# WHAT IS O-MINIMALITY?

by
Harvey M. Friedman*
Ohio State University
friedman@math.ohio-state.edu
http://www.math.ohio-state.edu/%7Efriedman/

November 8, 2006
Revised September 6, 2007
Revised October 29, 2007
Revised November 30, 2007

Abstract. We characterize the o-minimal expansions of the ring of real numbers, in mathematically transparent terms. This should help bridge the gap between investigators in o-minimality and mathematicians unfamiliar with model theory, who are concerned with such notions as non oscillatory behavior, tame topology, and analyzable functions. We adapt the characterization to the case of o-minimal expansions of an arbitrary ordered ring.

## 1. PRELIMINARIES.

We give a new characterization of o-minimal expansions of the ring of real numbers, in particularly transparent mathematical terms. The reader may wish to compare the characterization here with the approach of [Dr98], p.13, which can be adapted to give a characterization in terms of relations instead of functions.

The main definition is that of a rich class (¬). The (¬) indicates that we are working over the ring of real numbers.

We say that V is a rich class (¬) if and only if

1. MULTIVARIATE (¬). All elements of V are functions f such that the domain of f is some $\neg^n$ and the range of f is a subset of some $\neg^m$.

2. POLYNOMIALS (¬). Every polynomial with real coefficients from any $\neg^n$ into any $\neg^m$ is an element of V.

3. COMPOSITION (¬).

i. If f:$\neg^n$  $\neg^m$  V, g:$\neg^m$  $\neg^p$  V, then h:$\neg^n$  $\neg^p$  V, where for all x  $\neg^n$, h(x) = g(f(x)).

ii. If f:$\neg^n$  $\neg^m$  V, g:$\neg^n$  $\neg^p$  V, then h:$\neg^n$  $\neg^{m+p}$  V, where for all x  $\neg^n$, h(x) = (f(x),g(x)).

4. ZERO SELECTION (¬). Let f:$\neg^{n+m}$  ¬  V. There exists g:$\neg^n$  $\neg^m$  V such that if f(x,y) = 0 then f(x,g(x)) = 0.

We use the adjective “special” if, in addition, we have

5. LIMIT (¬). Every bounded f:¬  ¬  V has a limit at infinity.

From model theory, an expansion of the ring of real numbers is a system (¬,<,0,1,+,-,•,V), where V obeys MULTIVARIATE (¬).

From model theory, an o-minimal expansion of the ring of real numbers is an expansion of the ring of real numbers in which every first order definable subset of ¬ is a finite union of open intervals and points. Here  and -  are allowed as endpoints. Here, and throughout the paper, "definable" means "definable with parameters by a formula in the first order predicate calculus with equality".

The above definition is a special case of the more general definition of o-minimal structure introduced in [PS86], which we use in section 4.

Specifically, a linearly ordered structure is a system (D,<,V), where (D,<) is a strictly linear ordering, and V is a set of constants from D, multivariate relations on D, and multivariate functions from D into D. An o-minimal structure is a structure M = (D,<,V), where every M definable subset of D is a finite union of open intervals and points, where the endpoints of the intervals lie in D  {- , } and the points lie in D.

We prove the following.

THEOREM A. Let M = (¬,<,0,1,+,-,•,V) be an expansion of the ring of real numbers. The following are equivalent.
i. M is o-minimal.
ii. V is a subset of some special rich class (¬).
ii. There is a least rich class (¬) containing V, and this rich class (¬) is special.

We conclude with Theorem B, which is a version for arbitrary ordered rings.

## 2. RICH CLASSES OF REAL FUNCTIONS.

The key to proving Theorem A is the characterization of rich classes (¬) in terms of first order definability. We now fix V to be a rich class (¬). Let M = (¬,<,0,1,+,-,•,V).

Let $f:\neg^{n+m}$ □ ¬. A zero selector of f is a g with the property in the ZERO SELECTION (¬) clause.

LEMMA 2.1. There exist $f_1,\ldots,f_8$:¬ □ ¬ □ V and $f_9,\ldots,f_{12}:\neg^4$ □ ¬ such that

i. $f_1(x) = 1/x$ if $x \neq 0$.
ii. $f_2(x) = 0$ if $x \neq 0$; 1 otherwise.
iii. $f_3(0) = 0$; $f_3(x) = 1$ elsewhere.
iv. $x \geq 0$ □ $(f_4(x) = \mathrm{sqrt}(x) \quad f_4(x) = -\mathrm{sqrt}(x))$.
v. $f_5(x) = 0$ □ $x \geq 0$.
vi. $f_6(x) = 0$ □ $x > 0$.
vii. $f_7(x) = 0$ if $x \geq 0$; 1 otherwise.
viii. $f_8(x) = 0$ if $x > 0$; 1 otherwise.
ix. $f_9(x,y,z,w) = z$ if x □ y; w otherwise.
x. $f_{10}(x,y,z,w) = z$ if $x < y$; w otherwise.
xi. $f_{11}(x,y,z,w) = z$ if $x \neq y$; w otherwise.
xii. $f_{12}(x,y,z,w) = z$ if $x = y$; w otherwise.

Proof: i. Let $f(x,y) = xy-1$. Let $f_1$ be a zero selector of f.
ii. Let $f_2(x) = 1-xf_1(x)$.
iii. Let $f_3(x) = 1-f_2(x)$.
iv. Let $f(x,y) = y^2-x$. Let $f_4$ be a zero selector for f.
v. Let $f_5(x) = f_4(x)^2-x$.
vi. Let $f_6(x) = f_5(x)^2 + f_2(x)^2$.
vii. $f_7(x) = f_3(f_5(x))$.
viii. Let $f_8(x) = f_3(f_6(x))$.
ix. Let $f_9(x,y,z,w) = z(1-f_7(y-x)) + w(f_7(y-x))$.
x. Let $f_{10}(x,y,z,w) = z(1-f_8(y-x)) + w(f_8(y-x))$.
xi. Let $f_{11}(x,y,z,w) = z(f_3(x-y)) + w(1-f_3(x-y))$.
xii. Let $f_{12}(x,y,z,w) = z(1-f_3(x-y)) + w(f_3(x-y))$.

QED

Let t be a term in the language of M = (¬,<,0,1,+,-,•,V), whose variables are among $v_1,\ldots,v_n$, $n \geq 0$. I.e., t is an expression built up from 0,1,+,-,•, the functions in V, and the variables $v_1,\ldots,v_n$, in the standard way. For $x_1,\ldots,x_n \in \mathbb{R}$, We write $\mathrm{Val}(M,t;x_1,\ldots,x_n)$ for the value of t in M, when we interpret the variables $v_1,\ldots,v_n$ as the real numbers $x_1,\ldots,x_n$.

Let $\varphi$ be a first order formula in the language of M whose free variables are among $v_1,\ldots,v_n$, $n \geq 0$. We write $\mathrm{Sat}(M,\varphi;x_1,\ldots,x_n)$ to indicate that the formula $\varphi$ is true in M, when we interpret $v_1,\ldots,v_n$ as the real numbers $x_1,\ldots,x_n$.

LEMMA 2.2. Let t be a term in the language of M, whose variables are among $v_1,\ldots,v_n$, $n \geq 1$. Then the function from $\mathbb{R}^n$ into $\mathbb{R}$ given by $\mathrm{Val}(M,t;x_1,\ldots,x_n)$ lies in V.

Proof: Fix $n \geq 1$. We prove by induction on terms with variables among $x_1,\ldots,x_n$, that $\mathrm{Val}(M,t;x_1,\ldots,x_n)$ lies in V.

case 1. t is $x_i$ or a constant. Note that $\mathrm{Val}(M,t;x_1,\ldots,x_n)$ is a polynomial.

case 2. t is $f(s_1,\ldots,s_k)$, where $f \in V$. By the induction hypothesis, each function $\mathrm{Val}(M,s_i;x_1,\ldots,x_n)$ lies in V. Hence the function

$$(\mathrm{Val}(M,s_1;x_1,\ldots,x_n),\ldots, \mathrm{Val}(M,s_k;x_1,\ldots,x_n))$$

lies in V. Hence the function

$$f(\mathrm{Val}(M,s_1;x_1,\ldots,x_n),\ldots, \mathrm{Val}(M,s_k;x_1,\ldots,x_n)) = \mathrm{Val}(M,t;x_1,\ldots,x_n)$$

lies in V. QED

LEMMA 2.3. Let $\varphi$ be a first order formula in the language of M whose free variables are among $v_1,\ldots,v_n$, $n \geq 1$. Then the function $f(x_1,\ldots,x_n) = 1$ if $\mathrm{Sat}(M,\varphi;x_1,\ldots,x_n)$; 0 otherwise, lies in V.

Proof: We prove the following by induction on $\varphi$. For all $n \geq 1$ and formulas $\varphi$ whose free variables are among $x_1,\ldots,x_n$, the function $f(x_1,\ldots,x_n) = 1$ if $\mathrm{Sat}(M,\varphi;x_1,\ldots,x_n)$; 0 otherwise, lies in V.

case 1. $s = t$. Let the (free) variables in $s = t$ be among $v_1,\ldots,v_n$. By Lemma 2.2, the functions

$$\mathrm{Val}(M,s;x_1,\ldots,x_n)$$

$$\mathrm{Val}(M,t;x_1,\ldots,x_n)$$

lie in V. The function in question is

$$f(x_1,\ldots,x_n) = 1 \text{ if } \mathrm{Val}(M,s;x_1,\ldots,x_n) = \mathrm{Val}(M,t;x_1,\ldots,x_n);\quad 0 \text{ otherwise.}$$

We can rewrite f as

$$f(x_1,\ldots,x_n) = f_{12}(\mathrm{Val}(M,s;x_1,\ldots,x_n),\mathrm{Val}(M,t;x_1,\ldots,x_n),1,0)$$

and apply Lemmas 2.1, 2.2.

case 2. $s < t$. Let the (free) variables in $s = t$ be among $v_1,\ldots,v_n$. By Lemma 2.2, the functions

$$\mathrm{Val}(M,s;x_1,\ldots,x_n)$$
$$\mathrm{Val}(M,t;x_1,\ldots,x_n)$$

lie in V. The function in question is

$$f(x_1,\ldots,x_n) = 1 \text{ if } \mathrm{Val}(M,s;x_1,\ldots,x_n) < \mathrm{Val}(M;t;x_1,\ldots,x_n);\quad 0 \text{ otherwise.}$$

We can rewrite f as

$$f(x_1,\ldots,x_n) = f_{10}(\mathrm{Val}(M,s;x_1,\ldots,x_n),\mathrm{Val}(M,t;x_1,\ldots,x_n),1,0)$$

and apply Lemmas 2.1, 2.2.

case 3. □□. Let the free variables in □□ be among $v_1,\ldots,v_n$. Then the free variables in □□ are among $v_1,\ldots,v_n$. Let n-ary f be given by the induction hypothesis. The function in question is

$$1-f(x_1,\ldots,x_n)$$

which lies in V by Lemma 2.2.

case 4. □ □ □. Let the free variables in □ □ □ be among $v_1,\ldots,v_n$. Then the free variables in □,□ are among $v_1,\ldots,v_n$. Let n-ary f,g be given by the induction hypothesis for □,□, respectively. The function in question is

$$f(x_1,...,x_n)\bullet g(x_1,...,x_n)$$

which lies in V by Lemma 2.2.

case 5. $(□v_p)(□)$. Let the free variables in $(□v_p)(□)$ be among $v_1,...,v_n$, where $p$ □ $n$. (The case $p > n$ will be handled later). Then the free variables in □ are among $v_1,...,v_n$. By the induction hypothesis, the function

$$f(x_1,...,x_n) = 1 \text{ if } Sat(M,□;x_1,...,x_n); \quad 0 \text{ otherwise}$$

lies in V. Define

$$g(x_1,...,x_n,y) = 0 \text{ if } f(x_1,...,x_{p-1},y,x_{p+1},...,x_n) = 1; \quad 1 \text{ otherwise.}$$

By Lemma 2.2, g □ V. Let $h:¬^n$ □ ¬ be a zero selector for g. Finally, let $f^*:¬^n$ □ ¬ be given by

$$f^*(x_1,...,x_n) = 1 \text{ if } f(x_1,...,x_{p-1},,h(x_1,...,x_n),x_{p+1},...,x_n) = 1; \quad 0 \text{ otherwise.}$$

Then $f^*$ □ V and $f^*$ is as required.

Now suppose $p > n$. Then the free variables in $(□v_p)(□)$ are among $v_1,...,v_p$. The argument given above can be repeated with n replaced by p to show that the function

$$h(x_1,...,x_p) = 1 \text{ if } Sat(M,(□v_p)(□);x_1,...,x_p); \quad 0 \text{ otherwise}$$

lies in V. The function in question,

$$f(x_1,...,x_n) = 1 \text{ if } Sat(M,(□x_p)(□);x_1,...,x_n)$$

can be written in the form

$$f(x_1,...,x_n) = h(x_1,...,x_n,x_n,...,x_n)$$

and so by Lemma 2.2, is also in V.

The remaining cases □, , □ , □ , are left to the reader.
QED

LEMMA 2.4. Let f:$\neg^n$  $\neg^m$ be first order definable in M. Then f  V.

Proof: Let f be as given. By Lemma 2.3, the function

$$g(x_1,...,x_{n+m}) = 1 \text{ if } f(x_1,...,x_n) \neq (x_{n+1},...,x_{n+m});$$
$$0 \text{ otherwise}$$

lies in V. Let h:$\neg^n$  $\neg^m$  V be a zero selector for g. Then h = f  V. QED

THEOREM 2.5. Let V obey MULTIVARIATE (¬), and M = (¬,<,0,1,+,-,•,V). The following are equivalent.
i. V is a rich class (¬).
ii. Every function f:$\neg^{n+m}$  ¬ definable in M has a zero selector in V.
iii. Every function f:$\neg^n$  $\neg^m$ definable in M is an element of V, and every function f:$\neg^{n+m}$  ¬ definable in M has a zero selector in V.

Proof: Let V,M be as given. We show i  ii  iii  i.

Assume i. Let V be a rich class (¬). Let f:$\neg^{n+m}$  ¬ be definable in M. By Lemma 2.4, f  V. By ZERO SELECTION (¬), f has a zero selector in V.

Assume ii. Let f:$\neg^n$  $\neg^m$ be definable in M. Let g:$\neg^{n+m}$  ¬ be defined by g(x,y) = 0 if y = f(x); 1 otherwise. Then g is definable in M. Let h:$\neg^n$  $\neg^m$  V be a zero selector for g. Then h = f, and so f  V.

Assume iii. Since all polynomials are definable in M, and the composition of functions definable in M is definable in M, we have POLYNOMIAL (¬) and COMPOSITION (¬). Also ZERO SELECTION (¬) is immediate. Hence i. QED

## 3. PROOF OF THEOREM A.

Recall the definition of a special rich class (¬):

V is a special rich class (¬) if and only if V is a rich class (¬) obeying LIMIT (¬).

In the proof of Theorem A, we shall use the following crucial fact about o-minimal expansions of the ring of real numbers.

MONOTONICITY THEOREM. Let M = (ℝ,<,0,1,+,-,•,V) be an o-minimal expansion of the ring of real numbers. For all $f:\mathbb{R} \to \mathbb{R} \in V$, there exists $x_1 < \ldots < x_k$, $k \geq 1$, such that on each of the complementary open intervals $(-\infty,x_1),(x_1,x_2),\ldots,(x_{k-1},x_k),(x_k,\infty)$, f is continuous, and f is either constant, strictly increasing, or strictly decreasing. In fact, this is true for any o-minimal structure M = (D,<,V).

This result was first proved in [PS86]. In fact, it was proved there for general o-minimal structures. It is fundamental to the entire theory of o-minimality. Also see [Dr98], p.3, 43-46. In the notation there, $x_i$ is allowed to be $\infty$ or $-\infty$.

We will not be using the continuity from the Monotonicity Theorem.

THEOREM A. Let M = (ℝ,<,0,1,+,-,•,W) be an expansion of the ring of real numbers. The following are equivalent.
i. M is o-minimal.
ii. W is a subset of some special rich class (ℝ).
iii. There is a least rich class (ℝ) containing W, and this rich class (ℝ) is special.

Proof: Let M be as given. We shall prove ii → i → iii → ii. Since iii → ii is trivial, it suffices to prove ii → i → iii.

Assume ii. Let W ⊆ V, where V is a special rich class (ℝ). Towards establishing i, let A ⊆ ℝ be definable in M. We must show that A is a finite union of open intervals and points.

Let $\chi_A$ be the characteristic function of A. Then $\chi_A(x)$ and $\chi_A(-x)$ are definable in M, and hence lie in V, by Theorem 2.5.

By LIMIT (ℝ) in V, $\chi_A(x)$ and $\chi_A(-x)$ are constant for x >> 0. Hence $\chi_A$ is also constant for x << 0.

We claim that $\chi_A$ is pointwise continuous except at finitely many points. To see this, suppose this is false. Then the points of discontinuity form a bounded infinite subset of ℝ, and therefore have a limit point x. If x is a limit from the left, then we can use an order preserving bijection $g:(x-1,x) \to \mathbb{R}$ that is definable in (ℝ,<,0,1,+,•) in order to transform this situation to an element of V with no limit at infinity. Also, if x is a limit from the right, then we can use an order reversing bijection g:(x,x+1) □ ¬ that is definable in (¬,<,0,1,+,-,•) in order to transform this situation to an element of V with no limit at infinity.

Since $□_A$ is pointwise continuous except at finitely many points, let $a_1,...,a_k$ □ ¬ be such that $□_A$ is pointwise continuous off of $a_1,...,a_k$, k ≥ 1. It is clear by the intermediate value theorem that $□_A$ is constant on each interval (- ,$a_1$),($a_1$,$a_2$),...,($a_{k-1}$,$a_k$),($a_k$, ).

It is now obvious that A is the union of these intervals where $□_A$ is constantly 1, together with the $a_1,...,a_k$ at which $f_A$ is 1. Thus the o-minimality of M is verified.

Assume i. Let V be the family of all functions definable in M. We will now show that V is a rich class (¬). By Theorem 2.5, V is the smallest rich class (¬) containing W. Finally, we will show that V is special. This will conclude the derivation of iii and the proof of Theorem A.

Conditions 1-3 (¬) in the definition of rich class (¬) obviously hold of V.

To check that ZERO SELECTION (¬) also holds for V, we first prove the following by induction on m ≥ 1. For all n ≥ 1, every f:$¬^n$ □ $¬^m$ □ ¬ □ V has a zero selector in V. (This sort of result is well known from the theory of o-minimal structures, but we include it here for the sake of completeness).

Suppose m = 1. For each x □ $¬^n$ let $E_x$ = {y □ ¬: f(x,y) = 0}. Then by the o-minimality of M, $E_x$ is the union of finitely many open intervals and finitely many points.

1. If $E_x$ is empty, return 0.
2. If $E_x$ is nonempty and finite, then return the least element of $E_x$.
3. Otherwise, there is a nonempty open interval contained in $E_x$. Hence there is a nonempty maximal open interval contained in $E_x$. Hence there is a leftmost nonempty maximal open interval J contained in $E_x$.
4. If J is of the form (a,b), return (a+b)/2.
5. If J is of the form (- ,b), return b-1.
6. If J is of the form (b, ), return b+1.

Suppose this is true for fixed m ≥ 1, and let f:¬$^{n}$ □ ¬$^{m+1}$ □ ¬ □ V. Let g:¬$^{n}$ □ ¬$^{m}$ □ ¬ be given by g(x,y) = 0 if (□z □ ¬)(f(x,y,z) = 0); 1 otherwise. Then g □ V. By the induction hypothesis, let h:¬$^{n}$ □ ¬$^{m}$ □ V be a zero selector of g. Now Let h*:¬$^{n+m}$ □ ¬ be a zero selector for f:¬$^{n+m}$ □ ¬ □ ¬ (rearranging f).

Suppose f(x,y,z) = 0, x □ ¬$^{n}$, y □ ¬$^{m}$, z □ ¬. Then g(x,y) = 0, and so g(x,h(x)) = 0. Let f(x,h(x),z) = 0. Then f(x,h(x),h*(x,h(x))) = 0.

Thus we see that the function H:¬$^{n}$ □ ¬$^{m+1}$ □ V given by

H(x) = (h(x),h*(x,h(x)))

is a zero selector for f. Hence V is a rich class (¬).

To show that V is special, let f:¬ □ ¬ □ V be bounded. By the Monotonicity Theorem, f is eventually strictly increasing, strictly decreasing, or constant. Thus we have

lim$_{x□1}$g(x) = sup(g) = lim$_{x□}$ f(x).

QED

## 4. ORDERED RINGS.

We will use the usual notion of a ring **R** = (R,0,1,+,-,•), where (R,0,+,-) is an Abelian group, (R,1,•) is associative with two sided unit 1, and we have both left and right distributivity. We do not assume the commutativity of •.

We will also use the usual notion of ordered ring **R** = (R,0,1,<,+,-,•), where

i. < is a strict linear ordering of R.
ii. 0 < 1.
iii. x < y □ x + z < y + z.
iv. x < y □ z > 0 □ xz < yz.

A field is a ring where the • is an Abelian group. An ordered field is an ordered ring where the • is an Abelian group.

We now use any ordered ring **R** = (R,<,0,1,+,-,•) in place of the ordered ring of real numbers. We will prove that Theorem A holds using a modification of LIMIT.

Fix an ordered ring **R** = (R,<,0,1,+,-,•). Note that R is obviously an ordered Abelian group, which allows us to work rather easily with linear inequalities. When • is involved, we have to be more careful.

We say that V is a rich class (**R**) if and only if

1. MULTIVARIATE (**R**). All elements of V are functions f such that the domain of f is some $R^n$ and the range of f is a subset of some $R^m$.

2. POLYNOMIALS (**R**). Every polynomial over **R** from any $R^n$ into any $R^m$ is an element of V.

3. COMPOSITION (**R**).

i. If $f:R^n \to R^m \in V$, $g:R^m \to R^p \in V$, then $h:R^n \to R^p \in V$, where for all $x \in R^n$, $h(x) = g(f(x))$.

ii. If $f:R^n \to R^m \in V$, $g:R^n \to R^p \in V$, then $h:R^n \to R^{m+p} \in V$, where for all $x \in ¬^n$, $h(x) = (f(x),g(x))$.

4. ZERO SELECTION (**R**). Let $f:R^{n+m} \to R \in V$. There exists $g:R^n \to R^m \in V$ such that if $f(x,y) = 0$ then $f(x,g(x)) = 0$.

We use the adjective “special” if, in addition, we have

5. MONOTONICITY (**R**). For all $f:R \to R \in V$ there exists $x_1 < \ldots < x_k \in R$, $k \geq 1$, such that on each of the complementary open intervals $(- ,x_1),(x_1,x_2),\ldots,(x_{k-1},x_k),(x_k, )$, f is either constant, strictly increasing, or strictly decreasing.

Recall the definition of an o-minimal structure (D,<,...) given in section 1. Also recall that the Monotonicity Theorem is stated in section 3 for all o-minimal structures.

The notion of an o-minimal ordered ring, **R** = (R,<,0,1,+,-,•), without any auxiliary functions, is already quite interesting.

According to [PS86], the o-minimal ordered rings are exactly the real closed fields. Also see [Dr98], p. 21.

We now focus on expansions M = (R,<,0,1,+,-,•,W) of an ordered ring **R** = (R,<,0,1,+,-,•). Here W obeys MULTIVARIATE (**R**).

We fix V to be a rich class (**R**), where **R** = (R,<,0,1,+,-,•) is an ordered ring.

In order to repeat the proof of Lemma 2.1, we make the following assumption. **R** is an ordered field in which every positive element has a square root.

LEMMA 4.1. There exist $f_1,...,f_8$:R □ R □ V and $f_9,...,f_{12}$:$R^4$ □ R □ V such that
i. $f_1(x) = 1/x$ if $x \neq 0$.
ii. $f_2(x) = 0$ if $x \neq 0$; 1 otherwise.
iii. $f_3(0) = 0$; $f_3(x) = 1$ elsewhere.
iv. $x \geq 0$ □ ($f_4(x)$ = sqrt(x)   $f_4(x)$ = -sqrt(x)).
v. $f_5(x) = 0$ □ $x \geq 0$.
vi. $f_6(x) = 0$ □ $x > 0$.
vii. $f_7(x) = 0$ if $x \geq 0$; 1 otherwise.
viii. $f_8(x) = 0$ if $x > 0$; 1 otherwise.
ix. $f_9(x,y,z,w) = z$ if x □ y; w otherwise.
x. $f_{10}(x,y,z,w) = z$ if $x < y$; w otherwise.
xi. $f_{11}(x,y,z,w) = z$ if $x \neq y$; w otherwise.
xii. $f_{12}(x,y,z,w) = z$ if $x = y$; w otherwise.

Proof: We can repeat the proof of Lemma 2.1 without change. QED

LEMMA 4.2. Let t be a term in the language of M, whose variables are among $x_1,...,x_n$, $n \geq 1$. Then the function from $R^n$ into R given by Val(M,t;$x_1,...,x_n$) lies in V.

Proof: By the proof of Lemma 2.2 without change. QED

LEMMA 4.3. Let □ be a first order formula in the language of M whose free variables are among $x_1,...,x_n$, $n \geq 1$. Then the function $f(x_1,...,x_n) = 1$ if Sat(M,□;$x_1,...,x_n$); 0 otherwise, lies in V.

Proof: By the proof of Lemma 2.3 without change. QED

LEMMA 4.4. Let g:$R^n$ □ $R^m$ be first order definable in M. Then g □ V.

Proof: By the proof of Lemma 2.4 without change. QED

THEOREM 4.5. Let V obey MULTIVARIATE (**R**), and M = (R,<,0,1,+,-,•,V), where **R** = (R,<,0,1,+,-,•) is an ordered field in which every positive element has a square root. The following are equivalent.
i. V is a rich class (**R**).
ii. Every function $f:R^{n+m} \to R$ definable in M has a zero selector in V.
iii. Every function $f:R^n \to R^m$ definable in M is an element of V, and every function $f:R^{n+m} \to R$ definable in M has a zero selector in V.

Proof: By the proof of Theorem 2.5 without change. QED

We now fix M = (R,<,0,1,+,-,•,W) be an expansion of an ordered ring **R** = (R,<,0,1,+,-,•). We assume that W ⊆ V, where V is a special rich class (**R**). We wish to show that **R** is an ordered field in which every positive element has a square root.

LEMMA 4.6. (∃!x)(2x = 1). Write 1/2 for this unique x. 1/2 > 0. For all x, (1/2)x is the unique y such that 2y = x. (1/2)x = x(1/2). 0x = x0 = 0. -x = (-1)x = x(-1). xy = 0 → x = 0 ∨ y = 0. < is dense.

Proof: For uniqueness in the first claim, let 2x = 2y = 1. Then 2(x-y) = 0. Since **R** is an ordered ring, x-y = 0, x = y.

For existence in the first claim, let $f:R^2 \to R \in V$ be given by f(x,y) = 2y-x. By ZERO SELECTION (**R**), let $g,h:R \to R \in V$ be such that

$$(\exists y)(2y = x) \to 2g(x)-x = 0.$$
$$(\exists y)(2y = x) \leftrightarrow 2g(x) = x.$$
$$h(x) = 2g(x)-x.$$

If n ∈ Z is even then h(n) = 0. If n ∈ Z is odd then write n = 2m+1. If 2m+1 = 2x then 2(m-x) = 1, in which case we have established existence in the first claim. Hence we can assume that if n ∈ Z is odd, then h(n) ≠ 0.

Now apply MONOTONICITY (**R**) to h to obtain a finite set K such that h is monotone on the complementary open intervals of K. One of these complementary intervals, J, must contain all sufficiently large even integers. Therefore h must be constantly zero on J. But this is impossible since J also contains all sufficiently large odd integers.

If $1/2 = 0$ then $2(1/2) = 0 = 1$. But $0 \neq 1$. Also if $1/2 < 0$ then $1 = 1/2 + 1/2 < 0 + 0 = 0$, contradicting $0 < 1$. Hence $1/2 > 0$.

Clearly $2(1/2)x = (1/2)x + (1/2)x = (1/2 + 1/2)x = 2(1/2)x = x$. For uniqueness, let $2y = 2z = x$. Then $2y-2z = 2(y-z) = 0$. Since R is an ordered ring, $y-z = 0$, $y = z$.

$2(x(1/2)) = x(1/2) + x(1/2) = x(1/2 + 1/2) = x$. By uniqueness, $x(1/2) = (1/2)x$.

$$0x + x = 0x + 1x = 1x = x. \quad 0x = 0.$$
$$x0 + x = x0 + x1 = x1 = x. \quad x0 = 0.$$

$$(-1)x + x = (-1)x + 1x = 0x = 0.$$
$$(-1)x = -x.$$

$$x(-1) + x = x(-1) + x1 = x0 = 0.$$
$$x(-1) = -x.$$

Suppose $xy = 0$. Suppose $x,y \neq 0$.

case 1. $x > 0$, $y > 0$. Then $xy > 0$, which is a contradiction.

case 2. $x > 0$, $y < 0$. Then $x0 > xy$, $0 > xy$. This is a contradiction.

case 3. $x < 0$, $y > 0$. Then $-x > 0$, $(-x)y > 0$, $-xy > 0$. This is a contradiction.

case 4. $x < 0$, $y < 0$. Then $-x > 0$, $(-x)0 > (-x)y$, $0 > (-x)y = -(xy)$. This is a contradiction.

Clearly $<$ is dense since $x < y$ implies

$$x = (1/2)x + (1/2)x < (1/2)x + (1/2)y$$
$$= (1/2)(x+y) < (1/2)y + (1/2)y = (1/2 + 1/2)y = y.$$

QED

LEMMA 4.7. (□!x)(4x = 1). Write 1/4 for this unique x. 1/4 = (1/2)(1/2). For all x, (1/4)x is the unique y such that 4y = x. (1/4)x = x(1/4).

Proof: 4(1/2) = 1/2 + 1/2 + 1/2 + 1/2 = 1 + 1 = 2.

4((1/2)(1/2)) = (4(1/2))(1/2) =
2(1/2) = 1/2 + 1/2 = 1.

This establishes existence for the first claim. For uniqueness for the first claim,

*) 4x = 4y □ 4(x-y) = 0 □
x-y = 0 □ x = y.

Hence 1/4 = (1/2)(1/2), 4(1/4) = 1.

4((1/4)x) = (4(1/4))x = 1x = x.

Uniqueness of the last claim follows from *).

x(1/4) + x(1/4) + x(1/4) + x(1/4)
= x(4(1/4)) = x = 4(x(1/4)).

Hence x(1/4) = (1/4)x by uniqueness in the previous claim. QED

LEMMA 4.8. Let f □ V. Assume (□x > 0)(f(x) = 0), (□x > 0)(f(x) = 0 □ f(4x) = 0). Then (□x > 0)(f(x) = 0).

Proof: Let f be as given. Let $x_1$ < ... < $x_k$, k ≥ 1, such that on each of the complementary open intervals (- ,$x_1$),($x_1$,$x_2$), ...,($x_{k-1}$,$x_k$),($x_k$, ), f is either constant, strictly increasing, or strictly decreasing. Let f(□) = 0, □ > 0. Then f is 0 at A = {□ < 4□ < 16□ < ...}. Hence some complementary open interval contains infinitely many positive zeros of f.

Let J be the first complementary open interval which contains infinitely many positive zeros of f. Then f is constantly zero on J.

Let J = (□,□), □ □ R, □ < □, □ □ R □ { }. If □ > 0 then f is 0 at B = {□ > (1/4)□ > (1/16)□ > ...}. (Here we are iterating left multiplication by 1/4.) Hence some complementary open interval to the left of □ contains infinitely many positive zeros of f. This contradicts the choice of J. Hence □ □ 0.

We have thus shown that f is constantly zero on (0,□), □ > 0. Also note that if □ □ R, then f(□) = f((1/4)□) = 0.

If □ =  then f is constantly zero on (0, ), and we are done. So we let □ = $x_i$. We have that f is constantly zero on (0,$x_i$].

Suppose f is constantly zero on (0,$x_j$], where i □ j < k. Then f is constantly zero on (0,4$x_j$]. Hence f is constantly zero on ($x_j$,$x_{j+1}$). As above, f($x_{j+1}$) = 0, and so f is constantly zero on (0,$x_{j+1}$].

It is now evident by induction that f is constantly zero on every (0,$x_j$], i □ j □ k. We can argue as in the induction step to obtain that f is constantly zero on ($x_k$, ), and that f($x_k$) = 0. Hence f is constantly zero on (0, ). QED

LEMMA 4.9. (□x ≠ 0)(□!y)(xy = 1).

Proof: For existence, it suffices to prove that (□x > 0)(□y)(xy = 1). Let f:$R^2$ □ R be given by xy-1. By ZERO SELECTION (**R**), let g:R □ R □ V be such that for all x □ R,

(□y)(xy = 1) □ xg(x) = 1.

Let h:R □ R □ V be given by h(x) = xg(x)-1. Then for all x □ R,

(□y)(xy = 1) □ h(x) = 0.

Suppose h(x) = 0. Let xy = 1. Then

4x((1/4)y) = 4(x(1/4))y = 4((1/4)x)y = xy = 1.

Hence h(4x) = 0. Now suppose h(4x) = 0. Let 4xy = 1. Then

x(4y) = x(y+y+y+y) = xy+xy+xy+xy = 4xy = 1.

Hence h(x) = 0. Also h(1) = 1. So we can apply Lemma 4.8. Hence h is constantly 0 on (0, ). This establishes existence.

For uniqueness, let xy = 1, xz = 1. Then x(y-z) = 0. Obviously x ≠ 0. Hence y-z = 0, y = z. QED

LEMMA 4.10. $(\forall x,y)(xy = yx)$.

Proof: Fix $r \in R$. Let $f:R \to R \in V$ be given by $f(x) = rx-xr$.

$$f(4x) = r(4x)-(4x)r = r(x+x+x+x)-(x+x+x+x)r = rx+rx+rx+rx-xr-xr-xr-xr = rx-xr+rx-xr+rx-xr+rx-xr = 4(rx-xr).$$

Obviously $f(4x) = 0 \leftrightarrow f(x) = 0$, and $f(1) = 0$. By Lemma 4.8, $(\forall x > 0)(rx-xr = 0)$. Hence $(\forall x)(rx-xr = 0)$. QED

LEMMA 4.11. R is an ordered field in which every positive element has a square root.

Proof: By Lemmas 4.8, 4.10, R is an ordered field. It now suffices to show that every positive element has a square root. Let $f:R^2 \to R \in V$ be given by $f(x,y) = x-y^2$. By ZERO SELECTION (**R**), let $g:R \to R \in V$ be such that for all x,

$$(\exists y)(x = y^2) \to x = g(x)^2.$$

Let $h:R \to R \in V$ be given by $f(x) = x-g(x)^2$. Then for all $x \in R$,

$$h(x) = 0 \leftrightarrow (\exists y)(x = y^2).$$

Suppose $h(x) = 0$. Let $x = y^2$. Then $4x = 4y^2 = (2y)^2$. Hence $h(4x) = 0$.

Suppose $h(4x) = 0$. Let $4x = y^2$. Then $x = (y/2)^2$. Therefore $h(x) = 0$. Now apply Lemma 4.8. QED

THEOREM B. Let $M = (R,<,0,1,+,-,\bullet,W)$ be an expansion of an ordered ring **R** $= (R,<,0,1,+,-,\bullet)$. The following are equivalent.
i. M is o-minimal.
ii. W is a subset of some special rich class (**R**).
iii. There is a least rich class (**R**) containing W, and this rich class (**R**) is special.

Proof: We follow the proof of Theorem A, with the following modifications. We will again show that ii $\to$ i $\to$ iii.

Assume ii. Let $W \subseteq V$, where V is a special rich class (**R**). By Lemma 4.11, Theorem 4.5 applies.

In the proof of ii → i in Theorem A, we heavily used the fact that we were working over ¬. Specifically, we used that bounded infinite subsets of ¬ have limit points.

Here we argue as follows. Let A ⊆ R be definable in (R,<,0,1,+,-,•,W). Let $\chi_A$ be the characteristic function of A. Then $\chi_A \in$ V. By MONOTONICITY (**R**), let $x_1 < ... < x_k$, $k \geq 1$, be such that $\chi_A$ is constant, strictly increasing, or strictly decreasing on each of the complementary open intervals $(-\infty,x_1),(x_1,x_2),...,(x_{k-1},x_k),(x_k,\infty)$.

Obviously $\chi_A$ is constant on each one of the complementary open intervals. Hence $\chi_A$ is 1 exactly on the complementary open intervals on which it is constantly 1, and on the endpoints in R at which it is 1. Therefore A is a finite union of open intervals and points, where the finite endpoints and points all lie in R.

Assume i. Let V be the family of all functions definable in M. We argue and quote the Monotonicity Theorem, as in the proof of Theorem A. QED

## 5. CONCLUDING REMARKS.

The notion of special in section 2 (LIMIT (¬)) is too weak to be used in section 4, even if **R** is assumed to be a real closed field **F**. For a counterexample, let **F** = (F,<,0,1,+,•) be any non Archimedean real closed field, and consider the structure M = (F,<,0,1,+,•,h), where h is the characteristic function of the set of all elements of **F** whose absolute value is less than some algebraic element of **F**.

Obviously, M is not o-minimal. But by a result of [CD83], every M definable f:F → F is piecewise **F** definable (with finitely many pieces). Hence f is eventually **F** definable. So if f is bounded, then f has a limit.

In other words, M is not o-minimal, but h lies in a class V which is rich (**F**), and special in the sense of section 2. (Take V to be the M definable multidimensional functions).

Readers outside model theory can get some feeling for the difficulties involved in proving Wilkie's theorem from [Wi96], asserting that the field (ring) of real numbers augmented with the exponential function $e^x$ from ¬ to ¬

forms an o-minimal structure. By Theorem A, the purely mathematical formulation asserts that $e^x$ lies in a rich class (¬).

We remark that any o-minimal expansion of an ordered field must satisfy a tremendous array of crucially important properties. The derivation from o-minimality of many of these properties is highly nontrivial. See [PS86], [DM96], and [Dr98].

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*This research was partially supported by NSF Grant DMS 0245349.