $D(b(a))$, where <u>a</u> , b are distinct.

4. " x , y are equal." The representations are $R(a,b)$, where a , b are distinct.

5. " f is a function from reals to reals." The representations are $(\forall x)(R(x) \rightarrow R(a(x))) \& (\forall x)(D(a(x)) \rightarrow R(x))$.

6. " $x < y$ iff $0 < y - x$." The representations are $(R(a,b) \leftrightarrow R(c,F(b,a)))$, where a , b are distinct.

7. "the set of zeros of f is compact." The representations are $R(F(a))$.

8. "if f is continuous then $f'(x)$ exists." The representations are $R(a) \rightarrow D(F(a)(b))$.

9. "the members of the domain of f are precisely the places at which

f is defined." The representations are $(\forall x)(R(x,F(a)) \leftrightarrow D(a(x)))$.

Optional additions may be made to U . For instance, we may allow introduction of special parameters a_R and variables x_R , which range over those $\underline{a}$ such that $R(a)$, and those x such that $R(x)$, where R is a unary relation symbol. Another optional addition is the use of variable binding operators, such as $\sum_{k=0}^{n} f(k)$, $\prod_{k=0}^{n} f(k)$. The former can be dispensed with in the standard way. The latter can be replaced by $(\Sigma f)(n)$, $(\Pi f)(n)$. From the point of view of determining intrinsic logical strength, these additions are unnecessary. Nevertheless, it is most convenient to use the first addition in Appendices B - E, and both additions will be used in future reports.

PART B

LOWER BOUND AXIOMS

We begin with a basic theory of real sequences (and combinatorial functions). We use variables n_j over natural numbers (N), a_j over integers (Z), x_j over reals (R), f_j over sequences of integers $(N \to Z)$, g_j over unary functions of integers $(Z \to Z)$, h_j over sequences of functions on the integers $(h_j : N \times Z \to Z)$, and F_j over sequences of reals $(F_j : N \to R)$. The f, g, h, and F are understood to be total. We have the constants $0, 1$, the operation symbols $+, -, \bullet, 1/, |\ |$, and the relation symbols $=, <$.

The terms, N-terms, and Z-terms are given as follows. Every N-term is a Z-term, and every Z-term is a term. $0, 1, n_j$ are N-terms. Each a_j is a Z-term, and each x_j is a term. If s, t are N-terms, so are $s+t$, $s \bullet t$. If s is a Z-term, then $|s|$ is an N-term. If s is an N-term, t a Z-term, then $f_j(s)$, $g_j(t)$, and $h_j(s,t)$ are Z-terms. If s is an N-term, then $F_j(s)$ is a term. If s, t are Z-terms, so are $s+t$, $-s$, $s \bullet t$ and $|s|$. If s, t are terms, so are $s+t$, $-s$, $s \bullet t$, $1/s$, and $|s|$.

The formulae are given by i) $s = t$, $s < t$, $D(s)$ are formulae for terms s, t ii) $\sim A$, $A \vee B$, $A \,\&\, B$, $A \to B$, $A \leftrightarrow B$ are formulae if A, B are iii) $(\forall\alpha)(A)$, $(\exists\alpha)(A)$ are formulae if A is, and α is any variable.

Here are the logical axioms and rules.

AXIOMS

1. all propositional tautologies. 2. $((\forall\alpha)(A) \,\&\, D(\alpha)) \to A_t^{\alpha}$.

3. $(\forall G)(A) \to A^G_H$. 4. $(A^\alpha_t \ \& \ D(t)) \to (\exists\alpha)(A)$. 5. $A^G_H \to (\exists G)(A)$.
6. $D(\alpha)$. 7. $D(s) \to D(t)$, for subterms t of s . 8. $(s < t \vee s = t) \to (D(s) \ \& \ D(t))$.

RULES

9. from $A, A \to B$ derive B . 10. from $A \to B$ derive $A \to (\forall\beta)(B)$.
11. from $B \to A$ derive $(\exists\beta)(B) \to A$.

In the above, α is a variable n_j , a_j , or x_j ; t is a term of the same sort as α ; G , H are function variables of the same sort; and β is any variable not free in A . We require that no free occurrence of α in A lie within the scope of a quantifier $(\forall\gamma)$ or $(\exists\gamma)$, where γ is a variable in t , and that no free occurrence of G in A lie within the scope of a quantifier $(\forall G)$ or $(\exists G)$.

It will be convenient to use m, n, p, q for natural numbers; a, b, c, d, e for integers; and u, v, w, x, y, z for reals; f, g, h, F for total $f : N \to Z$, $g : Z \to Z$, $h : N \times Z \to Z$, $F : N \to R$; and G, H for partial $G : R \to R$, $H : N \times R \to R$.

We have the axioms NAOF (normed Archimedean ordered field).

1. $x = x$, $x = y \to y = x$, $(x = y \ \& \ y = z) \to x = z$.
2. $(x = y \ \& \quad = w \ \& \ x \neq 0) \to (x + z = y + w \ \& \ x \cdot z = y \cdot w \ \& \ -x = -y \ \& \ |x| = |y| \ \& \ (x < z \leftrightarrow y < w) \ \& \ (x \neq 0 \to 1/x = 1/y))$.
3. $x + y = y + x$, $x + (y + z) = (x + y) + z$, $0 + x = x$, $x + (-x) = 0$, $x \cdot y = y \cdot x$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $1 \cdot x = x$, $x \neq 0 \to x \cdot 1/x = 1$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, $0 \leq x \to |x| = x$, $x < 0 \to |x| = -x$.

4. $\sim(x < x)$, $(x < y \ \& \ y < z) \rightarrow x < z$, $x < y \vee y < x \vee x = y$, $x < y \rightarrow x+z < y+z$, $(x < y \ \& \ 0 < z) \rightarrow x \cdot z < y \cdot z$, $(\forall n)(n \neq 0 \leftrightarrow 1 \leq n)$.

5. $(0 < x \ \& \ 0 < y) \rightarrow (\exists n)(x < n \cdot y))$.

We will use the additional axioms of identity $n = m \rightarrow f(n) = f(m)$, $a = b \rightarrow g(a) = g(b)$, $(n = m \ \& \ a = b) \rightarrow h(n,a) = h(m,b)$, $n = m \rightarrow F(n) = F(m)$.

We will use the following ontological axioms. $(\exists a)(a = n)$, $(\exists n)(n = a \leftrightarrow a \geq 0)$, $(\exists a)(a = f(n))$, $(\exists a)(a = g(b))$, $(\exists a)(a = h(n,b))$, $D(f(x)) \rightarrow (\exists n)(n = x)$, $D(g(x)) \rightarrow (\exists a)(a = x)$, $D(h(x,y)) \rightarrow (\exists n)(\exists a)(n = x \ \& \ a = y)$, $D(F(x)) \rightarrow (\exists n)(n = x)$.

We next consider axioms of explicit definition. The simplest way to present these is by the definitional schemes $(\forall n)(D(s)) \rightarrow (\exists f)(\forall n)(f(n) = s)$, $(\forall a)(D(s)) \rightarrow (\exists g)(\forall a)(g(a) = s)$, $(\forall n)(\forall a)(D(s)) \rightarrow (\exists h)(\forall n)(\forall a)(h(n,a) = s)$, $(\forall n)(D(t)) \rightarrow (\exists F)(\forall n)(F(n) = t)$, where s is any Z-term, and t is any term. We wish to avoid schemes like the plague, for they are a logician's fiction. One alternative and equivalent way is to consider only those terms such that every proper subterm is of the form $f_1(m)$, $g_1(b)$, $h_1(m,b)$, $F(m)$, $|f_1(m)|$, $|g_1(b)|$, m , b , x , $|b|$. This ensures finite axiomatizability. The finite number of instances, alternatively, can be explicitly written down.

Next comes the axioms of variable summation and product. $(\forall F)(\exists F_1)(\exists F_2 : N \rightarrow R)(F_1(0) = F(0) \ \& \ (\forall n)(F_1(n+1) = F_1(n) + F(n+1)) \ \& \ F_2(0) = F(0) \ \& \ (\forall n)(F_2(n+1) = F_2(n) \cdot F(n+1)))$, $(\forall h)(\exists h_1)(\exists h_2)(h_1(0,a) = h(0,a) \ \& \ (\forall n)(h_1(n+1,a) = h_1(n,a) + h(n+1,a)) \ \& \ h_2(0,a) = h(0,a) \ \& \ (\forall n)(h_2(n+1,a) = h_2(n,a) \cdot h(n+1,a)))$.

Finally, we take elementary induction. $(\forall F)((F(0) = 0 \ \& \ (\forall n)(F(n) = 0 \rightarrow F(n+1) = 0)) \rightarrow (\forall n)(F(n) = 0))$, $(\forall F)(F(n) = 0 \rightarrow (\exists m)(F(m) = 0 \ \&$

$(\forall r)(F(r) = 0 \rightarrow m \leq r)))$.

We call the above system $\Sigma\Pi_0$.

We now introduce variables $G_j : R \rightarrow R$, and $H_j : N \times R \rightarrow R$, for the system $\Sigma\Pi_1$. It is understood that the G , H are <u>partial</u> function symbols.

We modify the clauses for term formation by adding the clause: if s is an N-term, t is a term, then $G_j(t)$, $H_j(s,t)$ are terms.

The axioms and rules of inference are carried over straightforwardly.

We add $x = y \rightarrow G(x) \simeq G(y)$, $(n = m \,\&\, x = y) \rightarrow H(n,x) \simeq H(m,y)$ to the axioms of identity.

We add $D(H(x,y)) \rightarrow (\exists n)(n = x)$ to the ontological axioms.

The axioms of explicit definition are expanded by referring to the new class of Z-terms s , and terms t , and also by adding the clauses $(\exists G)(\forall x)(G(x) \simeq t)$, $(\exists H)(\forall n)(\forall x)(H(n,x) \simeq t)$, t any term. The same subterm criterion is applicable. We also add the special axiom $(\exists G)(\forall x)(\forall y)(G(x) = y \leftrightarrow (0 \leq x \,\&\, y = 0))$.

The axioms of variable summation and product are extended by the following. $(\forall H)(\exists H_1)(\forall n)(\forall x)((D(H_1(n,x)) \leftrightarrow (\forall m \leq n)(D(H(m,x)))) \,\&\, H_1(0,x) \simeq H(0,x) \,\&\, H_1(n+1,x) \simeq H_1(n,x) + H(n+1,x))$, $(\forall H)(\exists H_2)(\forall n)(\forall x)((D(H_1(n,x)) \leftrightarrow (\forall m \leq n)(D(H(m,x)))) \,\&\, H_2(0,x) \simeq H(0,x) \,\&\, H_2(n+1,x) \simeq H_2(n,x) \bullet H(n+1,x))$.

Elementary induction remains unchanged.

The resulting system is called $\Sigma\Pi_1$.

To summarize, $\Sigma\Pi_1$ has nine sorts of objects: N , Z , R , $N \rightarrow Z$, $Z \rightarrow Z$, $N \times Z \rightarrow Z$, $N \rightarrow R$, $R \rightarrow R$, and $N \times R \rightarrow R$. The 4th, 5th, 6th and 7th are to be total. The last two, partial. $N \times Z \rightarrow Z$ is to be thought of as sequences of total unary maps on Z . $N \times R \rightarrow R$ is to be thought of as sequences of partial unary maps on R . The theory $\Sigma\Pi_0$ has only the first seven sorts.

We now introduce some additional axioms.

Definition 1. A sequence of real numbers is an $f : N \to R$, and is written $\{x_n\}$. We say $\{x_n\}$ is Cauchy just in case $(\forall\epsilon > 0)(\exists k)(\forall p,q > k)(|x_p - x_q| < \epsilon)$. We write $\{x_n\} \to y$ for $(\forall\epsilon > 0)(\exists k)(\forall n > k)(|x_n - y| < \epsilon)$. We say $\{x_n\}$ converges if $\{x_n\} \to y$ for some y . We say $\{x_n\}$ is explicitly Cauchy just in case for some f we have $(\forall n > 0)(\forall p,q > f(n))(|x_p - x_q| < 1/n)$.

Definition 2. A sequence of real functions is an H , and is written $\{H_n\}$. We say $\{H_n\}$ is explicitly Cauchy just in case $D(H_n(x)) \leftrightarrow D(H_0(x))$, and for some F we have $(\forall n > 0)(\forall p,q > g(n))(\forall x)(D(H_0(x)) \to |H_p(x) - H_q(x)| < 1/n)$. We write $\{H_n\} \to G$ to mean $(\forall x)(\forall y)(G(x) = y \leftrightarrow (\{H_n(x)\} \to y))$, where if $\{H_n(x)\}$ does not define a sequence, it is not considered to converge.

I. Explicit Cauchy completeness. Every explicitly Cauchy sequence of reals converges.

II. Cauchy completeness. Every Cauchy sequence converges.

III. Monotone completeness. If $(\forall n)(x_n < y)$, $(\forall n < m)(x_n \leq x_m)$, then $\{x_n\}$ converges.

IV. Explicit Cauchy limit. If $\{H_n\}$ is explicitly Cauchy, $(\forall x)(D(H_0(x)))$, then $(\exists G)(\{H_n\} \to G)$.

V. Step function. $(\exists f : R \to R)(\forall x)((x \leq 0 \to f(x) = 0) \& (x > 0 \to f(x) = 1))$.

VI. Pointwise explicit limit. If for all x , $\{H_n(x)\}$ is explicitly Cauchy, then $(\exists G)(\{H_n\} \to G)$.

VII. Bolzano-Weierstrass. $(\forall x)(D(G(x)) \to 0 \leq x \leq 1) \to ((\exists y)(\forall\epsilon > 0)(\exists x)$

$(D(G(x)) \;\&\; |y - G(x)| < \epsilon) \vee (\exists F)(\exists n)(\forall x)(D(G(x)) \rightarrow (\exists m < n)(F(m) = x)))$.

VIII. Inverse. Assume $G(x) = G(y) \rightarrow x = y$. Then $(\exists G_1)(\forall x)(\forall y)(G_1(x) = y \leftrightarrow G(y) = x)$.

IX. Least upper bound. If $(D(G(x)) \;\&\; (\forall y)(D(G(y)) \rightarrow y \leq z))$ then $(\exists w)(\forall y)(D(f(y)) \rightarrow y \leq w) \;\&\; (\forall u < w)(\exists y)(D(f(y)) \;\&\; u < y))$.

X. Complementation. $(\forall G)(\exists G_1)(\forall x)(D(G(x)) \leftrightarrow \sim D(G_1(x)))$.

XI. Range. $(\forall G)(\exists G_1)(\forall x)(D(G_1(x)) \leftrightarrow (\exists y)(x = G(y))$.

PART C

LOWER BOUND RESULTS

Equiconsistent with Peano arithmetic.

1. $\Sigma\Pi_0$ + Cauchy completeness.
2. $\Sigma\Pi_0$ + Monotone completeness.
3. $\Sigma\Pi_1$ + Explicit Cauchy limit + Step function.

Equiconsistent with $ATR(< \omega^{\omega})$.

4. $\Sigma\Pi_1$ + Pointwise explicit limit + Bolzano-Weierstrass.

Equiconsistent with $ATR(< \Gamma_0)$.

5. $\Sigma\Pi_1$ + Pointwise explicit limit + Inverse + Complementation.

Equiconsistent with $ID(< \omega)$.

6. $\Sigma\Pi_1$ + Pointwise explicit limit + Least upper bound.

Equiconsistent with 2nd order arithmetic.

7. $\Sigma\Pi_1$ + Pointwise explicit limit + Least upper bound + Complementation + Range.

PART D

UPPER BOUND AXIOMS

We have variables x_k ranging over real numbers, and n-ary function variables f_k^n ranging over partial n-ary functions from reals to reals.

In addition, we have the relation symbols $=$, $<$, N , the constants 0 , 1 , and the partial function symbols $+$, $\bullet$, $-$, $1/$, $| \ |$. We also have the special symbol D , for "being defined".

The terms are given as follows. Every variable x_k is a term. The constants 0 , 1 are terms. If s , t are terms, so are $s+t$, $s \bullet t$, $-s$, $1/s$, and $|s|$. If $s_1,\ldots,s_n$ are terms, then $f_k^n(s_1,\ldots,s_n)$ is a term.

The formulae are given by i) $s_1 = s_2$, $s_1 < s_2$, $N(s)$, and $D(s)$ are formulae. ii) $\sim A$, $A \vee B$, $A \& B$, $A \rightarrow B$, $A \leftrightarrow B$ are formulae if A , B are formulae. iii) $(\forall\alpha)(A)$, $(\exists\alpha)(A)$ are formulae if A is, and α is any variable.

The purely logical axioms and rules of inference common to all systems considered here are as follows.

AXIOMS

1. all propositional tautologies. 2. $((\forall y)(A) \& D(t)) \rightarrow A_t^y$. 3. $(\forall f)(A) \rightarrow A_g^f$. 4. $(A_t^y \& D(t)) \rightarrow (\exists y)(A)$. 5. $A_g^f \rightarrow (\exists f)(A)$. 6. $D(y)$.
7. $D(s) \rightarrow D(t)$, for subterms t of s . 8. $(s < t \vee s = t) \rightarrow (D(s) \& D(t))$.

RULES

9. from $A, A \rightarrow B$ derive B . 10. from $A \rightarrow B$ derive $A \rightarrow (\forall\alpha)(B)$.

11. from $B \to A$ derive $(\exists\alpha)(B) \to A$.

In the above, y is a real variable (i.e., an x_k), f , g are function variables of the same number of arguments, α is any variable, A^y_t is the result of replacing each free occurrence of y in A by t , A^f_g is the result of replacing each free occurrence of f in A by g . We require that no free occurrence of y in A be within the scope of a quantifier $(\forall x)$ or $(\exists x)$, where x is a variable in t , and that no free occurrence of f in A be within the scope of a quantifier $(\forall g)$ or $(\exists g)$. We also require that α is not free in A .

We now present the theories. The predicate symbol N is to denote "being a natural number"; it will be convenient to use $a, b, c, d, e, i, j, k, m, n, p, q, r, s, t$ for natural numbers. We will use $u, v, w, x, y, z, \epsilon, \delta$ for real numbers, and f, g, h, F, G, H for partial functions, where we leave off superscripts as long as no ambiguity arises. We use $s \neq t$ for $D(s)\ \&\ D(t)\ \&\ \sim s = t$. We use $s \simeq t$ for $(D(s) \vee D(t)) \to s = t$.

We first consider NAOF (normed Archimedean ordered field).

1. $x = x$, $x = y \to y = x$, $(x = y\ \&\ y = z) \to x = z$.
2. $(x = y\ \&\ z = w) \to (x + z = y + w\ \&\ x \cdot z = y \cdot w\ \&\ -x = -y\ \&\ |x| = |y|\ \&$ $(x < z \leftrightarrow y < w)\ \&\ (x \neq 0 \to 1/x = 1/y)\ \&\ (N(x) \leftrightarrow N(y)))$, $(x_1 = y_1\ \&\ \ldots$ $\&\ x_k = y_k) \to f(x_1, \ldots, x_k) \simeq f(y_1, \ldots, y_k)$.
3. $x + y = y + x$, $x + (y + z) = (x + y) + z$, $0 + x = x$, $x + (-x) = 0$, $x \cdot y = y \cdot x$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $1 \cdot x = x$, $x \neq 0 \to x \cdot (1/x) = 1$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, $0 \leq x \to |x| = x$, $x < 0 \to |x| = -x$.
4. $\sim(x < x)$, $(x < y\ \&\ y < z) \to x < z$, $x < y \vee y < x \vee x = y$, $x < y \to$ $x + z < y + z$, $(x < y\ \&\ 0 < z) \to x \cdot z < y \cdot z$, $N(0)$, $N(x) \to N(x + 1)$, $(\forall x)(N(x) \to (x \neq 0 \leftrightarrow 1 \leq x))$.

5. $(0 < x \,\&\, 0 < y) \rightarrow (\exists z)(N(z) \,\&\, x < z \cdot y)$.

We will use $s > t$, $s \leq t$, and $s \geq t$ as abbreviations.

We next consider the axioms $\Sigma\Pi$.

1. $(\exists f)(\forall x_1,\ldots,x_k)(f(x_1,\ldots,x_k) = y)$, $(\exists f)(\forall x_1,\ldots,x_k)(f(x_1,\ldots,x_k) = x_m)$, $(\exists f)(\forall x)(\forall y)(f(x,y) = x+y)$, $(\exists f)(\forall x)(\forall y)(f(x,y) = x \cdot y)$, $(\exists f)(\forall x)(f(x) = -x)$, $(\exists f)(\forall x)(f(x) \simeq 1/x)$, $(\exists f)(\forall x)(\forall y)(f(x) = y \leftrightarrow (0 \leq x \,\&\, y = 0))$.

2. $(\exists h)(\forall x_1,\ldots,x_k)(h(x_1,\ldots,x_k) \simeq f(g_1(x_1,\ldots,x_k),\ldots,g_p(x_1,\ldots,x_k)))$.

3. $(\exists g)(\forall x_1,\ldots,x_{k-1})(g(x_1,\ldots,x_{k-1},0) \simeq f(x_1,\ldots,x_{k-1},0) \,\&\, (\forall n)(g(x_1,\ldots,x_{k-1},n+1) \simeq g(x_1,\ldots,x_{k-1},n) + f(x_1,\ldots,x_{k-1},n))) \,\&\, (\forall y)(D(g(x_1,\ldots,x_{k-1},y)) \leftrightarrow (N(y) \,\&\, (\forall z \leq y)(D(f(x_1,\ldots,x_{k-1},z)))))$.

4. $(\exists g)(\forall x_1,\ldots,x_{k-1})(g(x_1,\ldots,x_{k-1},0) \simeq f(x_1,\ldots,x_{k-1},0) \,\&\, (\forall n)(g(x_1,\ldots,x_{k-1},n+1) \simeq g(x_1,\ldots,x_{k-1},n) \cdot f(x_1,\ldots,x_{k-1},n))) \,\&\, (\forall y)(D(g(x_1,\ldots,x_{k-1},y)) \leftrightarrow (N(y) \,\&\, (\forall z \leq y)(D(f(x_1,\ldots,x_{k-1},z)))))$.

PR (primitive recursion) will consist of 1, 2 above plus $(\exists h)(\forall x_1,\ldots,x_{k-1})(h(x_1,\ldots,x_{k-1},0) \simeq f(x_1,\ldots,x_{k-1}) \,\&\, (\forall n)(h(x_1,\ldots,x_{k-1},n+1) \simeq g(x_1,\ldots,x_{k-1},n,h(x_1,\ldots,x_{k-1},n))) \,\&\, (\forall y)(D(h(x_1,\ldots,x_{k-1},y)) \leftrightarrow (N(y) \,\&\, (\forall z \leq y)(D(f(x_1,\ldots,x_{k-1},y)))))$.

In the above if $k = 0$ then $f(x_1,\ldots,x_{k-1})$ is replaced by 'z' .

Finally, we consider the induction axioms

$$(\varphi^x_0 \,\&\, (\forall x)(N(x) \rightarrow \varphi^x_{x+1})) \rightarrow (\forall x)(N(x) \rightarrow \varphi) .$$

If we require φ to be quantifier free, then we call this "quantifier free induction". If we require all bound variables in φ to be relativized to the natural numbers, we call this "arithmetic induction". If we require all bound variables in φ to be real variables, we call this "real quantifier induction". If we put no restriction on φ , we call this "function quantifier induction".

Two other forms of induction will be considered, the first of which we call "elementary induction".

$((\forall x)(N(x) \rightarrow D(f(x))) \ \& \ f(0) = 0 \ \& \ (\forall x)(N(x) \rightarrow (f(x) = 0 \rightarrow f(x+1) = 0)))$ $\rightarrow (\forall x)(N(x) \rightarrow f(x) = 0)$, $((\forall x)(N(x) \rightarrow D(f(x))) \ \& \ (N(y) \ \& \ f(y) = 0)) \rightarrow$ $(\exists x)(N(x) \ \& \ f(x) = 0 \ \& \ (\forall y)((N(y) \ \& \ f(y) = 0) \rightarrow x \leq y))$.

The second is called "domain induction".

$(D(f(0)) \ \& \ (\forall x)((N(x) \ \& \ D(f(x)) \rightarrow D(f(x+1)))) \rightarrow (\forall x)(N(x) \rightarrow D(f(x)))$, $(D(f(x)) \ \& \ N(x)) \rightarrow (\exists y)(N(y) \ \& \ D(f(y) \ \& \ (\forall z)((D(f(z)) \ \& \ N(z)) \rightarrow y \leq z))$.

In Appendix C, all systems will include NAOF + $\Sigma\Pi$ + elementary induction.

We now consider several additional axioms.

Domain existence axioms.

AXIOM I. Complementation. $(\exists g)(\forall x_1,\ldots,x_n)(D(g(x_1,\ldots,x_n)) \leftrightarrow {\sim}D(f(x_1,\ldots,x_n)))$.

AXIOM II. Countable union. $(\exists g)(\forall x_1,\ldots,x_n)(D(g(x_1,\ldots,x_n) \leftrightarrow (\exists k)(D(f(k,x_1,\ldots,x_n))))$.

AXIOM III. Countable intersection. $(\exists g)(\forall x_1,\ldots,x_n)(D(g(x_1,\ldots,x_n)) \leftrightarrow (\forall k)(D(f(k,x_1,\ldots,x_n))))$.

AXIOM IV. σ-algebra. I & II & III.

AXIOM V. Range. $(\exists g)(\forall x_1,\ldots,x_n)(D(g(x_1,\ldots,x_n) \leftrightarrow (\exists y_1,\ldots,y_m)(f_1(y_1,\ldots,y_m) = x_1 \ \& \ \ldots \ \& \ f_n(y_1,\ldots,y_m) = x_n))$.

AXIOM VI. Uniformization. $(\exists g)(\forall x_1,\ldots,x_n)(\forall y_1,\ldots,y_m)((D(g(x_1,\ldots,x_n,y_1,\ldots,y_m)) \leftrightarrow (\exists z_1,\ldots,z_m)(D(f(x_1,\ldots,x_n,z_1,\ldots,z_m)))) \ \& \ (D(g(x_1,\ldots,x_n,y_1,\ldots,y_m)) \rightarrow D(f(x_1,\ldots,x_n,y_1,\ldots,y_m))) \ \& \ (\forall z_1,\ldots,z_m)(D(f(x_1,\ldots,x_n,z_1,\ldots,z_m)) \rightarrow (y_1 = z_1 \ \& \ \ldots \ \& \ y_m = z_m)))$.

Function existence axioms.

AXIOM VII. Inverse. Let $f_1,\ldots,f_n$ be n-ary. Suppose $(f_1(x_1,\ldots,x_n) = f_1(y_1,\ldots,y_n)$ & $\ldots$ & $f_n(x_1,\ldots,x_n) = f_n(y_1,\ldots,y_n)) \rightarrow (x_1 = y_1$ & $\ldots$ & $x_n = y_n)$. Then there are n-ary $g_1,\ldots,g_n$ such that $g_j(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)) = x_j$, and $f_j(g_1(x_1,\ldots,x_n),\ldots,g_n(x_1,\ldots,x_n)) = x_j$.

AXIOM VIII. Left inverse. Let $f_1,\ldots,f_n$ be m-ary. Then there are n-ary $g_1,\ldots,g_m$ such that $g_1(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)) = x_1$ & $\ldots$ & $g_m(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)) = x_m$.

AXIOM IX. Total left inverse. Let $f_1,\ldots,f_n$ be m-ary, $(\forall x_1,\ldots,x_m)(D(f_j(x_1,\ldots,x_m)))$. Then there are n-ary $g_1,\ldots,g_m$ such that $g_i(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)) = x_i$.

AXIOM X. Infinite join. $(\forall p,q,x_1,\ldots,x_n)((D(f(p,x_1,\ldots,x_n))$ & $D(f(q,x_1,\ldots,x_n))) \rightarrow f(p,x_1,\ldots,x_n) = f(q,x_1,\ldots,x_n)) \rightarrow (\exists g)(\forall x_1,\ldots,x_n,y)(g(x_1,\ldots,x_n) = y \leftrightarrow (\exists p)(f(p,x_1,\ldots,x_n) = y))$.

AXIOM XI. Integral inverse. Let $f_1,\ldots,f_n$ be m-ary, $(\forall x_1,\ldots,x_m)(D(f_j(x_1,\ldots,x_m)) \leftrightarrow (N(|x_1|)$ & $\ldots$ & $N(|x_m|)))$, $(\forall x_1,\ldots,x_m)(D(f_j(x_1,\ldots,x_m)) \rightarrow N(|f_j(x_1,\ldots,x_m)|))$. Then there are n-ary $g_1,\ldots,g_m$ such that $g_i(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)) = x_i$.

Completeness Axioms.

Definition 1. A sequence (of real numbers) is a unary function f defined on exactly the natural numbers, and is written $\{x_n\}$. A sequence $\{x_n\}$ is Cauchy just in case $(\forall \epsilon > 0)(\exists k)(\forall p,q > k)(|x_p - x_q| < \epsilon)$. We write $\{x_n\} \rightarrow y$ for $(\forall \epsilon > 0)(\exists k)(\forall n > k)(|x_n - y| < \epsilon)$. We say $\{x_n\}$ converges if $\{x_n\} \rightarrow y$ for some y.

AXIOM XII. Double monotone completeness. If $x_n \leq x_{n+1}$, $y_{n+1} \leq y_n$, $x_n \leq y_m$, then there is a z such that $x_n \leq z \leq y_m$.

AXIOM XIII. Cauchy completeness. Every Cauchy sequence converges.

AXIOM XIV. Double cut completeness. If $D(f(x))$ & $(\forall y)(\forall z)((D(f(y))$ & $z \leq y) \rightarrow D(f(z)))$ & $D(g(w))$ & $(\forall y)(\forall z)((D(g(y))$ & $y \leq z) \rightarrow D(g(z)))$ & $(\forall y)(\sim(D(f(y))$ & $D(g(y))))$, then $(\exists u)(\forall y)(\forall z)$ $((D(f(y)) \rightarrow y \leq u)$ & $(D(g(z)) \rightarrow u \leq z))$.

AXIOM XV. Left cut completeness. If $(D(f(x))$ & $(\forall y)(D(f(y)) \rightarrow y \leq z))$, and $(\forall y)(\forall z)((D(f(y))$ & $z \leq y) \rightarrow D(f(z)))$, then $(\exists w)((\forall y)$ $(D(f(y)) \rightarrow y \leq w)$ & $(\forall u < w)(\exists y)(D(f(y))$ & $u < y))$.

AXIOM XVI. Least upper bound. If $(D(f(x))$ & $(\forall y)(D(f(y)) \rightarrow y \leq z))$ then $(\exists w)(\forall y)(D(f(y)) \rightarrow y \leq w)$ & $(\forall u < w)(\exists y)(D(f(y))$ & $u < y))$.

Function completeness axioms.

Definition 2. Any $k+1$-ary partial function f may be regarded as a sequence $\{f_n\}$ of k-ary partial functions, where $f_n(x_1,\ldots,x_k) \simeq f(n,x_1,\ldots,x_k)$. We write $\{f_n\} \rightarrow g$ to mean $(\forall x_1,\ldots,x_k,y)(g(x_1,\ldots,x_k) = y \leftrightarrow (\{f_n(x_1,\ldots,x_k)\} \rightarrow y))$. Here it is understood that if $\{f_n(x_1,\ldots,x_k)\}$ does not define a (total) sequence, then it does not converge.

AXIOM XVII. Partial limit. $(\exists f)(\{f_n\} \rightarrow f)$.

Definition 3. Let $\rho_r(x_1,\ldots,x_r,y_1,\ldots,y_r) = |x_1 - y_1| + |x_2 - y_2| + \ldots + |x_n - y_n|$. We say $\{f_n\}$ has common domain just in case $(\forall x_1,\ldots,x_k)(D(f_0(x_1,\ldots,x_k)) \leftrightarrow D(f_n(x_1,\ldots,x_k)))$.

Definition 4. Let $\{f_n\}$ have common domain, where f is $r+1$-ary. Let $F_1,\ldots,F_r,G$ have domain exactly N. We say $F_1,\ldots,F_r,G$ is a sequential open covering for $\{f_n\}$ just in case $(\forall x_1,\ldots,x_r)(D(f_0(x_1,\ldots,x_r)) \rightarrow (\exists n)(\rho_r(x_1,\ldots,x_r,F_1(n),\ldots,F_r(n)) < G(n)))$. We say that $\{f_n\}$ is explicitly

Cauchy just in case there is a sequential open covering $F_1,\ldots,F_r,G$ of $\{f_n\}$, and a $g : N \times N \to N$ such that $(\forall m)(\forall n > 0)(\forall p,q > g(m,n))$ $(\forall x_1,\ldots,x_r)((D(f_0(x_1,\ldots,x_r)) \;\&\; \rho_r(x_1,\ldots,x_r,F_1(m),\ldots,F_r(m)) < G(m)) \to$ $|f_p(x_1,\ldots,x_r) - f_q(x_1,\ldots,x_r)| < 1/n)$.

AXIOM XVIII. Explicitly uniform limit. If $\{f_n\}$ is explicitly Cauchy, then $(\exists g)(\{f_n\} \to g)$.

Choice axioms.

AXIOM XIX. Countable choice. $(\forall x)(N(x) \to (\exists y_1,\ldots,y_k)(D(f(x,y_1,\ldots,y_k))))$ $\to (\exists g_1,\ldots,g_k)(\forall x)(N(x) \to D(f(x,g_1(x),\ldots,g_k(x))))$.

AXIOM XX. Dependent choice. $(\forall x_1,\ldots,x_n)(\exists y_1,\ldots,y_n)(D(f(x_1,\ldots,x_n,$ $y_1,\ldots,y_n))) \to (\forall x_1,\ldots,x_n)(\exists g_1,\ldots,g_n)(g_1(0) = x_1 \;\&\; \ldots \;\&$ $g_n(0) = x_n \;\&\; (\forall k)(D(f(g_1(k),\ldots,g_n(k),g_1(k+1),\ldots,g_n(k+1)))))$.

PART E

UPPER BOUND RESULTS

Provably consistent in Peano arithmetic.

1. NAOF + PR + Elementary induction + σ-algebra + Range + Uniformization + Integral inverse + Double monotone completeness + Explicitly uniform limit.
2. NAOF + PR + Quantifier free induction + Countable union + Range + Uniformization + Integral inverse + Double monotone completeness + Explicitly uniform limit.

Equiconsistent with Peano arithmetic.

3. NAOF + $\Sigma\Pi$ + Elementary induction + Countable union + Countable intersection + Left inverse + Cauchy completeness + Double cut completeness + Partial limit + Countable choice.
4. NAOF + $\Sigma\Pi$ + Elementary induction + σ-algebra + Range + Uniformization + Total left inverse + Cauchy completeness + Partial limit.
5. NAOF + $\Sigma\Pi$ + Arithmetic induction + σ-algebra + Infinite join + Cauchy completeness + Left cut completeness + Partial limit + Countable choice.
6. NAOF + PR + Elementary induction + Countable union + Countable intersection + Range + Uniformization + Integral inverse + Cauchy completeness + Double cut completeness + Explicitly uniform limit + Countable choice.
7. NAOF + PR + Elementary induction + σ-algebra + Range + Uniformization + Integral inverse + Cauchy completeness + Explicitly uniform limit.
8. NAOF + PR + Arithmetic induction + σ-algebra + Integral inverse + Cauchy completeness + Left cut completeness + Explicitly uniform limit + Countable choice.
9. NAOF + PR + Real quantifier induction + σ-algebra + Range + Uniformi-

zation + Integral inverse + Double monotone completeness + Explicitly uniform limit.

10. NAOF + PR + Function quantifier induction + Countable union + Range + Uniformization + Integral inverse + Double monotone completeness + Explicitly uniform limit.

<u>Equiconsistent with ATR</u> ($< \omega^{\omega}$) .

11. NAOF + PR + Elementary induction + σ-algebra + Range + Uniformization + Total left inverse + Cauchy completeness + Partial limit.

12. NAOF + PR + Domain induction + Countable union + Countable intersection + Left inverse + Cauchy completeness + Double cut completeness + Partial limit + Dependent choice.

13. NAOF + PR + Arithmetic induction + σ-algebra + Infinite join + Cauchy completeness + Left cut completeness + Partial limit + Dependent choice.

14. NAOF + PR + Real quantifier induction + σ-algebra + Range + Total left inverse + Cauchy completeness + Partial limit.

15. NAOF + PR + Function quantifier induction + σ-algebra + Infinite join + Cauchy completeness + Left cut completeness + Partial limit.

<u>Equiconsistent with ATR</u> ($< \epsilon_0$) .

16. NAOF + PR + Real quantifier induction + σ-algebra + Range + Uniformization + Total left inverse + Cauchy completeness + Partial limit.

17. NAOF + PR + Function quantifier induction + Countable union + Countable intersection + Left inverse + Cauchy completeness + Double cut completeness + Partial limit + Dependent choice.

18. NAOF + PR + Function quantifier induction + σ-algebra + Infinite join + Cauchy completeness + Left cut completeness + Partial limit + Dependent choice.

Equiconsistent with ATR $(< \Gamma_0)$.

19. NAOF + PR + Arithmetic induction + σ-algebra + Inverse + Cauchy completeness + Left cut completeness + Partial limit + Countable choice.

Equiconsistent with ATR.

20. NAOF + PR + Function quantifier induction + σ-algebra + Inverse + Cauchy completeness + Left cut completeness + Partial limit + Countable choice.

Intermediate between ATR and ID (1).

21. NAOF + PR + Function quantifier induction + σ-algebra + Inverse + Cauchy completeness + Left cut completeness + Partial limit + Dependent choice.

Equiconsistent with ID $(< \omega)$.

22. NAOF + PR + Arithmetic induction + Countable union + Countable intersection + Range + Inverse + Cauchy completeness + Least upper bound + Partial limit + Dependent choice.

23. NAOF + PR + Arithmetic induction + σ-algebra + Inverse + Cauchy completeness + Least upper bound + Partial limit + Dependent choice.

Equiconsistent with Π^1_1-CA.

24. NAOF + PR + Function quantifier induction + Countable union + Countable intersection + Range + Inverse + Cauchy completeness + Least upper bound + Partial limit + Dependent choice.

25. NAOF + PR + Function quantifier induction + Range + Inverse + Cauchy completeness + Least upper bound + Partial limit + Dependent choice.

Equiconsistent with second order arithmetic.

26. NAOF + PR + Arithmetic induction + σ-algebra + Left inverse + Cauchy completeness + Least upper bound + Partial limit + Dependent choice.

<u>Just beyond second order arithmetic</u>.

27. NAOF + PR + Function quantifier induction + σ-algebra + Left inverse + Cauchy completeness + Least upper bound + Partial limit + Dependent choice.