THE ANALYSIS OF MATHEMATICAL TEXTS, AND THEIR CALIBRATION IN TERMS OF INTRINSIC STRENGTH IV

by

Harvey Friedman *
State University of New York at Buffalo
August 15, 1975

This is the fourth in a series of reports.

In report III, the logical strength of numerous combinations of basic principles were calculated. This was done without regard to the detailed development of actual real variables, and so many systems which have no direct connection with this development were considered.

Since then, we have discovered what we think is the right "base theory" for real variables (in the sense of analysis on Euclidean space). A description of the program as it relates to real variables, including desiderata for a base theory, is given in the ensuing paragraphs.

As outlined in report III, one can associate sentences in a formal calculus $\mathfrak{U}$ (which is an extension of the first order predicate calculus) to assertions made in real analysis. This association preserves the logical structure of the deductive relations among the assertions. Then one can speak of the logical strength of assertions or bodies of assertions in real analysis as the logical strength of the formal systems resulting from their translations into $\mathfrak{U}$.

Theoretically, this is the program: to calculate the logical strength of statements, or groups of statements, of real analysis. However, taking

* This research was partially supported by NSF grant # MPS 75 - 05856.

statements out of context, in their raw form, and calculating their logical strength, would lead to a subject which is unmanageably complex and technical.

A smooth and workable theory is obtained by considering statements of real analysis in the context of a base theory. One then considers theorems only in the context of this base theory. Thus, in effect, we calculate the logical strength only of groups of statements which include the base theory.

Features of the base theory (RV_1) for real analysis that we have chosen include the following.

i) All of the axioms of RV_1 have clear mathematical meaning. This is in contrast to traditional systems logicians normally study, which are based on axioms of a metamathematical character.

ii) The system RV_1 is equiconsistent with Peano arithmetic (PA) ; and for theorems A , the logical strength of $RV_1 + A$ is independent of minor changes in the precise way A is formulated.

iii) All of the axioms of RV_1 represent fundamental principles basic to the practice of real analysis. If a weaker theory than RV_1 is chosen as the base theory, then the subject becomes unmaneagably complex; e.g., ii) above fails.

iv) A very substantial body of real analysis can be proved in RV_1 -- usually, by proofs which are the same as those customarily given. When the same proof cannot be used, some other proof can be found in RV_1 which is mathematically natural.

v) In general, the logical strength of RV_1 together with a finite set of statements is the same as the maximum of the strengths of RV_1 with the statements taken individually. Such is the case in each of the examples below.

We have also discovered that the strength of RV_1 appended with the basic theorems of Lebesgue theory on Euclidean space is that of Peano arithmetic. However, many difficult questions seem to arise -- in particular in connection with the strength of the existence of nonmeasurable sets and the Hahn-Banach theorem for certain Banach spaces. These matters will be taken up in the next report.

RV_1

(base theory; equiconsistent with PA)

RV_1 has variables α_k ranging over nonempty finite sequences of real numbers ($\mathbb{R}^*$), variables A_k ranging over subsets of $\mathbb{R}^*$, and n-ary partial function variables f_k^n ranging over n-ary partial functions on $\mathbb{R}^*$. In addition, we have the relation symbols $=,<,\in,N$, the constant symbols $0,1$, and the partial function symbols $+,\cdot,-,1/,|\,|,\ell th$. We also have the special symbol D, for "being defined".

The terms are given as follows. Every variable α_k is a term. The constants $0,1$ are terms. If s,t are terms, so are $s+t$, $s\cdot t$, $-s$, $1/s$, $|s|$, and $\ell th(s)$. If $s_1,\ldots,s_n$ are terms, then $f_k^n(s_1,\ldots,s_n)$ is a term. If s,t are terms, so is $s(t)$.

The formulae are given by i) $s_1 = s_2$, $s_1 < s_2$, $s \in A_n$, $N(s)$, and $D(s)$ are formulae. ii) $\sim\varphi$, $\varphi \vee \psi$, $\varphi \,\&\, \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ are formulae if φ,ψ are formulae. iii) $(\forall\lambda)(\varphi)$, $(\exists\lambda)(\varphi)$ are formulae if φ is, and λ is any variable.

The purely logical axioms and rules of inference are as follows.

1. all propositional tautologies. 2. $((\forall\alpha)(\varphi) \,\&\, D(t)) \to A_t^\alpha$.
3. $(\forall f)(\varphi) \to \varphi_g^f$. 4. $(\forall A)(\varphi) \to \varphi_B^A$. 5. $(\varphi_t^\alpha \,\&\, D(t)) \to (\exists\alpha)(\varphi)$.
6. $\varphi_g^f \to (\exists f)(\varphi)$. 7. $\varphi_B^A \to (\exists A)(\varphi)$. 8. $D(\alpha)$. 9. $D(s) \to D(t)$, for subterms t of s. 10. $(s < t \vee s = t \vee s \in A \vee N(s)) \to (D(s) \,\&\, D(t))$.
11. from $\varphi, \varphi \to \psi$ derive ψ. 12. from $\varphi \to \psi$ derive $\varphi \to (\forall\lambda)(\psi)$.
13. from $\psi \to \varphi$ derive $(\exists\lambda)(\psi) \to \varphi$.

The above comes with the usual restrictions. λ is any variable.

We now present the proper axioms for the system RV_1. It will be convenient to use $s \simeq t$ to abbreviate $(D(s) \vee D(t)) \rightarrow s = t$, and let p,q,r,n,m, etc. range over those α with $N(\alpha)$, and let x,y,z range over α with $\ell th(\alpha) = 1$. $N(\alpha)$ means "α is a natural number."

Equality axioms. $\alpha = \alpha$, $\alpha = \beta \rightarrow (\varphi \leftrightarrow \varphi^{\alpha}_{\beta})$, $\alpha = \beta \rightarrow (s \simeq s^{\alpha}_{\beta})$, $\alpha = \beta \leftrightarrow (\forall\gamma)(\alpha(\gamma) \simeq \beta(\gamma))$.

Miscellaneous. $N(\alpha) \rightarrow \ell th(\alpha) = 1$, $N(\ell th(\alpha))$, $\ell th(\alpha) \neq 0$, $D(\alpha(\beta)) \leftrightarrow (N(\beta)\ \&\ \beta \neq 0\ \&\ \beta \leq \ell th(\alpha))$, $D(\alpha + \beta) \leftrightarrow (\ell th(\alpha) = \ell(\beta) = 1) \leftrightarrow D(\alpha \cdot \beta)$, $D(-\alpha) \leftrightarrow \ell th(\alpha) = 1$, $D(1/\alpha) \leftrightarrow (\ell th(\alpha) = 1\ \&\ \alpha \neq 0)$, $\alpha < \beta \rightarrow (\ell th(\alpha) = \ell th(\beta) = 1)$, $D(\alpha(\beta)) \rightarrow \ell th(\alpha(\beta)) = 1$.

Explicit definition. $(\exists f)(\forall\alpha_1)\ldots(\forall\alpha_n)(f(\alpha_1,\ldots,\alpha_n) \simeq s)$, $(\exists f)(\forall\alpha_1)\ldots(\forall\alpha_n)(\forall\beta)(f(\alpha_1,\ldots,\alpha_n) = \beta \leftrightarrow (\forall m)(\beta(m) \simeq g(\alpha_1,\ldots,\alpha_n,m)))$, $(\exists A)(\forall x)(x \in A \leftrightarrow D(f(x)))$, $(\exists f)(\forall\alpha)((\alpha \in A \rightarrow f(\alpha) = 0)\ \&\ (\alpha \notin A \rightarrow \sim D(f(\alpha))))$, $(\exists f)(\forall\alpha_1)\ldots(\forall\alpha_n)((D(g(\alpha_1,\ldots,\alpha_n)) \rightarrow f(\alpha_1,\ldots,\alpha_n) = g(\alpha_1,\ldots,\alpha_n))\ \&\ (\sim D(g(\alpha_1,\ldots,\alpha_n) \rightarrow f(\alpha_1,\ldots,\alpha_n) = 0))$.

Normed Archimedean ordered field. $x + y = y + x$, $x + (y + z) = (x + y) + z$, $0 + x = x$, $x + (-x) = 0$, $x \cdot y = y \cdot x$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $1 \cdot x = x$, $x \neq 0 \rightarrow x \cdot (1/x) = 1$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, $0 \leq x \rightarrow |x| = x$, $x < 0 \rightarrow |x| = -x$, $\sim(x < x)$, $(x < y\ \&\ y < z) \rightarrow x < z$, $x < y \vee y < x \vee x = y$, $x < y \rightarrow x + z < y + z$, $(x < y\ \&\ 0 < z) \rightarrow x \cdot z < y \cdot z$, $N(0)$, $N(x) \rightarrow N(x + 1)$, $(\forall x)(N(x) \rightarrow (x \neq 0 \leftrightarrow 1 \leq x))$, $(0 < x\ \&\ 0 < y) \rightarrow (\exists n)(x < n \cdot y)$.

$\Sigma\Pi$. $(\exists g)(\forall\alpha_1)\ldots(\forall\alpha_{k-1})(g(\alpha_1,\ldots,\alpha_{k-1},0) \simeq f(\alpha_1,\ldots,\alpha_{k-1},0)\ \&\ (\forall n)(g(\alpha_1,\ldots,\alpha_{k-1},n+1) \simeq g(\alpha_1,\ldots,\alpha_{k-1},n) + f(\alpha_1,\ldots,\alpha_{k-1},n+1))\ \&\ (\forall\beta)(D(g(\alpha_1,\ldots,\alpha_{k-1},\beta)) \leftrightarrow (N(\beta)\ \&\ (\forall n \leq \beta)(D(f(\alpha_1,\ldots,\alpha_{k-1},n))))))$.
Also, with the first $+$ replaced by $\cdot$.

Sequential induction. $(\forall n)(D(f(n))\ \&\ f(0) = 0\ \&\ (\forall n)(f(n) = 0 \rightarrow f(n + 1) = 0)) \rightarrow (\forall n)(f(n) = 0)$.

We call a sequence $\{x_n\}$ of reals Cauchy just in case $(\forall n > 0)(\exists m)(\forall p,q > m)(|x_p - x_q| < 1/n)$. We say $\{x_n\} \to y$ just in case $(\forall n > 0)(\exists m)(\forall p > m)(|x_p - y| < 1/n)$. We say $\{x_n\}$ converges if $(\exists y)(\{x_n\} \to y)$.

Cauchy completeness. Every Cauchy sequence converges.

Pointwise limit. $(\exists g)(\forall \alpha_1)\ldots(\forall \alpha_k)(\forall x)(g(\alpha_1,\ldots,\alpha_k) = x \leftrightarrow (\{f(n,\alpha_1,\ldots,\alpha_k)\} \to x))$.

We now develop analysis on Euclidean space in RV_1, following Rudin, Principles of Mathematical Analysis.

I. The Real and Complex Number Systems.

THEOREM 1.1. Let $A \subset \mathbb{R}$, bounded above, and $x \leq y \in A \to x \in A$. Then A has a least upper bound. For each $x > 0$ there is a unique $y \geq 0$ such that $y^n = x$, where $n \geq 1$. Define $x^{n/m} = (x^{1/m})^n$, for $x \geq 0$. For $x > 0$, $\alpha \geq 0$, there is a unique $y = x^\alpha$ such that $x^{n/m} < y < x^{p/q}$, if $n/m < \alpha < p/q$. For $x > 0$, $\alpha \geq 0$, let $x^{-\alpha} = 1/x^\alpha$. If $A \subset \mathbb{R}$ is countable, $x \in A \to x > 0$, $\alpha \in \mathbb{R}$, then the function $f\colon A \to \mathbb{R}$ given by $f(x) = x^\alpha$ exists.

Later on, when we consider power series, we will prove the existence of x^α as a function of two arguments.

We let $\mathbb{C}$ be the complex numbers, which is just $\mathbb{R}^2$ with (a,b) written as $a + bi$, and with the definitions $(a + bi) + (c + di) = (a + c) + (b + d)i$, $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

For $x \in \mathbb{R}^n$ we let $|x| = (\sum_{k=1}^{n} x_k^2)^{\frac{1}{2}}$.

THEOREM 1.2. The complex numbers form a field. The real numbers form a subfield of the complex numbers. For complex x,y, we have $|x||y| = |xy|$,

$|x + y| \le |x| + |y|$. For complex $a_1,\dots,a_n,b_1,\dots,b_n$, we have $|\sum_{j=1}^{n} a_j\bar{b}_j|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$. Here $\bar{b} = x - yi$, where $b = x + yi$.

Let $x + y = (x_1 + y_1,\dots,x_k + y_k)$, where $x = (x_1,\dots,x_k)$, $y = (y_1,\dots,y_k)$. Let $\alpha x = (\alpha x_1,\dots,\alpha x_k)$. Let $x\cdot y = \sum_{i=1}^{k} x_i y_i$.

THEOREM 1.3. $|x + y| \le |x| + |y|$, for $x,y \in \mathbb{R}^k$.

III. Elements of Set Theory.

We call a set A finite just in case A is the range of a one-one function from some $\{j\colon j < n\}$, in which case we write $|A| = n$. We call a set infinite if it is not finite. We call A denumerable just in case A is the range of a one-one function from N . We call A countable just in case A is either finite or denumerable. An uncountable set is a set which is not countable.

THEOREM 2.1. If A is finite then there is a unique n such that $|A| = n$. If A is denumerable, A is infinite. Every subset of a finite set is finite. Every subset of a countable set is countable. The unbounded subsets of N are precisely the infinite subsets of N . The range of every function with countable domain exists, and is countable. The range of every function with finite domain is finite. The inverse of every one-one function with countable domain exists. The set of all finite sequences of $A \subset \mathbb{R}$ is countable if A is countable. $\mathbb{R}$ is uncountable.

A nhbd of x is a set of the form $\{y\colon |x - y| < \epsilon\}$, where $\epsilon > 0$. $A \subset \mathbb{R}^n$ is open if every point in A has a nhbd included in A . A is closed if its complement is open. $A \subset \mathbb{R}^n$ is called sequentially open if A is the union of a sequence of nhbds. A is called sequentially closed if its complement is sequentially open. A limit point of $A \subset \mathbb{R}^n$ is a point x every nhbd of which contains a point from A other than x. A is called bounded if it is contained in some nhbd. A is called perfect if it is closed and every point in A is a limit point of A. $A \subset B \subset \mathbb{R}^n$ is called dense in B just in case every point of B is a limit point of A or in A. $A \subset \mathbb{R}^n$ is called separable just in case it has a countable dense subset. $A \subset \mathbb{R}^n$ is called compact if it is closed and bounded. $A \subset \mathbb{R}^n$ is called a compactum if it is compact and separable.

THEOREM 2.2. Every sequentially open set is open. The union of a sequence of open sets is open. $A \subset \mathbb{R}^n$ is closed if and only if A includes all its limit points. The union of a finite sequence of closed sets is closed. Every bounded sequentially closed, or sequentially open subset of $\mathbb{R}$ has a least upper bound; in the former case, included in the set.

THEOREM 2.3. Every sequence of open sets covering a compactum, has an initial segment covering it also. If A is compact then every denumerable subset of A has a limit point in A. Every nonempty perfect set is uncountable. The separable closed sets are just the sequentially closed sets. The compacta are just the bounded sequentially closed sets. All open sets are separable.

III. <u>Numerical Sequences and Series</u>.

A sequence $\{x_n\}$ in $\mathbb{R}^k$ is said to converge just in case $(\exists y)(\forall \epsilon > 0)(\exists m)(\forall k > m)(|y - x_k| < \epsilon)$, and we write $\{x_n\} \to y$,, or $\lim_n x_n = y$. We say $\{x_n\}$ is Cauchy just in case $(\forall \epsilon > 0)(\exists m)(\forall p,q > m)(|x_p - x_q| < \epsilon)$.

THEOREM 3.1. Every convergent sequence is bounded, and has exactly one limit. The limit points of a set $A \subset \mathbb{R}^n$ are exactly those points which are limits of a sequence of distinct elements from A. Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence. Every Cauchy sequence in $\mathbb{R}^n$ converges, and vice versa. A monotone sequence in $\mathbb{R}$ converges if and only if it is bounded.

THEOREM 3.2. Let $\{s_n\}$ be bounded above. Then there is a least upper bound for $\{s_n\}$, written $\sup\{s_n\}$. There is also a sequence $\{a_m\}$, where $a_m = \sup\{s_{m+n}\}$. In addition, $\{a_m\}$ is monotonic and bounded. We write $\limsup\{s_n\}$ for $\lim\{a_m\}$. Similarly, if $\{s_n\}$ is bounded below, for inf and lim inf.

THEOREM 3.3. Let $\{s_n\}$ be bounded above. Then $\limsup\{s_n\} = a$ is the unique real such that there is a subsequence of $\{s_n\}$ converging to a, and for all $x > a$, $\{s_n\}$ is eventually smaller than x. In addition, no number smaller than a is the limit of a convergent subsequence of $\{s_n\}$. Similarly for $\{s_n\}$ bounded below, and lim inf.

THEOREM 3.4. If $p > 0$, $\lim n^{-p} = 0$. If $p > 0$, $\lim p^{1/n} = 1$. And $\lim n^{1/n} = 1$. If $p > 0$ then $\lim n^{\alpha}/(1+p)^n = 0$. If $|x| < 1$, $\lim x^n = 0$.

We use $\sum_{n=1}^{\infty} a_n$ for complex a_n.

THEOREM 3.5. Σa_n converges if and only if for every $\epsilon > 0$ there is an N such that $\left|\sum_{k=n}^{m} a_k\right| < \epsilon$ for $m \geq n \geq N$. If Σa_n converges then $\lim \{a_n\} = 0$. A series of nonnegative terms converges if and only if the partial sums are bounded.

THEOREM 3.6. If $|a_n| \le c_n$ eventually, and $\Sigma\, c_n$ converges, then $\Sigma\, a_n$ converges. If $a_n \ge d_n \ge 0$ eventually, and $\Sigma\, d_n$ diverges, then $\Sigma\, a_n$ diverges. If $0 \le x < 1$, then $\sum_{n=0}^{\infty} x^n = 1/1 - x$. If $x \ge 1$, the series diverges.

THEOREM 3.7. Suppose $a_1 \ge a_2 \ge a_3 \ge \ldots \ge 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. The series $\Sigma\, n^{-p}$ converges if $p > 1$ and diverges if $p \le 1$.

It is easy to see that $\Sigma\, 1/n!$ converges by the comparison test. Let e be its value.

THEOREM 3.8. $\lim(1 + 1/n)^n = e$. e is irrational.

THEOREM 3.9. Given $\Sigma\, a_n$, put $\alpha = \lim\sup|a_n|^{1/n}$. Then if $\alpha < 1$, Σa_n converges. If $\alpha > 1$, $\Sigma\, a_n$ diverges. Put $\beta = \lim\sup\, |a_{n+1}/a_n|$. If $\beta < 1$, $\Sigma\, \alpha_n$ converges. If $|a_{n+1}/a_n| \ge 1$ eventually, $\Sigma\, a_n$ diverges.

THEOREM 3.10. For any sequence $\{c_n\}$ of positive numbers, $\lim\inf(c_{n+1}/c_n) \le \lim\inf(c_n)^{1/n}$, $\lim\sup(c_n)^{1/n} \le \lim\sup(c_{n+1}/c_n)$.

THEOREM 3.11. Given the power series $\Sigma\, c_n z^n$, put $\alpha = \lim\sup|c_n|^{1/n}$, $R = 1/\alpha$. (If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$). Then $\Sigma\, c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

THEOREM 3.12. Suppose the partial sums of $\Sigma\, a_n$ form a bounded sequence, $b_0 \ge b_1 \ge \ldots$, and $\lim b_n = 0$. Then $\Sigma\, a_n b_n$ converges. Suppose $|c_1| \ge |c_2| \ge |c_3| > \ldots$, $c_{2m-1} \ge 0$, $c_{2m} \le 0$, $\lim c_n = 0$. Then $\Sigma\, c_n$

converges. Suppose $\Sigma c_n z^n$ has radius of convergence 1, and $c_0 \geq c_1 \geq c_2 \geq \ldots$, $\lim c_n = 0$. Then $\Sigma c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly $z = 1$.

The series Σa_n converges absolutely if the series $\Sigma |a_n|$ converges.

THEOREM 3.13. If Σa_n converges absolutely, Σa_n converges. If $\Sigma a_n = A$, $\Sigma b_n = B$, then $\Sigma (a_n + b_n) = A + B$ and $\Sigma ca_n = cA$.

Given Σa_n, Σb_n, put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, and call Σc_n the product of the two given series.

THEOREM 3.14. If Σa_n converges absolutely, Σb_n converges, then the product series converges to $\Sigma a_n \Sigma b_n$. If Σa_n is not absolutely convergent, and $\alpha \leq \beta$ (in the extended reals), where each a_n real, then there exists a rearrangement $\Sigma a_n'$ with partial sums s_n', such that $\liminf s_n' = \alpha$, $\limsup s_n' = \beta$.

THEOREM 3.15. Σa_n converges absolutely if and only if all rearrangements converge to the same sum. If all rearrangements of Σa_n converge, they converge to the same sum.

IV. Continuity.

We write $f\colon A \to B$ if $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, $(\forall x \in A)(f(x) \in B)$, and $(\forall x)(D(f(x)) \to x \in A)$. We say f is continuous at $x \in A$ just in case $(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in A)(|y - x| < \delta \to |f(y) - f(x)| < \epsilon)$. If f is continuous at every point of A, then f is said to be continuous (on A).

THEOREM 4.1. Let $f\colon A \to B$, $g\colon B \to C$, $x \in A$. If f is continuous at x

and g is continuous at $f(x)$, then $f \circ g$ is continuous at x . The range of a continuous function on a union of a sequence of compacta exists. The range of a continuous function on a compactum is a compactum. Every continuous function on a compactum into $\mathbb{R}$ assumes a maximum value. Every continuous one-one function on a separable domain has an inverse; if the domain is a compactum, then the inverse is continuous.

Let $f: A \to B$. Then f is said to be uniformly continuous just in case $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x,y \in A)(|x - y| < \delta \to |f(x) - f(y)| < \epsilon)$.

THEOREM 4.2. A continuous function on a compactum is uniformly continuous. Continuous functions on $\mathbb{R}$ map intervals onto intervals. Let $f_1,\ldots,f_k$ be real functions on $A \subset \mathbb{R}^n$. Then $f(x) = (f_1(x),\ldots,f_k(x))$ is continuous if and only if each $f_1,\ldots,f_k$ is continuous.

THEOREM 4.3. The set of discontinuities of a monotonic function is countable, and may be any prescribed countable set. Let $A \subset \mathbb{R}^m$ be separable and let $B \subset A$ be a countable dense subset of A . Then any uniformly continuous $f: B \to \mathbb{R}^n$ extends uniquely to a uniformly continuous $g: A \to \mathbb{R}^n$.

V. <u>Differentiation</u>.

Let $f: [a,b] \to \mathbb{R}$. For any $x \in [a,b]$ form the quotient

$\phi(t) = \dfrac{f(t) - f(x)}{t - x}$, $(a < t < b,\ t \neq x)$, and define $f'(x) = \lim_{t \to x} \phi(t)$, provided this limit exists.

THEOREM 5.1. Let $f: [a,b] \to \mathbb{R}$ be continuous. Then the function g given by $g(x) = f'(x)$ if $f'(x)$ exists; undefined o.w. exists. Suppose

$f'(x)$, $g'(x)$ exist. Then f is continuous at x. Also $(f+g)'(x) = f'(x) + g'(x)$, $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, $(f/g)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$, where in the latter, $g(x) \neq 0$.

THEOREM 5.2. Let $f\colon [a,b] \to [c,d]$ be continuous, $f'(x)$ exists, and $g\colon [c,d] \to \mathbb{R}$, $g'(f(x))$ exists. If $h(t) = g(f(t))$ for $a \le t \le b$, then $h'(x) = g'(f(x))f'(x)$.

THEOREM 5.3. Let $f\colon [a,b] \to \mathbb{R}$. If f has a local maximum at $x \in (a,b)$ and $f'(x)$ exists, then $f'(x) = 0$. If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which $(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$.

THEOREM 5.4. Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$.

THEOREM 5.5. Suppose $f\colon [a,b] \to \mathbb{R}$, $\{g_k\}$ is a sequence of functions $g_k\colon [a,b] \to \mathbb{R}$, and assume $n > 0$, g_{n-1} is continuous, and $g_{k+1}(x) = g_k'(x)$, for all $x \in [a,b]$, $k \in n-1$, $g_n(x) = g_{n-1}'(x)$ for all $x \in (a,b)$. Let α, β be distinct points of $[a,b]$. Then there is a point $x \in (\alpha,\beta)$ such that $f(\beta) = \sum_{k=0}^{n-1} \frac{g_k(\alpha)}{k!}(\beta - \alpha)^k + \frac{g_n(x)}{n!}(\beta - \alpha)^n$.

Let $f\colon [a,b] \to \mathbb{R}^k$. We let $f'(x)$ be such that $\lim_{t\to x} \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = 0$.

THEOREM 5.6. Let $f\colon [a,b] \to \mathbb{R}^k$. Then $f'(x) = (x_1,\ldots,x_k)$ if and only if $f_1'(x) = x_1$, $f_2'(x) = x_2, \ldots, f_k'(x) = x_k$.

THEOREM 5.7. Suppose f is a continuous mapping of $[a,b]$ into $\mathbb{R}^k$ and

f is differentiable in (a,b). Then there is an $x \in (a,b)$ such that $|f(b) - f(a)| \leq (b - a)|f'(x)|$.

THEOREM 5.8. Suppose f is differentiable on an interval, $f'(x) > 0$. Then f is strictly increasing, and has a differentiable inverse function g. Moreover, $g'(f(x)) = 1/f'(x)$.

VI. <u>The Riemann Integral</u>.

Fix $f\colon [a,b] \to \mathbb{R}$. A partition P is a finite sequence $a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b$. A P-sum is any number of the form $\sum_{k=0}^{n-1} f(y_k)(x_{k+1} - x_k)$, where $x_k \leq y_k \leq x_{k+1}$. We write mesh(P) $< \delta$ if all $x_{k+1} - x_k < \delta$. We write $\int f = \int_a^b f = \int_a^b f(x)\,dx = \alpha$ just in case for all $\epsilon > 0$ there is a $\delta > 0$ such that for all P-sums β with mesh(P) $< \delta$, we have $|\beta - \alpha| < \epsilon$. If $\int f = \alpha$ for some α, we say f is Riemann integrable over $[a,b]$, and we write $f \in \mathcal{R}([a,b])$.

THEOREM 6.1. If f is continuous on $[a,b]$, or if f is monotonic on $[a,b]$, then $f \in \mathcal{R}([a,b])$. If $f_1, f_2 \in \mathcal{R}([a,b])$, then $\int f_1 + f_2 = \int f_1 + \int f_2$, $\int cf = c\int f$, $\int_a^b f + \int_b^c f = \int_a^c f$. If $f_1 \leq f_2$, then $\int f_1 \leq \int f_2$. If $|f| \leq M$ then $|\int f| \leq M(b - a)$. If $m \leq f \leq M$, ϕ is continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$, then $\phi \circ f \in \mathcal{R}([a,b])$.

THEOREM 6.2. If $f,g \in \mathcal{R}([a,b])$ then $fg, |f| \in \mathcal{R}([a,b])$, and $|\int f| \leq \int |f|$. If $f \in \mathcal{R}([a,b])$, then the function $F(x) = \int_a^x f(t)\,dt$ exists, and F is continuous on $[a,b]$. Furthermore, if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

THEOREM 6.3. If $f \in \mathcal{R}([a,b])$ and if there is a differentiable function

F on $[a,b]$ such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Let $f_1,\dots,f_k$ be real functions on $[a,b]$, and let $f = (f_1,\dots,f_k)$. Define $f \in \mathcal{R}([a,b])$ just in case each $f_j \in \mathcal{R}([a,b])$, and $\int f = (\int f_1,\dots,\int f_k)$.

THEOREM 6.4. If $f,F\colon [a,b] \to \mathbb{R}^k$, if $f \in \mathcal{R}([a,b])$, and if $F' = f$, then $\int_a^b f(t)dt = F(b) - F(a)$. If $f \in \mathcal{R}([a,b])$, then $|\int f| \le \int |f|$.

Let $f\colon [a,b] \to \mathbb{R}^k$. For any partition P of $[a,b]$, let $V(f,P)$ be $\sum_{k=0}^{n-1} |f(x_{k+1} - f(x_k)|$. We say f is of bounded variation if the $V(f,P)$ have a least upper bound.

THEOREM 6.5. The continuous functions of bounded variation into $\mathbb{R}$ are precisely the differences between monotonically increasing functions. If $f_1,\dots,f_k$ are of bounded variation into $\mathbb{R}$, then $(f_1,\dots,f_k)$ is of bounded variation. The continuous functions of bounded variation are precisely those whose coordinate functions are differences of continuous monotone increasing functions.

THEOREM 6.6. If f is continuous and real, then there is an $x \in (a,b)$ such that $\int_a^b f = f(x)(b - a)$.

A continuous mapping γ of an interval $[a,b]$ into $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$. We call γ rectifiable if γ is of bounded variation, and we define the length of γ to be the least upper bound to all $V(\gamma,P)$.

THEOREM 6.7. If γ' is continuous on $[a,b]$, then γ is rectifiable and has length $\int_a^b |\gamma'(t)|dt$.

VII. Sequences and Series of Functions.

We will confine attention to functions from $\mathbb{R}^k$ into $\mathbb{C}$.

A sequence of functions is given by a binary function f on $\mathbb{R}^*$, where f_n is given by $f_n(x) \simeq f(n,x)$. In this section, by a sequence $\{f_n\}$ of functions, we mean a sequence of functions $f_n : \mathbb{R}^k \to \mathbb{C}$.

We say that a sequence of functions $\{f_n\}$ converges uniformly on E to a function f if for all $\epsilon > 0$ there is an N such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$, for all $x \in E$. Similarly for $\Sigma f_n(x)$.

THEOREM 7.1. $\{f_n\}$ converges uniformly on E if and only if $(\forall\epsilon > 0)$ $(\exists N)(\forall m,n > N)(\forall x \in E)(|f_n(x) - f_m(x)| < \epsilon)$. If $|f_n(x)| \leq M_n$, all $n, x \in E$, then Σf_n converges uniformly on E if ΣM_n converges. If $f_n \to f$ uniformly on E, and each f_n is continuous on E, then f is continuous on E. If $f_n \geq f_{n+1}$ on a compactum E, $f_n \to f$ on E, and f_n, f are continuous on E, then $f_n \to f$ uniformly on E.

THEOREM 7.2. Suppose f_n are Riemann integrable on $[a,b]$, and $f_n \to f$ uniformly on $[a,b]$. Then f is Riemann integrable on $[a,b]$, and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$. If each f_n is Riemann integrable, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, the series converging uniformly on $[a,b]$, then $\int_a^b f = \sum_n \int_a^b f_n$.

THEOREM 7.3. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$. There exists a continuous $f: \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.

$\{f_n\}$ is said to be equicontinuous on $E \subset \mathbb{R}^k$ if $(\forall\epsilon > 0)(\exists\delta > 0)$ $(\forall x,y \in E)(\forall n)\, |x - y| < \delta \to |f_n(x) - f_n(y)| < \epsilon)$.

THEOREM 7.4. Let K be a compactum. If $\{f_n\}$ is a uniformly convergent sequence of continuous functions on k, then $\{f_n\}$ is equicontinuous. If $\{f_n\}$ is pointwise bounded and equicontinuous on K, then $\{f_n\}$ contains a uniformly convergent subsequence, and $\{f_n\}$ is uniformly bounded in K.

A family of functions is a binary function f on $\mathbb{R}^*$, together with an index set A; interpreted as $\{f_x : x \in A\}$, where $f_x(y) \simeq f(x,y)$.

A family of complex functions defined on a set E is called an algebra if it is closed under sums, products, and complex scalar multiples. A real algebra is a family of real functions defined on a set E, closed under sums, products, and real scalar multiples.

THEOREM 7.5. Let E be a compactum in $\mathbb{R}^k$, and let $\mathfrak{a}$ be any real algebra of continuous functions on E such that i) for all $x \in E$ there is an n with $f_n(x) \neq 0$ ii) for all $x \neq y$ in E there is an n with $f_n(x) \neq f_n(y)$. Then for all continuous $f: E \to \mathbb{R}$ and $\epsilon > 0$, there is an n such that $(\forall x \in E)(|f(x) - f_n(x)| < \epsilon)$.

THEOREM 7.6. Let E be a compactum in $\mathbb{R}^k$, and let $\mathfrak{a}$ be any complex algebra of continuous functions on E such that i) for all $x \in E$ there is an n with $f_n(x) \neq 0$ ii) for all $x \neq y$ in E there is an n with $f_n(x) \neq f_n(y)$ iii) if $f + gi \in \mathfrak{a}$ then $f - gi \in \mathfrak{a}$. Then the conclusion of theorem 7.5 holds.

VIII. Further Topics in the Theory of Series.

We begin with the section on the exponential and logarithmic functions. We use x,y for reals, z,w for complex numbers. Define $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

THEOREM 8.1. $E(x) = e^x$. e^x is continuous and differentiable for all x. $(e^x)' = e^x$. e^x is strictly increasing, $e^x > 0$. $e^{x+y} = e^x e^y$, $e^x \to +\infty$ as $x \to +\infty$, $e^x \to 0$ as $x \to -\infty$. $\lim_{x \to +\infty} x^n e^{-x} = 0$, for every n.

THEOREM 8.2. Let $L: \mathbb{R} \to \mathbb{R}$ be the inverse of $E: \mathbb{R} \to \mathbb{R}$. Then $L'(y) = 1/y$, for $y > 0$. Also $L(y) = \int_1^y \frac{dx}{1}$, $y > 0$. $L(xy) = L(x) + L(y)$, $x^y = e^{yL(x)}$, for $x > 0$. $(x^y)' = yx^{y-1}$. Also, $\lim_{x \to \infty} x^{-y} \log x = 0$, for $y > 0$.

Define $C(x) = \frac{1}{2}(E(ix) + E(-ix))$, $S(x) = \frac{1}{2i}(E(ix) - E(-ix))$.

THEOREM 8.3. $C(x)$, $S(x)$ are real. $E(ix) = C(x) + iS(x)$. $|E(ix)| = 1$. $C(0) = 1$, $S(0) = 0$. $C'(x) = -S(x)$, $S'(x) = C(x)$. There is a smallest positive number x such that $C(x) = 0$.

We write π for $2x$.

THEOREM 8.4. $C(\pi/2) = 0$, $S(\pi/2) = 1$, $E(\pi i/2) = i$, $E(\pi i) = -1$, $E(2\pi i) = 1$, $E(z + 2\pi i) = E(z)$.

THEOREM 8.5. The function E is periodic with period $2\pi i$. The functions C, S are periodic, with period 2π. If $0 < t < 2\pi$, then $E(it) \neq 1$. If $|z| = 1$, there is a unique $t \in [0, 2\pi)$ with $E(it) = z$, Let $\gamma(t) = E(it)$, $0 \le t \le 2\pi$. Then γ is a simple closed curve whose range is the unit circle in the plane. Its length is 2π.

THEOREM 8.6. Suppose $a_0, \ldots, a_n$ are complex numbers, $n \ge 1$, $a_n \neq 0$, $P(z) = \sum_0^n a_n z^k$. Then $P(z) = 0$ for some complex number z.

A trigonometric polynomial is a finite sum of the term $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx), \ x \text{ real}, \ a_0, \ldots, a_N, b_1, \ldots, b_N \text{ are}$$

complex. This can also be written in the forms $\sum_{-N}^{N} c_n e^{inx}$.

A trigonometric series is a series of the form $\sum_{-\infty}^{\infty} c_n e^{inx}$, x real, where the Nth partial sum is as above. If $f \in \mathcal{R}([-\pi,\pi])$, the numbers $c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx$ are called the Fourier coefficients of f , and $\sum_{-\infty}^{\infty} c_n e^{inx}$ is called the Fourier series of f . (We can prove the existence of $\{c_n\}$ as a sequence).

Let $\{\phi\}$, $n > 0$, be a sequence of complex functions on $[a,b]$, such that $\int_a^b \phi_n(x)\overline{\phi_m(x)}dx = 0$, $n \neq m$. Then $\{\phi_n\}$ is called orthogonal on $[a,b]$. If in addition, $\int_a^b |\phi_n(x)|^2 dx = 1$ for all n, $\{\phi_n\}$ is said to be orthonormal.

If $\{\phi_n\}$ is orthonormal on $[a,b]$ and if $c_n = \int_a^b f(t)\overline{\phi_n(t)}dt$, we call c_n the nth Fourier coefficient of f relative to $\{\phi_n\}$. We write $f(x) \sim \sum_1^{\infty} c_n\phi_n(x)$ and call this the Fourier series of f (relative to $\{\phi_n\}$) .

THEOREM 8.7. Let $\{\phi_n\}$ be orthonormal on $[a,b]$. Let $s_n(x) = \sum_{m=1}^{n} c_m\phi_m(x)$, $t_n(x) = \sum_{m=1}^{n} \gamma_m\phi_m(x)$. Then $\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx$, and equality holds if and only if $\gamma_m = c_m$. Also $\sum_{n=1}^{\infty} |c_n|^2 \le \int_a^b |f(x)|^2 dx$.

Let $D_n(x) = \sum_{-n}^{n} e^{ikx}$, $K_n(x) = \frac{1}{n+1}\sum_0^n D_m(x)$.

THEOREM 8.8. For $n \ge 0$ we have $D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin(x/2)}$, $K_n(x) = \frac{1}{n+1} \cdot \frac{1-\cos(n+1)x}{1-\cos x}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x)dx = 1$. Also $K_n(x) \ge 0$, $K_n(x) \le \frac{2}{(n+1)(1-\cos\delta)}$, $(0 < \delta \le |x| \le \pi)$.

THEOREM 8.9. Let f be Riemann integrable over $[-\pi,\pi]$. Then

$s_n(x) = s_n(f;x) = \sum_{-n}^{n} c_m e^{imx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt$. If $0 < \delta < \pi$, then $\lim_{n\to\infty} (\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}) f(x-t)D_n(t)dt = 0$.

THEOREM 8.10. If f is continuous and if $\{\sigma_n\}$ is the sequence of arithmetic means of the partial sums of the Fourier series for f , then $\lim_{n\to\infty} \sigma_n(x) = f(x)$, uniformly for all x in $[-\pi,\pi]$. If two continuous functions have the same Fourier series, then they are equal in $[-\pi,\pi]$.

THEOREM 8.11. Suppose f and g are continuous and have period 2π, with Fourier coefficients c_n , γ_n . Then $\lim_{n\to\infty} \int_{-\pi}^{\pi} |f - s_n|^2 dx = 0$,

$\sum_{-\infty}^{\infty} c_n \bar{\gamma}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}$, $\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$.

THEOREM 8.12. Suppose $|f(y) - f(x)| < M|y - x|$, for some fixed M,x, and all $|y - x| < \delta$. Then the Fourier series of f converges to $f(x)$. Suppose f is of bounded variation on $[-\pi,\pi]$. If for some x , $s = \frac{1}{2}[f(x+) + f(x-)]$, then the Fourier series of f converges to s at x.

IX. Functions of Several Variables.

All sequences of elements of $\mathbb{R}^k$ used here are to be finite and have distinct terms, unless otherwise mentioned.

THEOREM 9.1. Let $x_1,\ldots,x_n$ be a sequence in $\mathbb{R}^k$. Then the set $\mathrm{Span}(x_1,\ldots,x_n) = \{y \in \mathbb{R}^k : (\exists c_1,\ldots,c_n \in \mathbb{R})(c_1x_1 + \ldots + c_nx_n = y)\}$ exists, and is sequentially closed.

A vector space is just a subset of some $\mathbb{R}^k$ of the form $\mathrm{Span}(x_1,\ldots,x_n)$. We say $x_1,\ldots,x_n \in \mathbb{R}^k$ is independent if $c_1x_1 + \ldots + c_nx_n = 0$ implies $c_1 = \ldots = c_n = 0$.

THEOREM 9.2. If two independent sequences in $\mathbb{R}^k$ have the same span, they have the same length. Every sequence in $\mathbb{R}^k$ has an independent subsequence with the same span. Any independent sequence from a vector space can be extended to an independent sequence whose span is the whole space.

A basis for a vector space is an independent spanning sequence. Its length is called the dimension of the vector space.

Let $f\colon V \to \mathbb{R}^k$, where V is a vector space. Then f is said to be linear if $f(x + y) = f(x) + f(y)$, $f(cx) = cf(x)$.

THEOREM 9.3. The range of a linear function $f\colon V \to \mathbb{R}^k$ is a vector space of dimension at most that of V. Also, f is one-one if and only if the range has the same dimension as V.

We write $A \in L(X,Y)$ to indicate that A is a linear map from the vector space X into the vector space Y, and $L(X)$ for $L(X,X)$.

THEOREM 9.4. If $f\colon V \to W$ is one-one and linear, so is $f^{-1}\colon W \to V$. If $A \in L(\mathbb{R}^n,\mathbb{R}^m)$, then the norm $\|A\|$ exists, given as the least upper bound of all numbers $|Ax|$, where x ranges over all vectors in $\mathbb{R}^n$ with $|x| \le 1$. Also, A is uniformly continuous. If $A,B \in L(\mathbb{R}^n,\mathbb{R}^m)$, then $\|A + B\| \le \|A\| + \|B\|$, $\|cA\| = |c|\|A\|$. Also $\|B \circ A\| \le \|B\|\|A\|$.

THEOREM 9.5. If $A \in L(\mathbb{R}^n)$, A is one-one, $\|A^{-1}\| = 1/\alpha$, $B \in L(\mathbb{R}^n)$, and $\|B - A\| < \alpha$, then B is one-one. If $A \in L(\mathbb{R}^n)$ is one-one, then there is an $\epsilon > 0$ such that for all $B \in L(\mathbb{R}^n)$, if $\|A - B\| < \epsilon$, then B is one-one.

Let $E \subset \mathbb{R}^n$ be open, $f\colon E \to \mathbb{R}^m$, $x \in E$. If there exists $A \in L(\mathbb{R}^n,\mathbb{R}^m)$ such that $\lim_{h\to 0} \dfrac{|f(x + h) - f(x) - Ah|}{|h|} = 0$, then we say f is differentiable at x, and we write $f'(x) = A$. If f is differentiable at every $x \in E$, f is differentiable in E.

THEOREM 9.6. The above definition agrees with the previous definition for $n = 1$. f is continuous at every point at which f is differentiable. f' is unique. Let $E \subset \mathbb{R}^n$ is open, $f: E \to \mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$, and g is differentiable at $f(x_0)$. Then the mapping $F = g \circ f$ is differentiable at x_0, and $F'(x_0) = g'(f(x_0))f'(x_0)$.

THEOREM 9.7. Let $f: E \to \mathbb{R}^m$, $E \subset \mathbb{R}^n$, E open, f continuous. Then the function $g(x,y)$ exists, where $g(x,y) \simeq f'(x)(y)$.

Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and has components $f_1,\ldots,f_m$. If $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbb{R}^n$, we define

$$D_j f_i(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_j(x)}{t}$$ provided the limit exists.

THEOREM 9.8. Let $f: E \to \mathbb{R}^m$, $E \subset \mathbb{R}^n$, E open, f continuous. Then for each $1 \le i \le m$, $1 \le j \le n$, the partial function $D_j f_i$ on E exists, given by $(D_j f_i)(x) \simeq D_j f_i(x)$. If $f'(x)$ exists, then $f'(x)(e_i)(j) = D_j f_i(x)$.

We say that $f \in C^1(E)$ if to every $x \in E$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|(f'(y) - f'(x))| < \epsilon$ if $y \in E$ and $|y - x| < \delta$.

THEOREM 9.8. Suppose f maps open $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ are continuous on E.

THEOREM 9.9. Suppose f is a C^1 mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is one-one for some $a \in E$, and $b = f(a)$. Then (a) there exist open sets $\mathcal{U}$ and V in $\mathbb{R}^n$ such that $a \in \mathcal{U}$, $b \in V$, f is one-one on $\mathcal{U}$, $f(\mathcal{U}) = V$. (b) Let g be the inverse of f restricted to $\mathcal{U}$. Then $g \in C^1(V)$, and $g'(y)$ is the inverse of $f'(g(y))$.

THEOREM 9.10. Suppose f is a C^1-mapping of an open set $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$. Suppose $(a,b) \in E$, $f(a,b) = 0$, $A = f'(a,b)$, and $(\forall h)(A(h,0) = 0 \rightarrow h = 0)$. Then there is a neighborhood W of b, $W \subset \mathbb{R}^m$, and a unique function $g \in C'(W)$, with values in $\mathbb{R}^n$, such that $g(b) = a$ and $f(g(y),y) = 0$, for $y \in W$.

The rank of a linear map is the dimension of its range.

THEOREM 9.11. Suppose p,q,r are nonnegative integers, X and Y are vector spaces, $\dim X = r + p$, $\dim Y = r + q$, and A is a linear transformation of X into Y, of rank r. Then there are vector spaces X_1,X_2 in X, Y_1,Y_2 in Y, of rank r, such that

(a) every $x \in X$ has a unique representation of the form $x = x_1 + x_2$, with $x_1 \in X_1$, $x_2 \in X_2$;

(b) every $y \in Y$ has a unique representation of the form $y = y_1 + y_2$, with $y_1 \in Y_1$, $y_2 \in Y_2$;

(c) $Ax_2 = 0$ for every $x_2 \in X_2$;

(d) the restriction of A to X_1 is a one-one mapping of X_1 onto Y_1;

(e) $\dim X_1 = \dim Y_1 = r$.

THEOREM 9.12. Suppose $X = \mathbb{R}^{r+p}$, $Y = \mathbb{R}^{r+q}$, F is a C-mapping of an open set $E \subset X$ into Y, and $F'(x)$ has rank r for every $x \in E$. Fix $a \in E$, put $A = F'(a)$, choose X_1,X_2,Y_1,Y_2 as in Theorem 9.11, and define

F_1, F_2 by $F(x) = F_1(x) + F_2(x)$, all $x \in E$, where $F_1(x) \in Y_1$, $F_2(x) \in Y_2$. Then there is an open set U in X such that $a \in U$, $U \subset E$, and (a) $F_1(U)$ is an open set in Y_1; (b) to each $y_1 \in F_1(U)$ there exists precisely one $y_2 \in Y_2$ such that $y_1 + y_2 \in F(U)$.

If f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, and for some j, $e_i \cdot f(x) = e_i \cdot x$ for all $i \neq j$, $x \in E$, then we say f is primitive.

THEOREM 9.13. Suppose f is a C^1-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $0 \in E$, $f(0) = 0$, $f'(0)$ is one-one. Then there is a neighborhood of 0 in $\mathbb{R}^n$ in which a representation $f(x) = g_n(B_n g_{n-1}(\ldots g_1(B_1 x)))$ is valid. Here each g_k is a primitive C^1-mapping in some neighborhood of 0, $g_k(0) = 0$, and each B_k is a linear map on $\mathbb{R}^n$ which either is the identity or merely interchanges some pair of coordinates.

Let A be a finite set, $f: A \to \mathbb{R}$. We define $\sum_A f$ to be $\sum_{k=0}^{n} f(g(k))$, where g is chosen to be a one-one function from $\{i: i \leq n\}$ onto A. It can be shown that this sum is independent of the choice of g. In addition, we can prove that the set of all functions on a finite set is finite. With these preliminaries, we may define the determinant of a linear operator A on $\mathbb{R}^n$. Fix $A \in L(\mathbb{R}^n)$. Let $a(i,j)$ be $A(e_j)(i)$, for $0 \leq i, j \leq n - 1$. If $j_1, \ldots, j_n$ is an ordered n-tuple of integers, define $s(j_1, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_p - j_q)$, where $\operatorname{sgn} x = 1$ if $x > 0$, $\operatorname{sgn} x = -1$ if $x < 0$, $\operatorname{sgn} 0 = 0$. Define $\det(A) =$ $\sum s(j_1, \ldots, j_n) a(1, j_1) a(2, j_2) \ldots a(n, j_n)$, where the sum is over all ordered n-tuples $(j_1, \ldots, j_n)$, where $j_i < n$.

THEOREM 9.14. $\det(B \circ A) = \det(B)\det(A)$. A is one-one if and only if

$\det(A) \neq 0$. Det is continuous, in the sense that $(\forall A \in L(\mathbb{R}^n))(\forall \epsilon > 0)$ $(\exists \delta > 0)(\forall B \in L(\mathbb{R}^n))(\|A - B\| < \delta \rightarrow |\det(A) - \det(B)| < \epsilon)$.

We omit the treatment of integration on p. 204-208 of Rudin, and instead use the treatment in Section 24 of Bartle, The Elements of Real Analysis.

Let $D \subset \mathbb{R}^p$ be a compactum, $f: D \rightarrow \mathbb{R}^q$ be bounded. Let I be a closed rectangle in $\mathbb{R}^p$ enclosing D . Define partitions and mesh analogously as in the 1-dimensional case. A Riemann sum $S(P;f)$ is a sum of the form $\sum_{k=1}^{n} f(x_k)A(J_k)$, where x_k is any point in the subinterval J_k , and $A(J_k)$ is the volume of J_k . An element L of $\mathbb{R}^q$ is defined to be the Riemann integral $\int_D f$ if, for all $\epsilon > 0$ there is a $\delta > 0$ such that $|S(P;f) - L| < \epsilon$ for all P with $\mathrm{mesh}(P) < \delta$. If $f(x_k)$ is not defined, it is taken to be 0 .

THEOREM 9.15. The function f is integrable on D if and only if $(\forall \epsilon > 0)$ $(\exists \delta > 0)(|S(P_1;f) - S(P_2;f)| < \epsilon$ if $\mathrm{mesh}(P_1), \mathrm{mesh}(P_2) < \delta)$.

THEOREM 9.16. If $\int_D f, \int_D g$ exists, then $\int_D (af + bg) = a\int_D f + b\int_D g$. If $f \geq 0$, $\int_D f$ exists, then $\int_D f \geq 0$.

A set $A \subset \mathbb{R}^p$ has zero content if for all $\epsilon > 0$ there is a finite sequence of closed rectangles covering A of total volume $< \epsilon$.

THEOREM 9.17. Suppose that f is defined on an interval J of $\mathbb{R}^p$. If f is continuous except on a subset E of J which has zero content, then f is integrable over J .

If $D \subset \mathbb{R}^p$, a boundary point of D is a point every neighborhood of which contains points both of D and its complement. The boundary of D is the set of all boundary points (if it exists).

THEOREM 9.18. The boundary B of any compactum D (in fact sequentially closed or open set) exists. Let $D \subset \mathbb{R}^p$ be a compactum, and let f be continuous on D. If the boundary of D has zero content, then f is integrable over D.

We say the compactum D has content iff the boundary of D has content zero. We define $A(D) = \int_D 1$.

THEOREM 9.19. Let D be a compactum in $\mathbb{R}^p$ which has content and let D_1, D_2 be compacta with $D = D_1 \cup D_2$, $D_1 \cap D_2$ of zero content. If g is integrable over D then g is integrable over D_1, D_2, and $\int_D g = \int_{D_1} g + \int_{D_2} g$.

THEOREM 9.20. Let D be a compactum in $\mathbb{R}^p$ which has content. Let f be integrable over D such that $|f(x)| \le M$ for x in D. Then $|\int_D f| \le MA(D)$. In particular, if f is real-valued and $m \le f(x) \le M$ for x in D, then $mA(D) \le \int_D f \le MA(D)$. If D is a compactum that is pathwise connected in $\mathbb{R}^p$ with content, and if f is continuous on D with values in $\mathbb{R}$, then there is a point p in D such that $\int_D f = f(p)A(D)$.

THEOREM 9.21. Let f be integrable over the rectangle $D \subset \mathbb{R}^{p+q}$ with values in $\mathbb{R}$, and suppose that, for each value x in the rectangle $E_1 \subset \mathbb{R}^p$, the integral $F(x) = \int_{E_2} f(x,y)\,dy$ exists, where $D = E_1 \times E_2$. Then F is integrable on E_2, and $\int_D f = \int_{E_2} F$.

THEOREM 9.22. If $\varphi \in L(\mathbb{R}^p)$ and if K is a cube in $\mathbb{R}^p$, then the set $\varphi(K)$ has content and $A(\varphi(K)) = |J_\varphi| A(K)$. If $\varphi \in C^1$ on an open set G in $\mathbb{R}^p$ to $\mathbb{R}^p$, and if D is a compactum lying in G which has content zero, then $\varphi(D)$ has content zero.

THEOREM 9.23. Let $\varphi \in C^1$ on an open set $G \subset \mathbb{R}^p$ to $\mathbb{R}^p$, whose Jacobian J_φ does not vanish on G . If D is a compactum lying in G with content, then $\varphi(D)$ is a compactum with content.

THEOREM 9.24. Suppose $\varphi \in C^1$ on an open set G and that J_φ does not vanish on G . If D is a compactum lying in G , $\epsilon > 0$, there exists $\delta > 0$ such that if K is a cube with center x in D and side length less than δ , then $|J_\varphi(x)|(1 - \epsilon)^p \le \frac{A[\varphi(K)]}{A(K)} \le |J_\varphi(x)|(1 + \epsilon)^p$.

THEOREM 9.25. Suppose that $\varphi \in C^1$ on an open subset G of $\mathbb{R}^p$ with values in $\mathbb{R}^p$ and that the Jacobian J_φ does not vanish on G . If D is a compactum lying in G which has content and if f is continuous on $\varphi(D)$ to $\mathbb{R}$, then $\varphi(D)$ has content and $\int_{\varphi(D)} f = \int_D (f\circ\varphi)|J_\varphi|$.

X. <u>Miscellaneous</u>.

THEOREM 10.1. Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$, and suppose $f_n(x) < B$. If $f_n \to f$ and f is continuous, then $\int_0^b f = \lim \int_a^b f_n$.

THEOREM 10.2. The intersection of every sequence of dense sequentially open subsets of $\mathbb{R}^p$ is dense in $\mathbb{R}^p$.

We call a set $E \subset \mathbb{R}$ explicitly of measure zero if there is a double sequence A_{ij} of open intervals such that each $\bigcup_j A_{ij}$ covers E , and for each i , the sum of the lengths of the A_{ij} is $< 2^{-i}$.

THEOREM 10.3. Let f be a monotone function on $[a,b]$. Then $\{x: f$ is not differentiable at $x\}$ is explicitly of measure zero.

Theorems of Strength PA

The theorems below are provable in RV_1 + sequential choice, which reads $(\forall n)(\exists\alpha)(f(n,\alpha) = 0) \rightarrow (\exists g)(\forall n)(f(n,g(n)) = 0)$, and is equiconsistent with PA.

THEOREM 1. Every limit point x of A is the limit of a sequence of elements from A other than x.

THEOREM 2. If a set contains arbitrarily large finite subsets, then it contains a denumerable subset.

THEOREM 3. Every nhbd of a limit point of A contains a denumerable subset from A.

THEOREM 4. Every sequence of open sets covering a compact set has an initial segment covering it also.

THEOREM 5. A continuous function on a compact set is uniformly continuous.

THEOREM 6. If $f_n \geq f_{n+1}$ on a compact set E, $f_n \rightarrow f$ on E, and f_n, f are continuous on E, then $f_n \rightarrow f$ uniformly on E.

THEOREM 7. Let K be a compact set. If $\{f_n\}$ is a uniformly convergent sequence of continuous functions on K, then $\{f_n\}$ is equicontinuous.

THEOREM 8. If $f: I^k \rightarrow \mathbb{R}$, f is continuous a.e., then f is Riemann integrable.

THEOREM 9. If $A \subset B$, A compact, B open, then there is a finite sequence of open rectangles from B which cover A.

In Theorem 8, cont. a. e. means that for each $\epsilon > 0$ there is a sequence of open rectangles, the sum of whose volumes is $< \epsilon$, such that the function is continuous off of these rectangles.

Theorems of Strength $ATR(< \omega^\omega)$

The theorems below are equiconsistent with $ATR(< \omega^\omega)$ when added to RV_1, and are provable in RV_1 + dependent choice, which is $(\forall\alpha)(\exists\beta)(f(\alpha,\beta) = 0) \rightarrow (\forall\alpha)(\exists g)(g(0) = \alpha \;\&\; (\forall n)(f(g(n),g(n+1)) = 0))$, and is also equiconsistent with $ATR(< \omega^\omega)$.

THEOREM 1. A set is infinite if and only if it contains a denumerable subset.

THEOREM 2. A set is infinite if and only if it contains arbitrarily large finite subsets.

THEOREM 3. If $A \subset \mathbb{R}^n$ is closed under addition and scalar multiplication, then A is a vector space.

THEOREM 4. The finite union of finite sets is finite.

Theorems of Strength $ATR(< \Gamma_0)$, ATR_0

The following theorems are equiconsistent with $ATR(< \Gamma_0)$ and ATR_0, when added to RV_1.

THEOREM 1. Every one-one function has an inverse.

THEOREM 2. Every graph determines a function.

THEOREM 3. Countable union of countable sets is countable.

THEOREM 4. The countable intersection of dense open subsets of $\mathbb{R}^n$ is dense.

Theorems of Strength $ID(<\omega)$

The theorems below are equiconsistent with $ID(<\omega)$ when added to RV_1.

THEOREM 1. Every nonempty bounded set of reals has a least upper bound.

THEOREM 2. Every open set is sequentially open.

THEOREM 3. Every bounded closed set of reals has a least upper bound.

THEOREM 4. Every indexed family of open sets covering the unit interval contains a finite subcover.

THEOREM 5. The set of interior points of every set exists.