TRANSFER PRINCIPLES IN SET THEORY

by

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TABLE OF CONTENTS

NOTE: Talks are based on Part A only

PART A. HIGHLIGHTS

INTRODUCTION

The results presented here establish unexpected formal relationships between the functions on N and the functions on On. (Here N is the set of all natural numbers and On is the class of all ordinal numbers). These results provide a reinterpretation of certain large cardinals axioms as extensions of known facts about functions on N to functions on On.

More specifically, the transfer principles assert that

any assertion of a certain logical form that holds of all functions on N holds of all functions on On.

These transfer principles are proved using certain large cardinal axioms.

In fact, we show that these transfer principles are equivalent to certain large cardinal axioms.

A1. TWO BASIC EXAMPLES OF TRANSFER PRINCIPLES

Let N = {0,1,...} and On be the class of all ordinals.

We begin by considering the sentences

$$*)\ (\forall f_1...f_p:N^k \to N)(\forall x_1...x_q)(\exists y_1...y_r) A(x_1...x_q,y_1...y_r)),$$

where A is a Boolean combination of inequalities between (possibly nested) terms involving the f's, x's, and y's. Constants for elements of N are allowed. The x's and y's range over N.

And consider the corresponding sentence

$$**)\ (\forall f_1...f_p:On^k \to On)(\forall x_1...x_q)(\exists y_1...y_r) (A(x_1...x_q,y_1...y_r)).$$

The x's and y's range over On. Note that **) is a sentence in class theory.

Now consider this transfer principle:

T_0) for all suitable k,p,q,r,A, * $\rightarrow$ **.

Unfortunately, it is easy to refute this transfer principle, even for k = 1 and no constants allowed.

We say that $f:N^k \rightarrow N$ is weakly regressive iff for all $x \in N^k$, $f(x) \leq \min(x)$. Here min(x) is the least coordinate of x.

Consider the following sentences.

$$*') (\forall wr\ f_1...f_p:N^k \rightarrow N)(\forall x_1...x_q)(\exists y_1...y_r) (A(x_1...x_q,y_1...y_r))$$

$$**') (\forall wr\ f_1...f_p:On^k \rightarrow On)(\forall x_1...x_q)(\exists y_1..y_r) (A(x_1...x_q,y_1,...y_r)).$$

Again, the x's and y's in the first form range over N, and the x's and y's in the second form range over On.

And the transfer principle:

T_1) for all suitable k,p,q,r,A, *' $\rightarrow$ **'.

Our first interesting transfer principle T_1 is likely to be equivalent to a large cardinal principle. We show that it outright proves one large cardinal principle and is derivable from a somewhat stronger large cardinal principle.

Here we use VB + AxC as the base theory.

We can even weaken this transfer principle to

T_1') for all suitable k,p,q,r,A, $*$ $\rightarrow$ **'

and obtain the same results.

We now introduce another modification of T_0 involving quantification over all functions on N.

Fix $E = \{2^n: n \in N\}$, and $E\^{} = \{2^\alpha:\alpha \in On\}$.

$$*\^{}\ (\forall f_1...f_p:N^k \rightarrow N)(\forall x_1...x_q)(\exists y_1...y_r \in E) (A(x_1...x_q,y_1...y_r))$$

$$**\^{}\ (\forall f_1...f_p:On^k \rightarrow On)(\forall x_1...x_q)(\exists y_1...y_r \in E\^{}) (A(x_1...x_q,y_1...y_r))$$

We were deliberately vague as to what kind of exponentiation is used in the definition of E^.

We can take it to be either ordinal exponentiation or cardinal exponentiation. The results are the same.

T_2) for all suitable k,p,q,r,A, *^ $\rightarrow$ **^.

This second transfer principle is equivalent to a class theoretic large cardinal axiom.

The same result applies even if we weaken the principle to

T_2') for all suitable k,p,q,r,A, *^ $\rightarrow$ **.

There is a decision procedure for the set of true sentences of the form *^ which provably works within RCA_0, and also has low computational complexity provably in EFA (exponential function arithmetic).

The results about T_2 and T_2' depend only on E being superpolynomial; i.e., that for all n,

$$E_{i+1} - E_i \geq i^n$$

for all sufficiently large i.

Let Y be the sentences produced by the transfer principle T_2'; i.e., Y is the set of all sentences ** such that *^ is true. (We could instead use the **^ such that *^ is true, but this is more natural).

A2. SOME FORMAL CONJECTURES

CONJECTURE 1. Let $\leq$ be the derivability relation between sentences in class *^ in RCA_0. Then $\leq$ is a quasi linear ordering. In fact, $\leq$ is a quasi well ordering. It has order type ϵ_0. It also has low computational complexity. The witness function for the proofs in RCA_0 is just beyond the $<\epsilon_0$-recursive functions. A < B if and only if RCA_0 + A proves Con(RCA_0 + B).

Let MAH be the formal system ZFC + {there exists an n-Mahlo cardinal}$_n$.

CONJECTURE 2. Let $\leq$ be the derivability relation between sentences in Y in VB + AxC. Then $\leq$ is a quasi linear ordering. In fact, $\leq$ is a quasi well ordering. It has order

type the provable ordinal of MAH. It also has low computational complexity. The witness function for the proofs in VBC is just beyond the provably recursive functions of MAH. A < B if and only if VB + AxC + A proves Con(VB + AxC + B).

CONJECTURE 3. These conjectures hold if we close these two class of sentences under all Boolean operations.

A3. SKETCH OF SOME PROOFS

Recall the sentences of the form

$$*)\ (\forall f_1 \ldots f_p : N^k \to N)(\forall x_1 \ldots x_q)(\exists y_1 \ldots y_r)(A(x_1 \ldots x_q, y_1 \ldots y_r)),$$

where A is a Boolean combination of inequalities between (possibly nested) terms involving f's, x's, and y's. Constants for elements of N are allowed. The x's and y's range over N.

And recall the corresponding sentences

$$**)\ (\forall f_1 \ldots f_p : On^k \to On)(\forall x_1 \ldots x_q)(\exists y_1 \ldots y_r)(A(x_1 \ldots x_q, y_1 \ldots y_r)).$$

The x's and y's range over On. Note that **) is a sentence in class theory.

Recall the transfer principle

$$T_0)\ \text{for all suitable } k,p,q,r,A,\ * \to **.$$

THEOREM 1. The transfer principle T_0 is refutable in VB + AxC. This refutation can be done for k = 1.

Proof: Note that

$$(\forall f : N \to N)((\forall x)(f(x) = x+1) \to (\forall z)(z \neq 0 \to (\exists w)(f(w) = z))).$$

We can put this in the form

$$(\forall f : N \to N)((\forall x)(\forall y)(x < f(x) \wedge (y \leq x \vee f(x) \leq y)) \to (\forall z)(z \neq 0 \to (\exists w)(f(w) = z))).$$

Now this statement is true for N. But this statement is false for On.

By predicate calculus manipulations, we can put it in the desired form.

To eliminate the constant 0, we use

$$(\forall f:N \to N)((\forall x)(\forall y)(x < f(x) \wedge (y \leq x \vee f(x) \leq y)) \to (\forall z)(\forall u)(u < z \to (\exists w)(f(w) = z))).$$

For $k \geq 0$, a cardinal is k-ineffable iff it is regular and every partition of the unordered k+1-tuples into two pieces has a stationary homogeneous set. (This is not the original definition, but is known to be equivalent).

A cardinal is called 0-Mahlo iff it is regular. A cardinal is called k+1-Mahlo if and only if every stationary subset has an element which is a k-Mahlo cardinal.

Recall the following sentences forms:

$$*\wedge)\ (\forall wr\ f_1...f_p:N^k \to N)(\forall x_1...x_q)(\exists y_1...y_r)(A(x_1...x_p,y_1...y_q))$$

$$**\wedge)\ (\forall wr\ f_1...f_p:On^k \to On)(\forall x_1...x_p)(\exists y_1...y_q)(A(x_1...x_p,y_1...y_q)).$$

Here the x's and y's in the first form range over N, and the x's and y's in the second form range over On.

And recall the transfer principle:

T_1) for all suitable k,p,q,r,A, $*' \to *''$.

THEOREM 2. (VBC). If for all $k \geq 0$, there exists arbitrarily large k-subtle cardinals, then T_1 holds.

We need the following combinatorial theorem:

THEOREM 3. (ZFC). Let $k \geq 1$ and l be a (k-1)-ineffable cardinal. Let $f_1,...,f_p:\lambda^k \to \lambda$ be weakly regressive, and $B \subseteq \lambda$ be finite. Then there exists $E \subseteq \lambda$ of order type ω such that each $f_i[E^k] \cup B \subseteq E$.

Proof of Theorem 2: We prove the contrapositive. Let

$$(\exists x_1...x_q)(\forall y_1...y_r)(A(x_1...x_q,y_1...y_r))$$

hold in $(On,<,f_1,...,f_p)$, where $f_1,...,f_p:On^k \to On$ are fixed weakly regressive functions. Then

$$(\exists x_1...x_q)(\forall y_1...y_r)(A(x_1...x_q,y_1...y_r))$$

holds in $(\lambda,<,f_1,...,f_p)$.

Fix $x_1,...,x_q < \lambda$ such that

$$(\forall y_1...y_r)(A(x_1...x_q,y_1...y_r)).$$

holds in $(\lambda,<,f_1,…,f_p)$.

Let B consist of $x_1,...,x_q$ together with all elements of N that are $\leq$ some constant appearing in A.

According to Theorem 3, we can choose a set $E \subseteq \lambda$ of order type ω such that each $f_i[E^k] \cup B \subseteq E$.

Now the relational structure $(E,<,f_1,...,f_p)$ is isomorphic to a unique relational structure $(\omega,<,g_1,...,g_p)$, and the isomorphism h is unique. Also h is the identity at all constants in A.

From this we conclude that

$$(\forall y_1...y_r)(A(hx_1...hx_q,y_1...y_r))$$

holds in $(\omega,<,g_1,...,g_p)$.

Hence

$$(\exists x_1...x_q)(\forall y_1...y_r)(A(x_1...x_q,y_1...y_r))$$

holds in $(\omega,<,g_1,...,g_p)$ as required.

Proof of Theorem 3: Without loss of generality, we can assume that p = 1. To see this, just throw in a suitably large number of elements of B as constants.

Let $f:\lambda^k \to \lambda$ be weakly regressive and λ be a k-ineffable cardinal and let B be any finite subset of λ. We choose $E = \{\alpha_1,\alpha_2,...\}_<$ such that for all $x,y \in E^k$ of the same order type, if $f(x) < \min(x)$ then $f(x) = f(y) < \alpha_1$.

Let T be the set of all terms involving f and elements of E and elements of B.

The depth of a term in T is defined by recursion as follows.

The depth of an element of $E \cup B$ is 1.
The depth of $f(s_1,...,s_k)$ is 1 + the maximum of the depths of $s_1,...,s_k$.

Let T′ be the set of all terms involving f and elements of $\{\alpha_1,...,\alpha_k\}$ and elements of B. We claim that

#) for all $t \in T$, the value of t is either in E or is a value of a term in T′ that is $< \alpha_1$.

This is proved by induction on the depth of the term t. The basis case is trivial.

Suppose it is true for terms of smaller depth than t. Let $t = f(s_1,...,s_k)$.

By the induction hypothesis, each s_i either has value in E or is the value of a term in T′ that is $< \alpha_1$.

By the regressivity of f, the value of t is at most the values of the s_i. If all of the s_i have values in E then the indiscernibility tells us that the value of t is either in E or is $< \alpha_1$.

And it also tells us that in the latter case, we can replace all of the s_i with elements of $\{\alpha_1,..., \alpha_k\}$. So t has the same value as a term in T′.

On the other hand, suppose the value of some s_i is $< \alpha_1$. Then by the regressivity of f, the value of t is smaller than α_1.

Also by the indiscernibility, we can move all of the s_i that lie in E to elements of $\{\alpha_1,..., \alpha_k\}$.

By the induction hypothesis, the remaining elements of s_i have the same values as terms in T′. Hence t has the same value as a term in T′. This establishes claim #.

In order to establish Theorem 3, it suffices to prove that there are at most finitely many values of terms in T′.

Now let M be the set of all terms in T′ that are minimal in the following sense. The value of t is different from the values of all terms in T′ of smaller depth.

Clearly, in order to establish Theorem 3 we only have to show that there are at most finitely many values of terms in M.

We now define a tree S of finite sequences of ordinals as follows.

$(\beta_1,...,\beta_n)$ is in S if and only if

i) $n \geq 1$;
ii) $\beta_1 \in \{\alpha_1,...,\alpha_k\} \cup B$;
iii) $\beta_1 > ... > \beta_n$;
iv) for $1 < i < n-1$, β_{i+1} is the value of a term in M of depth i+1 of the form $f(s_1,...,s_k)$, where some s_j has value β_i.

We claim that the value of every term in M appears in S.

We actually establish that for each $n \geq 1$, the values of terms in M of depth n all appear in the n-th level of S. We prove this by induction on depth.

For the basis case, the values of terms in M of depth 1, which is just $\{\alpha_1,...,\alpha_k\} \cup B$, all appear in the first level of S.

Suppose that all values of terms in M of depth n all appear in the n-th level of S.

Let $t = f(s_1,...,s_k)$ be a term in M of depth n+1. Choose $s_1',...,s_k' \in M$ with the same values as $s_1,...,s_k$. Then $f(s_1',...,s_k')$ has the same value as t, and so has depth n+1.

Let s_i' have depth n. Then by induction hypothesis, s_i' appears in the n-th level in S. Also since $t \in M$, the value of t is not the same as the value of s_i', and hence by regressivity of f, value of t is < value of s_i'. Therefore, t appears in the n+1-st level in S.

Obviously S is finitely branching, since there are at most finitely many terms in T of any given depth.

By the construction of S, clearly S cannot have an infinite path (this would create an infinite descending sequence of ordinals).

Hence S is a finite tree, and so we have established that there are at most finitely many values of terms in M, as required.

The kind of indiscernibles used here are those that you get from ineffable cardinals. This was needed in order to establish claim #.

We know that Theorem 3 is roughly best possible:

THEOREM 4. (ZFC). Let $k \geq 1$. The combinatorial property in Theorem 3 implies that λ is a (k-1)-subtle cardinal.

This suggests that Theorem 2 is also roughly best possible. However, we haven't been able to carry this out.

THEOREM 5. (VB + AxC). Suppose T_1 holds. Then for all $k \geq 0$, there exists arbitrarily large k-Mahlo cardinals.

Proof: Consider the following statement:

for every wr $f:N^k \to N$ there exists an r element set E such that the values of f on $E^{k<}$ depend only on their first term.

This is a true statement. It is a variant of Paris/ Harrington, and is essentially the same as the statement studied in Kanamori/McAloon.

Now adjust this as follows:

for every wr $f:N^k \to N$ there exists an r element set E above any given number such that the values of f on $E^{k<}$ depend only on their first term.

Under T_1, this transfers to:

for every wr $f:On^k \to On$ there exists an r element set E above any given ordinal such that the values of f on $E^{k<}$ depend only on their first term.

Now suppose that for some k, there is only a bounded number of k-Mahlo cardinals. This amounts to an explicit failure of On being (k+1)-Mahlo.

By using Schmerl like combinatorics (Ph.D. thesis with Silver), if λ is a cardinal such that

> **for every wr $f:\lambda^{k+3} \to \lambda$ there exists an r element set E above any given ordinal < λ such that the values of f on $E^{k<}$ depend only on their first term,**

then λ is a (k+1)-Mahlo cardinal. Now this same argument can be applied to On instead of to λ.

Recall the following sentence forms:

$$*\wedge)\ (\forall f_1\ldots f_p:N^k \to N)(\forall x_1\ldots x_p)(\exists y_1\ldots y_q) \in E)\ (A(x_1\ldots x_p,y_1\ldots y_q))$$

$$**\wedge)\ (\forall f_1\ldots f_p:On^k \to On)(\forall x_1\ldots x_p)(\exists y_1\ldots y_q) \in E\wedge)\ (A(x_1\ldots x_p,y_1\ldots y_q)).$$

Here E is the ordinal powers of 2, and the x's and y's in the first form range over N, and the x's and y's in the second form range over On. One could also use the cardinal powers of 2. And recall the transfer principle:

$$T_2)\ \text{for all suitable } k,p,q,r,A,\ *' \to *''.$$

THEOREM 6. (VBC). If for all k $\geq$ 0, there exists arbitrarily large k-Mahlo cardinals, then T_2 holds.

To prove this Theorem, we need the following result in combinatorial set theory:

THEOREM 7. Let $f:\lambda^k \to \lambda$, $B \subseteq \lambda$ be finite, and $U \subseteq \lambda$ be unbounded, where λ is k-Mahlo. There exists infinite ordinals $\max(B) < \beta_1 < \beta_2 < \ldots < \lambda$ from U such that for sufficiently large i, $f[B \cup \{\beta_1,\beta_2,\ldots\}]$ has $\leq (k+i)^i$ elements below β_i; furthermore this image is included in the limit of the β_i's.

Proof of Theorem 7: By Schmerl combinatorics. He proves the existence of an infinite (in fact unbounded) set $E \subseteq \lambda$ such that the truth values of first order properties over $(\lambda,<,f)$ at k-tuples of the β's, with parameters lower than the first β, depends only on the order type, the first term, and the parameters. From this, it is easy to see that every term $f(x_1,\ldots,x_k)$, where the x's are either elements of B or among the β's, must be equaled to a term where the β's

appearing that are higher than $f(x_1,...,x_k)$ have been moved down to occupy consecutive positions above the first β that is $= f(x_1,...,x_k)$. A simple counting argument completes the proof.

Proof of Theorem 6: We prove the contrapositive. Fix $f:On^k \to On$ and ordinals $x_1,...,x_q$. Assume that

$$(\forall y_1,...,y_q)(A(x_1...x_p,y_1...y_q)).$$

Let $I = \{\beta_1 < \beta_2 ...\}$ be a set of ordinals of type ω according to Theorem 7, where B is $\{x_1,...,x_q\}$ together with the constants appearing in A together with $[0,i]$ for a sufficiently large chosen i. We let β be the limit of the β_i's.

According to Theorem 7, we can choose a set $S \subseteq \beta$ and finite j such that

i) $I \cup B \cup f[I \cup B]^k \cup [0,2^j] \subseteq S$;
ii) $|S \cap (I_t,I_{t+1}]| = 2^{j+t}$
iii) $|S \cap (2^j,I_1]| = 2^j$.

Now let $g:S^k \to S$ be the extension of f using the default value β_1.

Now g is order isomorphic to a function $g':N^k \to N$, via the unique order isomorphism h from S onto N.

Note that the inverse image of every power of 2 is in $I \cup [0,j]$, and hence in $I \cup B$. Also h is the identity at all constants in A.

Recall that

$$(\forall y_1...y_q \in E)(A(x_1...x_p,y_1...y_q))$$

where the quantifiers range over λ. Hence

$$(\forall y_1...y_q \in E)(A(x_1...x_p,y_1...y_q))$$

holds where the quantifiers range over N.

THEOREM 8. (VBC). If T_1 holds then for all $k \geq 0$, there is a stationary class of k-Mahlo cardinals.

Proof: Consider the following statement:

for every $f:N^k \to N$ there exists an r element subset of E such that values of f on $E^{k<}$ depend only on their first term.

This is a variant of Paris/Harrington, essentially studied by Kanamori/McAloon.

Now adjust this as follows:

for every $f:N^k \to N$ there exists an r element subset of E above any given number such that values of f on $E^{k<}$ depend only on the first term.

Under T_1, this transfers to:

for every $f:On^k \to On$ there exists an r element subset of E above any given ordinal such that values of f on $E^{k<}$ depend only on their first term.

Using Schmerl combinatorics, one obtains that the class of appropriately Mahlo cardinals is stationary in On.

A4. RAMSEY CARDINALS

We now present transfer principles corresponding to Ramsey cardinals. κ is Ramsey iff for all partitions of the finite subsets of κ into two parts, there exists a set of power κ which is simultaneously homogenous in each exponent.

There is a weakening of Ramsey cardinals which is relevant here, and is also incompatible with the axiom of constructibility.

We say that κ is almost Ramsey iff for every partition of the finite subsets of κ into two parts, there exists sets of every cardinality < κ which are simultaneously homogenous in each exponent.

It can be shown that for every Ramsey cardinal κ, the set of almost Ramsey cardinals < κ is stationary in κ. Almost Ramsey cardinals are incompatible with the axiom of constructibility.

We now introduce the sentences $U(N,wr,k,A_1,A_2,...)$ written

$$(\forall wr\ f:N^k \to N)(\exists \text{ unbounded } Y)(A_1 \wedge A_2\ ...),$$

where the A's are the result of placing zero or more universal quantifiers ranging over Y in front of a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants for elements of N are allowed.

Here "unbounded Y" means that Y is an unbounded subset of N.

Now consider the corresponding sentences $U(N,wr,k,A_1,A_2,\ldots)$ written

$$(\forall wr\ f:On^k \to On)(\exists \text{ unbounded } Y)(A_1 \wedge A_2 \ldots).$$

Here "unbounded Y" means that Y is an unbounded subclass of On.

Note that infinite conjunctions of qf formulas universally quantified into Y are allowed instead of a single such universally quantified formula.

THEOREM. The following are provably equivalent in VB + AxC.
i) $U(N,wr) \Rightarrow U(On,wr)$;
ii) there are arbitrarily large almost Ramsey cardinals.

To get full Ramseyness, we replace "unbounded" by "stationary." Thus we write:
$S(N,wr,k,A_1,A_2,\ldots)$,
$S(On,wr,k,A_1,A_2,\ldots)$, and $S(N,wr) \Rightarrow S(On,wr)$.

THEOREM. (VB + AxC). If the transfer principle $S(N,wr) \Rightarrow S(On,wr)$ holds then there are is a stationary class of Ramsey cardinals. If there are arbitrarily large ineffably Ramsey cardinals then $S(N,wr) \Rightarrow S(On,wr)$.

A5. TOWARDS A NEW VIEW OF SET THEORY

We begin with a discussion of some current views about set theory, and their drawbacks.

One focal point on which people have widely differing views is the following.

Is the concept of set sufficiently clear to fix the truth value of basic set theoretic assertions such as the continuum hypothesis?

Bear in mind that we are talking about the truth value being determined independent-ly of whether or not we know what the truth value is.

Now, at one extreme, there is the view that the concept of set is sufficiently clear to fix the truth value of every first order assertion about sets. Under this view, the inability to the determine the truth value of, say, the continuum hypothesis, is to be expected when mathematicians try to work on hard problems. After all, it took a long time to determine the truth value of Fermat's last theorem, and we still don't know the truth value of the Riemann hypothesis. Under this view, there is no essential difference between the continuum hypothesis, Fermat's last theorem, and the Riemann hypothesis. Admittedly, some particular set theoretic axioms don't determine the truth value of the continuum hypothesis under the axioms of rules of predicate calculus, but so what? There is no essential difference between finding additional set theoretic axioms and finding new proofs. The particular axioms, say, of ZFC, are of course evident, but are just some ad hoc stopping point - a mere drop in the bucket of what we can see is true. For that matter, an instance of induction with a clever induction hypothesis is also evident. And they are evident in the same way. The distinction is bogus, having to do with the history of mathematical logic and what mathematical logicians find interesting. This process of seeing the truth is essentially the same when it goes beyond ZFC as it is when it is within ZFC. This kind of thinking is done every day in every mathematics department, or for that matter, in theoretical science generally.

And at the other extreme, there is the view that the concept of set is a mirage - there are only formalisms that people find interesting or useful for various purposes. Under this view, when, by accident, somebody discovers an inconsistency which renders the formalism useless (although even this can be argued), people adjust the system to get rid of the discovered inconsistency - until the next inconsistency arrives. Under this view, the next inconsistency, if any, cannot be predicted, and is largely a function of the amount of effort people put into finding one. This is the attitude under this view to Russell's paradox, and also to the more modern and technical inconsistency involving Reinhardt's elementary embed-ding axiom. And under this view, the independence of the continuum hypothesis from ZFC completely solves the problem - until one changes formal systems. Once the formal system is changed, the problem of the continuum hypothesis is thereby changed.

Here are some problems with these views, which we will refer to as Platonism and formalism.

The major problem for Platonism in set theory has been the history of the continuum hypothesis. The history of large cardinals has been mixed for Platonism.

At the present time, there is no promising proposal for settling the continuum hypothesis consistent with spirit of realism. All large cardinal axioms have been shown to be insufficient for deciding the continuum hypothesis. The only proposals for answering the continuum hypothesis consistent with the spirit of realism are as follows:

i) postulating that the set theoretic universe is generated by an inductive process from data associated with a large cardinal. This goes under the name “the set theoretic universe is an inner model of a large cardinal.” This implies the continuum hypothesis.

ii) postulating the existence of a nontrivial countably additive measure on the reals. This refutes the continuum hypothesis.

iii) postulating that lots of generic sets exist. This goes under the name of Martin’s axiom, or Martin’s maximum. This refutes the continuum hypothesis.

All three of these proposals have serious drawbacks in the context of Platonism. This context demands that any additional postulates be self evident.

The drawback with i) is that it is a mixture of a limitation on the set theoretic universe and a large cardinal axiom. They are normally viewed as inconsistent in spirit, and so how could their combination be evident? The ulti-mate axiom of limitation is the axiom of constructibility, which asserts that all sets are built up from nothing by an inductive process. Now this is well known to be formally incompatible with large cardinals such as measurable cardinals (Dana Scott, 1960s). So why is i) evident at the same time that the axiom of constructibility is not evident? From the Platonist viewpoint, where there is only one objective reality of the set theoretic universe, this appears to be incoherent. On top of all this, it is very difficult to defend the idea that the existence of large cardinals such as measurable cardinals are self evident. Certainly they seem completely different in this respect than most if not all of the axioms of ZFC, which at least have good stories and pictures.

The drawback with ii) and iii) is that both of them, especially iii), are too technical to be regarded as self evident. But there is perhaps an even more telling objection. This comes from the known fact that they are inconsistent with each other. So how can they both be self evident? And there doesn't seem to be any better reason why one of them is more self evident than the other; i.e., it is not clear how one argues for the self evidence of one without being able to modify the argument and argue for the self evidence of the other with roughly equal force.

Now i) is based on some nice, canonical models of large cardinals with pleasing properties, including the continuum hypothesis (pleasing or not). Woodin has succeeded is constructing some nice, canonical models of large cardinals with pleasing properties, including the negation of the continuum hypothesis. But no one has put forth an argument of self evidence in connection with his construction. In any case, such a proposed new axiom would likely be too technical to pass for a self evident principle.

The die hard Platonist can still maintain that the present impasse regarding the continuum hypothesis is not so worrisome. That the continuum hypothesis, despite its simplicity and the fact that it is the first problem left open in the field (except the axiom of choice, before it become accepted as an axiom), is a very very difficult problem. But then the Platonist should at least indicate what might constitute evidence that he/she is wrong. After all, in all fairness, if the Platonist is wrong, then exactly the sort of thing that has been going on with the continuum hypothesis would be very natural and expected.

Now there is a closely related view which should be thought of as less radical than Platonism. This is realism. Realism has the view in common with Platonism that the reality of the set theoretic universe is, in some sense, on a par with physical reality; i.e., there is an external reality that guides us. But it falls short of accepting the idea of unique truth values to set theoretic statements.

Realism instead takes its cue from physical theories, which people long ago had to stop thinking are self evident (relativity, quantum mechanics?? - one has to learn how to stop thinking that their falsity is self evident!!). The principal reason that physical theories get accepted, or perhaps get accepted as "true," is because of their consequences. (There also is the important idea of "simplest possible coherent explanation.") There is a whole culture of

confirmation. Its part of the requirement for the Nobel Prize.

The set theoretic realists insist that this process has already lead to the acceptance of the current axioms of set theory as well as of some large cardinals, because of the variety of consequences. But unfortunately for the realists, nothing of this sort has happened for the continuum hypothesis or its negation, or for that matter for any axiom that might settle the continuum hypothesis. Recall, as said earlier, that large cardinal axioms are known to neither prove or refute the continuum hypothesis.

I am not really convinced by the analogy drawn by the set theoretic realists between

i) the experimental confirmations of, say, general relativity and quantum mechanics;
ii) the “confirmations” of, say, large cardinal axioms through their consequences for the projective hierarchy of sets of real numbers.

First of all, there is the obvious difference is that the experiments are generally regarded as unassailable. One cannot argue with them. Also, often there is quantitative information, so in the spectrum of all possible theories, almost none would get the right numerical prediction - this makes the theory confirmed with such numerical data have a special status. Nothing of this kind happens in ii).

In fact, the argument that the consequences for the projective hierarchy established from large cardinals is any kind of confirmation is itself not entirely convincing. The realists like to use words like “pleasing” for these consequences. They certainly don’t use words like “evident.” But in i), the idea is that the experiment is supposed to be designed so that the result of the experiment is evident.

It’s even more problematic than that. There is another hypothesis, the axiom of constructibility, which is incompatible with the relevant large cardinal axioms, which also gives a rather complete picture of the elementary properties of the projective hierarchy of sets of real numbers. And this picture is completely at odds with the picture obtained from the large cardinal axioms. Which is more pleasing?

For instance, the picture of the projective sets obtained from the axiom of constructibility certainly looks duller than the one obtained from large cardinals. It is also

easier to prove. But since when is dullness such a major factor?

The realists' best case is with the regularity conditions. The large cardinal axioms prove that all projective sets are Lebesgue measurable, whereas the axiom of constructibility proves that not all projective sets are Lebesgue measurable, and gives explicit counter-examples.

We again make the objection that experimental confirmation in physical theories cannot be attacked assuming the experiment has been designed properly, yet here we simply assert that Lebesgue measurability is desirable.

In fact, it is hard to give any good reason why Lebesgue measurability of projective sets is more likely to be true than not.

Furthermore, in the case of physical theories, we really do want to make numerical predictions. This is usually a major reason for formulating physical theories. In the case of Lebesgue measurability of projective sets, mathematicians have a different attitude. Lebesgue measurability is rarely an issue. It most commonly appears as an hypothesis on theorems. That is, one normally assumes Lebesgue measurability, and so the Lebesgue measurability of sets does not occur as an issue.

There is another objection to using the measurability of projective sets that does not apply to the confirmation of physical theories. Specifically, in virtually all of mathematics involving projective sets, all of the projective sets are in fact Borel sets or analytic sets, in which case Lebesgue measurability is outright provable with no additional axioms needed whatsoever. Lebesgue measurability is problematic only if the sets are higher up in the projective hierarchy, and more remote than normal.

Now on to the criticism of formalism. The main criticism is that it doesn't account for why we have settled on certain axiomatizations of set theory. Or why we are so successful in working within certain formal systems of set theory.

In defense, the formalist might say that we fiddle around and experiment with various formal systems before we pick certain ones. But a problem with this view is that these formalisms behave well and continue to be natural and easy to work in long after they were initially chosen. The Platonist and realist say that it is because of what these

formalisms say about the world, and not because of any of their syntactic properties.

Of course, there is a more extreme kind of formalism which says that the formalist is under no onus to explain why we have settled on certain axiomatizations of set theory. Such a formalist can simply say that such formalisms and the intellectual activity associated with it is not only meaningless but pointless. That perhaps only certain very very weak formalisms are either meaningful or fruitful, connected with the most basic levels of arithmetic and finite set theoretic reasoning.

This is not the place to start arguing that set theoretic reasoning is both meaningful and fruitful. I want to focus attention now on an emerging view that doesn't appear to be subject to the drawbacks raised above in connection with Platonism, realism, and formalism. This view is only viable in light of the formal discoveries reported on here. Further formal discoveries will add immeasurably to the development of this view.

Generally speaking, the view is that set theory can be taken to be a purely formal extension of certain known facts in finite set theory by simply formally adding the axiom of infinity. One selects a convenient collection of known facts so that the resulting formal system is consistent. What we have is really a transfer principle from the finite to the transfinite. This is because the resulting system is supposed to be about the transfinite since it contains the axiom of infinity. Since we are not changing the assertion in the integers, we regard this as a transfer.

Now this view cannot be fully supported by the formal discoveries that have been made yet. And there is a whole list of plausible conjectures that need to be verified in order for this view to be fully supportable.

Imagine a fully documented view like this. An anticipated discovery is that the resulting formal systems settle such well known set theoretic hypothesis as the continuum hypothesis (in the negative), and the existence of large cardinals (in the positive). Furthermore, that no such resulting formal system will settle such hypotheses in the opposite direction.

This solves the main problem with Platonism and realism - their apparent inability to deal with the continuum hypothesis in any viable way, and also their failure to do some really convincing with large cardinal axioms.

Now you might object that there is nothing particularly evident about such transfer principles. So we have the same problem that the Platonists did - there is nothing evident about the continuum hypothesis or its negation.

Now here comes the formal aspect of this view. We deny ever saying or needing to say that there is anything evident about such transfer principles.

Then what is the criteria by which we select such transfer principles under this view, if not their evidence?

Assuming the appropriate formal conjectures, we say that there is exactly one ground for selection: consistency.

In fact, the compatibility conjecture states that any two transfer principles that are individually formally consistent, are formally consistent with each other. This assumes an appropriate formalization capturing the notion of a natural transfer principle; this seems reasonable in light of the fact that the existing studied transfer principles are such basic low complexity.

Under the compatibility conjecture, there is a single powerful axiom of set theory. It asserts that any consistent transfer principle is true. According to the conjectures, it would imply the existence of large cardinals as well as the negation of the continuum hypothesis, and settle allied questions.

PART B. FULL LIST OF CLAIMS (Based on an abstract of May, 1996)

1.TRANSFER PRINCIPLES FROM N TO ON

Let N be the set of all nonnegative integers, and On be the class of all ordinals.

In analyzing transfer principles from N to On, we will use the von Neumann Bernays class theory VBC with the axiom of choice as the base theory (in the form that there is a class function that chooses an element from every nonempty set).

We obtain the provable equivalence in VBC of various transfer principles with the following three class theoretic large cardinal axioms (all terminology explained at the appropriate place):

- **a)** for all $n \in N$, the class of n-Mahlo cardinals is stationary in On;
- **b)** On is weakly compact;
- **c)** On is ineffable;
- **d)** On is Ramsey;
- **e)** On is ineffably Ramsey.

These statements are not provably equivalent to purely set theoretic statements. At the appropriate places we indicate the appropriate strong set theoretic consequences of these statements.

The main point of section 3 is to avoid the use of class theory. The large cardinal axioms are stated set theoretically in section 3.

A. MAHLO CARDINALS

We begin by considering the formulas

$$*)\ (\forall f:N^k \rightarrow N)(\exists x_1,\ldots,x_p)(A(f,x_1,\ldots,x_p)),$$

where A is a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and additional free variables for elements of N are allowed. The existential quantifiers range over N.

And consider the corresponding formula

$$**)\ (\forall f:On^k \rightarrow On)(\exists x_1,\ldots,x_p)(A(f,x_1,\ldots,x_p)).$$

The existential quantifiers and free variables range over On. Note that this is a formula in class theory.

We say that * or ** is valid if and only if it is true for all appropriate assignments to the free variables.

Now consider the transfer principle:

for all k,p,A, if * is valid then ** is valid.

Unfortunately, it is easy to refute this transfer principle.

But consider the following modification. We say that $f:N^k \rightarrow N$ is weakly regressive (wr) if and only if for all $x \in N^k$,

$$f(x) \leq \min(x).$$

Here min(x) is simply the minimum coordinate of x.

We make the same definition for k-ary functions on any linearly ordered set - in particular, for On.

The odd sounding terminology "weakly regressive" is now standard, and has legitimate origins in combinatorial set theory.

Now let M(N,wr,k,p,A) be

$$(\forall \text{wr } f:N^k \to N)(\exists x_1,\ldots,x_p)(A(f,x_1,\ldots,x_p)),$$

where A is a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and additional free variables for elements of N are allowed. The existential quantifiers range over N. The M indicates that this form is connected with Mahlo cardinals, the N indicates the use of the natural numbers N in the assertion, and wr indicates the restriction to weakly regressive functions.

And let M(On,wr,k,p,A) be the corresponding formula

$$(\forall \text{wr} f:On^k \to On)(\exists x_1,\ldots,x_p)(A(f,x_1,\ldots,x_p)).$$

Here the existential quantifiers and free variables range over On.

We now consider the transfer principle M(N,wr) $\Rightarrow$ M(On,wr), which asserts that

for all k,p,A, if M(N,wr,k,p,A) is valid then M(On,wr,k,p,A) is valid.

We say that a cardinal κ is 0-Mahlo if and only if it is strongly inaccessible. We say that a cardinal κ is (n+1)-Mahlo if and only if $\{\alpha:\alpha < \kappa$ and α is an n-Mahlo cardinal$\}$ is stationary in κ. See [1] for more information about Mahlo cardinals.

We say that $f:S_n(\kappa) \to S(\kappa)$ is regressive if and only if for all $A \in S_n(\lambda)$, $f(A) \subseteq \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \kappa$ and for all $C,D \in S_n(E)$, we have

$$f(C) \cap \min(C \cup D) = f(D) \cap \min(C \cup D).$$

Let n > 0. κ is n-subtle if and only if

i) κ is a cardinal;
ii) For all closed unbounded $C \subseteq \lambda$ and regressive $f:S_n(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \in S_{n+1}(C)$.

We say that κ is 0-subtle if and only if κ is an uncountable regular cardinal.

THEOREM 1A.1. Each of the following implies the next in VBC.
i) for all $n \in N$, there is a proper class of n-subtle cardinals;
ii) $M(N,wr) \Rightarrow M(On,wr)$;
iii) for all $n \in N$, there is a proper class of n-Mahlo cardinals.

B. WEAKLY COMPACT CARDINALS

In this section we replace the M classes given in section 1A by the W classes, and form the analogous transfer principles.

Let W(N,wr,k,p,A) be the formula

$$(\forall wr\ f{:}N^k \to N)(\exists \text{ unbounded } Y)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)),$$

where A is a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and additional free variables for elements of N are allowed. Here "unbounded Y" means Y is an unbounded subset of N. The W indicates that this form is connected with weakly compact cardinals, and the N indicates the use of the natural numbers N in the assertion.

Now consider the corresponding assertions W(On,wr,k,p,A) written

$$(\forall wr\ f{:}On^k \to On)(\exists \text{ unbounded } Y)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Here "unbounded Y" means that Y is an unbounded subclass of On.

A weakly compact cardinal is an uncountable cardinal such that any partition of the two element subsets into two pieces has an unbounded homogenous set. This is not the original definition of weak compactness, but rather is a well known equivalent. See [1] for a discussion of weakly compact cardinals.

Now it turns out that the relevant large cardinal axiom for us will be "On is weakly compact." This has the obvious meaning in class theory (VBC).

It is desirable to have a reasonably strong purely set theoretic consequence of "On is weakly compact." Straightforward arguments show that it implies that the class of cardinals k that are k-Mahlo is unbounded (in fact, stationary) in.

There is a stronger notion of Mahloness called greatly Mahlo. VBC proves that if On is weakly compact, then the class of greatly Mahlo cardinals is stationary in On. See [2].

THEOREM 1B.1. Each of the following implies the next in VBC.

i) for all $n \in N$, there is a proper class of n-subtle cardinals, and On is weakly compact;

ii) W(N,wr) $\Rightarrow$ W(On,wr);

iii) On is weakly compact;

iv) the class of greatly Mahlo cardinals is stationary in On.

C. INEFFABLE CARDINALS

In this section we use the W classes for N, but replace the W classes for On by the I classes, and form the analogous transfer principles.

Let I(On,wr,k,p,A) be the formula

$$(\forall \text{wr } f{:}On^k \to On)(\exists \text{ stationary } Y)$$
$$(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

We follow the same conventions as in section 1B. Here "stationary Y" means that Y is a stationary subclass of On.

For $n \in N$, an infinite cardinal is n-ineffable if and only if it is regular and any partition of the (k+1)-element subsets into two pieces has a stationary homogenous set. This is not Baumgartner's original definition, but is a nice equivalent due to him (see [4]). There is a discussion of Baumgartner's work in [5].

The relevant large cardinal axiom for us will be "On is n-ineffable for all $n \in N$." This has the obvious meaning in class theory (VBC). Straightforward arguments show that "On is n-ineffable for all $n \in N$" implies that "for all n in N, the class of all n-ineffable cardinals is unbounded in On (in fact, is stationary in On)."

THEOREM 1C.1. The following are provably equivalent in VBC.

i) W(N,wr) $\Rightarrow$ I(On,wr);

ii) On is n-ineffable for all $n \in N$.

D. RAMSEY CARDINALS

We now present transfer principles from N to On which correspond to Ramsey cardinals. We say that a cardinal κ is Ramsey if and only if for every partition of the finite subsets of k into two parts, there exists a set of power κ which is simultaneously homogenous in each exponent. The existence of a Ramsey cardinal is well known to be incompatible with the axiom of constructibility. Ramsey cardinals are discussed in [1].

There is a weakening of Ramsey cardinals which is relevant here, and is also incompatible with the axiom of constructibility.

We say that κ is almost Ramsey if and only if for every partition of the finite subsets of κ into two parts, there exists sets of every cardinality $< \kappa$ which are simultaneously homogenous in each exponent.

It can be shown that for every Ramsey cardinal κ, the set of almost Ramsey cardinals $< \kappa$ is stationary in κ. Almost Ramsey cardinals are still well into the region of large cardinals which are incompatible with the axiom of constructibility.

Analogously, one can show that in VBC, "On is Ramsey" implies the class of almost Ramsey cardinals is unbounded (even stationary) in On.

We now introduce the formulas $R(N,wr,k,A_1,\ldots)$ written

$$(\forall wr\ f:N^k \to N)(\exists \text{ unbounded } Y)(A_1 \ \&\ A_2 \ \ldots),$$

where the A's are the result of placing zero or more universal quantifiers ranging over Y in front of a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and finitely many additional free variables for elements of N are allowed. The R indicates that this form is connected with Ramsey cardinals, and the N indicates the use of the natural numbers N in the assertion. These conventions are the same as those made in section 1B. Here "unbounded Y" means that Y is an unbounded subset of N.

Now consider the corresponding formulas $R(On,wr,k,A_1\ldots)$ written

$$(\forall wr\ f:On^k \to On)(\exists \text{ unbounded } Y)(A_1 \ \&\ A_2 \ \ldots).$$

Here "unbounded Y" means that Y is an unbounded subclass of On.

Note that these new forms differ from the earlier only in that an infinite conjunction of quantifier free formulas universally quantified into Y are allowed instead of a single such universally quantified formula.

THEOREM 1D.1. The following are provably equivalent in VBC.

i) $R(N,wr) \Rightarrow R(On,wr)$;
ii) On is Ramsey.

E. INEFFABLY RAMSEY CARDINALS

In this section we use the R classes for N, but replace the R classes for On by the IR classes, and form the analogous transfer principles.

We say that a cardinal κ is ineffably Ramsey if and only if for every partition of the finite subsets of κ into two parts, there exists a stationary subset of κ which is simultaneously homogenous in each exponent.

It can be shown that for every ineffably Ramsey cardinal κ, the set of Ramsey cardinals < κ is stationary in κ.

Analogously, one can show that in VBC, "On is ineffably Ramsey" implies the class of Ramsey cardinals is unbounded (even stationary) in On.

Now consider the assertions IR(On,wr,k,A_1…) written

$$(\forall wr\ f{:}On^k \rightarrow On)(\exists \text{ stationary } Y)(A_1 \ \&\ A_2 \ \ldots),$$

We follow the same conventions as in section 1D. Here "stationary Y" means Y is a stationary subclass of On.

THEOREM 1E.1. The following are provably equivalent in VBC.

- **i)** R(N,wr) $\Rightarrow$ IR(On,wr);
- **ii)** On is ineffably Ramsey.

F. SUBTLE CARDINALS

Let S(N,wr,k,p,A_1,…) be

$$(\forall wr\ f{:}N^k \rightarrow N)(\exists x_1,\ldots,x_p)(A_1 \wedge A_2 \wedge \ldots)$$

where each A_i is a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and additional free variables for elements of N are allowed. The existential quantifiers range over N. The S indicates that this form is connected with subtle cardinals, the N indicates the use of the natural numbers N in the assertion, and wr indicates the restriction to weakly regressive functions.

And let S(On,wr,k,p,A_1,…) be the corresponding formula

$$(\forall wrf{:}On^k \rightarrow On)(\exists x_1,\ldots,x_p)(A_1 \wedge A_2 \wedge \ldots).$$

Here the existential quantifiers and free variables range over On.

We now consider the transfer principle S(N,wr) $\Rightarrow$ S(On,wr), which asserts that for all k,p,A_1,…, if S(N,wr,k,p,A_1,…) is valid then S(On,wr,k,p,A_1,…) is valid.

THEOREM. 1F. The following are provably equivalent in VBC.
i) S(N,wr) $\Rightarrow$ S(On,wr);
ii) for all $n \in N$, there is a proper class of n-subtle cardinals.

G. FROM N TO <On

In this section we discuss the analogous transfer principles from N to <On. Here <On signifies that the statements involve functions on functions on ordinals instead of functions on On. We can now use ZFC as the base theory.

We define M(<On,wr,k,p,A) as

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall \text{wr } f:\lambda^k \to \lambda)$$
$$(\exists x_1,\ldots,x_p)(A(f,x_1,\ldots,x_p)).$$

Here the existential quantifiers range over λ.

Let W(<On,wr,k,p,A) be the assertion

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\exists \text{ unbounded } Y)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Here "unbounded Y" means that Y is an unbounded subset of λ.

Let I(<On,wr,k,p,A) be the assertion

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall \text{wr } f:\lambda^k \to \lambda)$$
$$(\exists \text{ stationary } Y)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Here "stationary Y" means that Y is a stationary subset of λ.

Let R(<On,wr,k,A_1,…) assert that

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall \text{wr } f:\lambda^k \to \lambda)$$
$$(\exists \text{ unbounded } Y)(A_1 \;\&\; A_2 \ldots),$$

where "unbounded Y" means Y is an unbounded subset of λ.

Let IR(<On,wr,k,A_1,…) assert that

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall \text{wr } f:\lambda^k \to \lambda)$$
$$(\exists \text{ stationary } Y)(A_1 \;\&\; A_2 \ldots),$$

where "stationary Y" means Y is a stationary subset of λ.

Finally, let S(<On,wr,k,p,A1,…) assert that

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall\text{wr } f:\lambda^k \to \lambda)$$
$$(\exists x_1,\ldots,x_p)(A_1 \;\&\; A_2 \ldots).$$

THEOREM 1G.1.Each of the following proves the next in ZFC.

i) for all $n \in N$, there exists an n-subtle cardinal;
ii) M(N,wr) $\Rightarrow$ M(<On,wr);
iii) for all $n \in N$, there exists an n-Mahlo cardinal.

THEOREM 1G.2. Each of the following proves the next in ZFC.

i) for all $n \in N$, there exists an n-subtle cardinal;
ii) W(N,wr) $\Rightarrow$ W(<On,wr);
iii) for all $n \in N$, there exists a weakly compact cardinal.

THEOREM 1G.3. The following are provably equivalent in ZFC.

i) for all $n \in N$, there exists an n-ineffable cardinal;
ii) I(N,wr) $\Rightarrow$ I(<On,wr).

THEOREM 1G.4. The following are provably equivalent in ZFC.

i) there exists a Ramsey cardinal;
ii) R(N,wr) $\Rightarrow$ R(<On,wr).

THEOREM 1G.5. The following are provably equivalent in ZFC.

i) there exists an ineffably Ramsey cardinal;
ii) IR(N,wr) $\Rightarrow$ IR(<On,wr).

THEOREM 1G.6. The following are provably equivalent in ZFC.

i) for all $n \in N$, there exists an n-subtle cardinal;
ii) S(N,wr) $\Rightarrow$ S(<On,wr).

G.CONVERSES

The converses all fail:

THEOREM 1G.1. The following are refutable in VBC:

i) M(On) $\Rightarrow$ M(N);
ii) I(On) $\Rightarrow$ W(N).

THEOREM 1G.2. The following are refutable in ZFC:

i) M(<On) $\Rightarrow$ M(N);

ii) $I(<On) \Rightarrow W(N)$.

2.TRANSFER PRINCIPLES FOR GENERAL FUNCTIONS

Here we consider transfer principles where we do not restrict the functions to be weakly regressive. This, however, requires us to restrict the existential quantifiers. The transfer principles are therefore somewhat more awkward, and the results are similar. Some additional issues do arise. However, the situation with the converses is entirely different.

A.EQUIVALENCE WITH MAHLONESS

We begin by considering the formulas M(N,k,p,E,A) written

$$(\forall f:N^k \rightarrow N)(\exists x_1,\ldots,x_p \in E)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq N$, and A is a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and additional free variables for elements of N are allowed. The M indicates that this form is connected with Mahlo cardinals, and the N indicates the use of the natural numbers N in the assertion. Note that there are continuumly many assertions under consideration because of the E.

Now consider the corresponding assertions M(On,k,p,E,A) written

$$(\forall f:On^k \rightarrow On)(\exists x_1,\ldots,x_p \in E)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq On$, and A is a Boolean combination of inequalities between (possibly nested) terms involving the f's and the x's. Note that M(On,k,p,E,A) is an assertion in class theory.

For each choice of $E \subseteq N$ and $E' \subseteq On$, we consider the transfer principle $M(N,E) \Rightarrow M(On,E')$:

for all k,p,A, if M(N,k,p,E,A) is valid then M(On,k,p,E',A) is valid.

We say that $E \subseteq N$ is adequate if and only if E is infinite, and for all n,

$$E_{i+1} - E_i > i^n$$

holds for all sufficiently large i. Here E_i is the i-th smallest element of E, starting from i = 1.

THEOREM 2A.1. The following are provably equivalent in VBC.

i) $M(N,E) \Rightarrow M(On,E')$ holds if E is adequate, E' is unbounded in On, and $E \subseteq E'$;
ii) $M(N,E) \Rightarrow M(On,On)$ holds for some infinite $E \subseteq N$;
iii) for all $n \in N$, the class of n-Mahlo cardinals is stationary in On.

We now consider a modification, in which the E are
quantified.

Let M(N,k,p,A) be the formula

$$(\forall f:N^k \to N)(\forall \text{ unbounded } E \subseteq N)(\exists x_1,\ldots,x_p \in E)(A(f,x_1,\ldots,x_p)).$$

Note that the "E" is missing in "M(N,k,p,A)."

Let M(On,k,p,A) be given by

$$(\forall f:On^k \to On)(\forall \text{ unbounded } E \subseteq On)$$
$$(\exists x_1,\ldots,x_p \in E)(A(f,x_1,\ldots,x_p)).$$

Thus we have the transfer principle $M(N) \Rightarrow M(On)$:

for all k,p,A, if M(N,k,p,A) is valid then M(On,k,p,A) is valid.

THEOREM 2A.2. The following are provably equivalent in VBC.
i) $M(N) \Rightarrow M(On)$;
ii) for all $n \in N$, the class of n-Mahlo cardinals is stationary in On.

B.EQUIVALENCE WITH WEAK COMPACTNESS

In this section we replace the M classes given in section 2A by the W classes, and form the analogous transfer principles.

Let W(N,k,p,E,A) be the formula

$$(\forall f:N^k \to N)(\exists \text{ unbounded } Y \subseteq E)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq N$, and A is a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and additional free variables for elements of N are allowed. Here "unbounded Y" means Y is a unbounded subset of N. The W indicates that this form is connected with weakly compact cardinals, and the N indicates the use of the natural numbers N in the assertion. Note that there are continuumly many assertions under consideration because of the E.

Now consider the corresponding formulas W(On,k,p,E,A) written

$$(\forall f:On^k \to On)(\exists \text{ unbounded } Y \subseteq E)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq On$. Here "unbounded Y" means that Y is unbounded in On.

THEOREM 2B.1. The following are provably equivalent in VBC.

i) $W(N,E) \Rightarrow W(On,E')$ holds if E is adequate and E' is unbounded in On;
ii) $W(N,E) \Rightarrow W(On,On)$ holds for some infinite $E \subseteq N$;
iii) On is weakly compact.

We now consider a modification, in which the E are quantified.

Let W(N,k,p,A) assert that

$$(\forall f:N^k \to N)(\forall \text{ unbounded } E \subseteq N)(\exists \text{ unbounded } Y \subseteq E)$$
$$(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Note that the "E" is missing from "W(N,k,A)."

Let W(On,k,p,A) assert that

$$(\forall f:On^k \to On)(\forall \text{ unbounded } E \subseteq On)(\exists \text{ unbounded } Y \subseteq E)$$
$$(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

THEOREM 2B.2. The following are provably equivalent in VBC.

i) $W(N) \Rightarrow W(On)$;
ii) On is weakly compact.

C.EQUIVALENCE WITH INEFFABILITY

In this section we use the W classes for N, but replace the W classes for On by the I classes, and form the analogous transfer principles.

Let I(On,k,p,E,A) be the formula

$$(\forall f:On^k \to On)(\exists \text{ stationary } Y \subseteq E)$$
$$(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq On$. Here "stationary Y" means that Y is stationary in On.

For $n \in N$, an infinite cardinal is n-ineffable if and only if it is regular and any partition of the (k+1)-element subsets into two pieces has a stationary homogenous set. This is not Baumgartner's original definition, but is a nice equivalent due to him (see [4]). There is a discussion of Baumgartner's work in [5].

The relevant large cardinal axiom for us will be "On is n-ineffable for all $n \in N$." This has the obvious meaning in class theory (VBC). Straightforward arguments show that "On is n-ineffable for all $n \in N$" implies that "for all n in N, the class of all n-ineffable cardinals is unbounded in On (in fact, is stationary in On)."

THEOREM 2C.1. The following are provably equivalent in VBC.

i) $W(N,E) \Rightarrow I(On,E')$ holds if E is adequate and E' is unbounded in On;

ii) $W(N,E) \Rightarrow I(On,On)$ holds for some infinite $E \subseteq N$;

iii) "On is n-ineffable for all $n \in N$."

We now consider a modification, in which the E are quantified.

Let I(On,k,p,A) be the formula

$$(\forall f:On^k \rightarrow On)(\forall \text{ closed unbounded } E \subseteq On)$$
$$(\exists \text{ stationary } Y \subseteq E)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Here "stationary Y" means "Y is stationary in On."

THEOREM 2C.2. The following are provably equivalent in VBC.

i) $W(N) \Rightarrow I(On)$;

ii) "On is n-ineffable for all $n \in N$."

D.EQUIVALENCE WITH RAMSEYNESS

We now present transfer principles from N to On which correspond to Ramsey cardinals. We say that a cardinal κ is Ramsey if and only if for every partition of the finite subsets of k into two parts, there exists a set of power κ which is simultaneously homogenous in each exponent. The existence of a Ramsey cardinal is well known to be incompatible with the axiom of constructibility. Ramsey cardinals are discussed in [1].

There is a weakening of Ramsey cardinals which is relevant here, and is also incompatible with the axiom of constructibility.

We say that κ is almost Ramsey if and only if for every partition of the finite subsets of κ into two parts, there exists sets of every cardinality < κ which are simultaneously homogenous in each exponent.

It can be shown that for every Ramsey cardinal κ, the set of almost Ramsey cardinals < κ is stationary in κ. Almost Ramsey cardinals are still well into the region of large cardinals which are incompatible with the axiom of constructibility.

Analogously, one can show that in VBC, "On is Ramsey" implies the class of almost Ramsey cardinals is unbounded (even stationary) in On.

We now introduce the assertions $R(N,k,E,A_1,\ldots)$ written

$$(\forall f:N^k \to N)(\exists \text{ unbounded } Y \subseteq E)(A_1 \,\&\, A_2 \ldots),$$

where $E \subseteq N$, and the A's are the result of placing zero or more universal quantifiers ranging over Y in front of a Boolean combination of inequalities between (possibly nested) terms involving f and the x's. Constants and finitely many additional free variables for elements of N are allowed. The R indicates that this form is connected with Ramsey cardinals, and the N indicates the use of the natural numbers N in the assertion. Note that there are continuumly many assertions under consideration because of the E as well as the A's. These conventions are the same as those made in section 2B. Here "unbounded Y" means that Y is an unbounded subset of N.

Now consider the corresponding formulas $R(On,k,E,A_1\ldots)$ written

$$(\forall f:On^k \to On)(\exists \text{ unbounded } Y \subseteq E)(A_1 \,\&\, A_2 \ldots),$$

where $E \subseteq On$. Here "unbounded Y" means that Y is an unbounded subclass of On.

Note that these new forms differ from the forms in section 2C only in that an infinite conjunction of quantifier free formulas universally quantified into Y are allowed instead of a single such universally quantified formula.

THEOREM 2D.1. The following are provably equivalent in VBC.

i) $R(N,E) \Rightarrow R(On,E')$ holds if E is adequate and E' is unbounded in On;
ii) $R(N,E) \Rightarrow R(On,On)$ holds for some infinite $E \subseteq N$;
iii) On is Ramsey.

Now let $R(N,k,A_1,\ldots)$ be written as

$$(\forall f:N^k \to N)(\forall \text{ unbounded } E \subseteq N)$$
$$(\exists \text{ unbounded } Y \subseteq E)(A_1 \,\&\, A_2 \ldots).$$

Note that the "E" is missing from "$R(N,k,A_1,\ldots)$."

Let $R(On,k,A_1,\ldots)$ be given by

$$(\forall f_1,\ldots,f_m:On^k \to On)(\forall \text{ unbounded } E \subseteq On)$$
$$(\exists \text{ unbounded } Y \subseteq E)(A_1 \,\&\, A_2 \ldots).$$

THEOREM 2D.2. The following are provably equivalent in VBC.

i) $R(N) \Rightarrow R(On)$;
ii) On is Ramsey.

E. EQUIVALENCE WITH INEFFABLE RAMSEYNESS

In this section we use the R classes for N, but replace the R classes for On by the IR classes, and form the analogous transfer principles.

We say that a cardinal κ is ineffably Ramsey if and only if for every partition of the finite subsets of κ into two parts, there exists a stationary subset of κ which is simultaneously homogenous in each exponent.

It can be shown that for every ineffably Ramsey cardinal κ, the set of Ramsey cardinals < κ is stationary in κ.

Analogously, one can show that in VBC, "On is ineffably Ramsey" implies the class of Ramsey cardinals is unbounded (even stationary) in On.

Now consider the formulas IR(On,k,E,A_1…) written

$$(\forall f{:}On^k \to On)(\exists \text{ stationary } Y \subseteq E)(A_1 \ \& \ A_2 \ \ldots),$$

where $E \subseteq On$. We follow the conventions from section 2D. Here "stationary Y" means "Y is a stationary subclass of On."

THEOREM 2E.1. The following are provably equivalent in VBC.

i) R(N,E) $\Rightarrow$ IR(On,E') holds if E is adequate and E' is unbounded in On;
ii) R(N,E) $\Rightarrow$ IR(On,On) holds for some infinite $E \subseteq N$;
iii) On is ineffably Ramsey.

Let IR(On,m,k,A_1,…) be given by

$$(\forall f_1,\ldots,f_m{:}On^k \to On)(\forall \text{ closed unbounded } E \subseteq On)$$
$$(\exists \text{ stationary } Y \subseteq E)(A_1 \ \& \ A_2 \ \ldots).$$

THEOREM 2E.2. The following are provably equivalent in VBC.

i) R(N) $\Rightarrow$ IR(On);
ii) On is ineffably Ramsey.

F.FROM N TO <On

In this section we develop transfer principles from N to <On. Here <On signifies that the statements involve functions on functions on ordinals instead of functions on On. We still use VBC as the base theory. In many cases, we can use ZFC as the base theory.

We define M(<On,k,p,E,A) as

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f{:}\lambda^k \to \lambda)$$

$$(\exists x_1,\ldots,x_p \in E)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq On$.

Let M(<On,k,p,A) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\forall \text{ unbounded } E \subseteq \lambda)(\exists x_1,\ldots,x_p \in E)(A(f,x_1,\ldots,x_p)).$$

Let W(<On,k,p,E,A) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\exists \text{ unbounded } Y \subseteq E)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq On$. Here “unbounded Y” means that Y is unbounded in λ.

Let W(<On,k,p,A) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\forall \text{ unbounded } E \subseteq \lambda)(\exists \text{ unbounded } Y \subseteq E)$$
$$(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Let I(<On,k,p,E,A) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\exists \text{ stationary } Y \subseteq E)(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)),$$

where $E \subseteq On$. Here “unbounded Y” means that Y is an unbounded subset of λ.

Let I(<On,k,p,A) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\forall \text{ closed unbounded } E \subseteq \lambda)(\exists \text{ stationary } Y \subseteq E)$$
$$(\forall x_1,\ldots,x_p \in Y)(A(f,x_1,\ldots,x_p)).$$

Let R(<On,k,E,A_1,…) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\exists \text{ unbounded } Y \subseteq E)(A_1 \,\&\, A_2 \ldots),$$

where “unbounded Y” means Y is an unbounded subset of λ.

Let R(<On,k,A_1,…) be written as

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$

$$(\forall \text{ unbounded } E \subseteq \lambda)(\exists \text{ unbounded } Y \subseteq E)(A_1 \& A_2 \ldots),$$

Let IR(<On,k,E,A_1,…) be the formula

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\exists \text{ stationary } Y \subseteq E)(A_1 \& A_2 \ldots),$$

where "stationary Y" means Y is a stationary subset of λ.

Finally, let IR(<On,k,A_1,…) be written as

$$(\exists \text{ an uncountable cardinal } \lambda)(\forall f:\lambda^k \to \lambda)$$
$$(\forall \text{ closed unbounded } E \subseteq \lambda)(\exists \text{ stationary } Y \subseteq E)(A_1 \& A_2 \ldots),$$

THEOREM 2F.1. The following are provably equivalent in ZFC.

i) M(N) $\Rightarrow$ M(<On);

ii) M(N,E) $\Rightarrow$ M(<On,E') holds if E $\subseteq$ N is adequate, E' is unbounded in On, and E $\subseteq$ E';

iii) M(N,E) $\Rightarrow$ M(<On,On) holds for some infinite E $\subseteq$ N;

iv) for all n $\in$ N, there exists an n-Mahlo cardinal.

Furthermore, this holds if we replace M by W,I,R,IR, drop E $\subseteq$ E' in ii), and replace iv) by "there exists a weakly compact cardinal," "there exists an ineffable cardinal," "there exists a Ramsey cardinal," "there exists an ineffably Ramsey cardinal," respectively.

G.CONVERSES

THEOREM 2G.1. The following are provable in VBC.

i) M(On) $\Rightarrow$ M(N);

ii) W(On) $\Rightarrow$ W(N);

iii) R(On) $\Rightarrow$ R(N).

Thus by the previous results, assuming the associated large cardinal axioms, we have the following bi-directional transfer principles:

$$M(N) \Leftrightarrow M(On),\ W(N) \Leftrightarrow W(On),\ W(N) \Leftrightarrow I(On),$$
$$R(N) \Leftrightarrow R(On),\ R(N) \Leftrightarrow IR(On).$$

THEOREM 2G.2. The following is provable in VBC. Let E $\subseteq$ N be infinite, E' $\cap$ N $\subseteq$ E, no element of E'\N is a limit point of E', and no two elements of E'\N have the same cardinality. Then

i) M(On,E') $\Rightarrow$ M(N,E);

ii) W(On,E') $\Rightarrow$ W(N,E);

iii) R(On,E’) ⇒ R(N,E).

THEOREM 2G.3. The following are provable in ZFC.

i) M(<On) ⇒ M(N);
ii) W(<On) ⇒ W(N);
iii) R(<On) ⇒ R(N).

THEOREM 2G.4. The following is provable in ZFC. Let E ⊆ N be infinite, E’ ∩ N ⊆ E, no element of E’\N is a limit point of E’, and no two elements of E’\N have the same cardinality. Then

i) M(<On,E’) ⇒ M(N,E);
ii) W(<On,E’) ⇒ W(N,E);
iii) R(<On,E’) ⇒ R(N,E).

Thus by the previous results, assuming the appropriate large cardinal axioms over ZFC, we have the following bi-directional transfer principles:

M(N) ⇔ M(<On),
W(N) ⇔ W(<On),
W(N) ⇔ I(<On),
R(N) ⇔ R(<On),
R(N) ⇔ IR(<On).

And by the previous results, assuming the appropriate large cardinal axioms over VBC, we have the following multi- directional transfer principles:

M(N) ⇔ M(<On) ⇔ M(On),
W(N) ⇔ I(<On) ⇔ I(On) ⇔ W(<On) ⇔ W(On),
R(N) ⇔ R(<On) ⇔ R(On) ⇔ IR(<On) ⇔ IR(On).

H.SOME NECESSARY CONDITIONS

We would like to know, for example, if we can use E = N and E’ = On in any of the transfer principles in section 2. The answer is no, as indicated by the following general result (which should be improved):

THEOREM 2H.1. The following is provable in VBC.

i) if M(N,E) ⇒ M(On,E’), then E is adequate, E’ is unbounded in On, and E’ ∩ N is infinite;
ii) if W(N,E) ⇒ W(On,E’), then N\E is infinite and E’ is unbounded in On;
iii) if R(N,E) ⇒ R(On,E’), then N\E is infinite and E’ is unbounded in On.

Also, we know that we cannot drop “weakly regressive” for any of the results in section 1.

3.TRANSFER PRINCIPLES WITH ARBITRARY ALTERNATIONS OF QUANTIFIERS

We have chosen to state the transfer principles here for rather explicit forms involving few alternations of quantifiers. However, the transfer principles of section 2 can be modified to involve arbitrary prenex formulas, with corresponding results. We now present these alternative formulations.

Let φ be a prenex formula in first order predicate calculus with infinitely many function, relation, and constant symbols, equality, and the distinguished binary relation symbol $<$. The formula may of course have free variables. We assume an appropriate effective indexation of the logical symbols.

We say that φ is N-valid if and only if it holds universally in every predicate calculus interpretation in which $<$ is correct on N. (It is automatically required without mention that a predicate calculus interpretation must correctly interpret =). More generally, let α be an ordinal or On. We say that φ is α-valid if and only if it holds universally in every predicate calculus interpretation in which = and $<$ are as usual on α.

Let us first review the facts from basic model theory. As long as α is infinite, and we do not use $<$, we get the same valid formulas, and the valid formulas are recursively enumerable, but not recursive. Once we use $<$, we see that
N-validity is complete Π-1-1, and general α-validity
(including On-validity) behaves even worse.

We now make the crucial definition. Let $E \subseteq N$. We say that φ is $(N,E,\exists)$-valid if and only if φ holds universally in all $<$-correct interpretations, **where the existential quantifiers range over E.**

Note that we are doing something unusual for model theory, where it would be more common to relativize all quantifiers in φ to the same set E. But it is crucial to just relativize the existential quantifiers to E.

We say that φ is $(N,\forall\text{unb},\exists)$-valid if and only if φ is $(N,E,\exists)$-valid for all unbounded $E \subseteq N$. The analogous definitions are made for $(\alpha,E,\exists)$-valid and $(\alpha,\forall\text{unb},\exists)$-valid, where $E \subseteq \alpha$ and α is an ordinal or On.

We can analogously define φ to be $(\alpha,E,\forall)$-valid if and only if φ holds universally in all $<$-correct interpretations, **where the universal quantifiers range over E.**

We say that φ is $(\alpha,\exists unb,E,\forall)$-valid if and only if φ is $(\alpha,Y,\forall)$-valid for some $Y \subseteq E$ that is unbounded in α. We also say that φ is $(\alpha,\exists stat,E,\forall)$-valid if and only if φ is $(\alpha,E,\forall)$-valid for some $Y \subseteq E$ that is stationary in α.

We say that φ is $(\alpha,\forall unb,\exists unb,\forall)$-valid if and only if for all unbounded $E \subseteq \alpha$ there exists $Y \subseteq E$ such that Y is unbounded in α and φ is $(\alpha,Y,\forall)$-valid.

We also say that φ is $(\alpha,\forall club,\exists stat,\forall)$-valid if and only if for all closed unbounded $E \subseteq \alpha$ there exists $Y \subseteq E$ such that Y is stationary in α and φ is $(\alpha,E,\forall)$-valid.

We now consider the following transfer principles.

1. $(N,E,\exists) \Rightarrow (On,E',\exists)$.
2. $(N,\forall unb,\exists) \Rightarrow (On,\forall unb,\exists)$.
3. $(N,\exists unb,E,\forall) \Rightarrow (On,\exists unb,E',\forall)$.
4. $(N,\exists unb,E,\forall) \Rightarrow (On,\exists stat,E',\forall)$.
5. $(N,\forall unb,\exists unb,\forall) \Rightarrow (On,\forall unb,\exists unb,\forall)$.
6. $(N,\forall unb,\exists unb,\forall) \Rightarrow (On,\forall club,\exists stat,\forall)$.

THEOREM 3.1. The following is provable in VBC. The above transfer principles, and their converses, are respectively equivalent to

1. $M(N,E) \Rightarrow M(On,E')$.
2. $M(N) \Rightarrow M(On)$.
3. $W(N,E) \Rightarrow W(On,E')$.
4. $W(N,E) \Rightarrow I(On,E')$.
5. $W(N) \Rightarrow W(On)$.
6. $W(N) \Rightarrow I(On)$.

and their converses.

So we obtain the respective connections with large cardinal axioms from Theorem 3.1 and the earlier results.

In addition, all of the results in section 2 involving M,W, and I can be modified in this way with these more general formulas. In particular, there are exponential time decision procedures for the $(N,\forall unb,\exists)$-valid, $(N,\forall unb,\exists unb,\forall)$-valid, and $(N,\forall club,\exists stat,\forall)$-valid formulas.

We now come to R and IR. We can extend the transfer principles 3 - 6 above in order to include more formulas phi. In particular, let us first simply include all (possibly infinite) conjunctions of formulas.

THEOREM 3.2. The following is provable in VBC. The above transfer principles 3 - 6, formulated for arbitrary conjunctions, as well as their converses, are respectively equivalent to

3'. $W(N,E) \Rightarrow R(On,E')$.

4’. W(N,E) ⇒ IR(On,E’).
5’. W(N) ⇒ R(On).
6’. W(N) ⇒ IR(On).
and their converses.

We can even extend the formulas to the infinitary language $L_{\infty\omega}$, where there are no restrictions on the ordinal depth or on the length of the conjunctions and disjunctions, other than the formula must be a set; and formulas must have at most finitely many free variables. But it turns out that we get nothing new this way.

4. DECIDABIITY OF STATEMENTS ON N

For the study of statements on N, it is appropriate to use the base theory ACA = arithmetic comprehension axiom schema with full induction.

THEOREM 4.1. The following is provable in ACA. The set of all valid formulas of the form M(N,k,p,A) is decidable. The set of all valid formulas of the form W(N,k,p,A) is also decidable. Furthermore, there is a specific decision procedure which, provably in ACA, works, and runs in exponential time.

THEOREM 4.2. The following is provable in ACA. Let E ⊆ N be adequate. The set of all valid formulas of the form M(N,k,p,E,A) is decidable relative to E. The set of all valid formulas of the form W(N,k,p,E,A) is also decidable relative to E. In particular, these are decidable for the case E = {1,2,4,8,…}, in which case, specific decision procedures exist which, provably in ACA, work, and run in exponential time.

The situation is quite different for M(N,wr,k,p,A):

THEOREM 4.3. The set of all valid formulas in M(N,wr,k,p,A) is complete r.e. The set of all valid formulas in W(N,wr,k,p,A) is complete Π-1-1.

We want to briefly touch on transfer principles for statements involving functions that are not on N, but rather on [n]. We expect to say much more about these in a later abstract.

Here it will suffice to consider the assertions M(n,k,p,A) given by

$$(\forall f:[n]^k \rightarrow [n])(\exists x_1,\ldots,x_p \in \{2^u: 0 \leq u \leq \log(u)\})$$
$$(A(f,x_1,\ldots,x_p)),$$

where E ⊆ [n].

THEOREM 4.4. The following is provable in PA + “PA is 1-consistent.” The following are problems are decidable (in exponential time). Given k,p,A, determine whether or not M(n,k,p,A) holds

i) for all n;
ii) for some n;

iii) for infinitely many n;
iv) for all sufficiently large n.

We do not know whether PA alone can be used in Theorem 4.3. However, we do know that PA alone cannot really handle these decision problems:

THEOREM 4.5. There is no specific algorithm such that can be proved in PA to give a correct decision procedure for any of the four decision problems in Theorem 4.3. In fact, for the natural presentations of the correct decision procedures, the correctness of the decision procedure is provably equivalent, in a weak fragment of PA, to the 1-consistency of PA.

Now let f(k,p,A) = card{n: M(n,k,p,A) holds}. Note that f is infinite valued. It follows from Theorem 4.3 that f is a recursive function.

THEOREM 4.6. f is not a provably recursive function of PA. In fact, f dominates every finitely valued provably recursive function of PA at infinitely many arguments at which f is finite.

We state the following connection with M(N,k,p,E,A):

THEOREM 4.7. Let k,p,A be given. Suppose that M(n,k,p,A) holds for all sufficiently large n. Then M(N,k,p,E,A) holds for E = {1,2,4,8,…}.

5.DEICIDABILITY OF STATEMENTS ON <On AND On

We first discuss the forms of sections 2 and 3. The following is clear from the multi-directional transfer principles stated in section 3:

THEOREM 5.1. The set of all valid formulas of each of the forms M(<On,k,p,A),W(<On,k,p,A),I(<On,k,p,A),M(On,k,p,A), W(On,k,p,A),I(On,k,p,A) are provably recursive using the associated large cardinal axiom. In fact, specific natural decision procedures will be provided which, provably using the associated large cardinal axiom, work, and run in exponential time.

There is an important asymmetry between true sentences and false sentences in these forms. Let EFA be exponential function arithmetic, which is a weak fragment of Peano Arithmetic.

THEOREM 5.2. For each of the four forms, one can prove in EFA that if a formula in the given form comes out “not valid” under the algorithm given by Theorem 5.1, then that formula (universally quantified) has a refutation within (a weak fragment of) ZFC (VBC) -

in fact whose size is linear in that of the formula. Here ZFC is used for <On and VBC is used for On.

THEOREM 5.3. Let one of the four forms be given. Then the following are provably equivalent in ZFC (VBC).

- **i)** the algorithm given by Theorem 5.1 is correct;
- **ii)** the large cardinal axiom heretofore associated with this form is true.

OPEN QUESTION: Are the appropriate large cardinal axioms required to prove decidability? In particular, can decidability be proved in ZFC or VBC?

We do know that a decision procedure cannot be **given** in ZFC or VBC:

THEOREM 5.4. (The following is provable in EFA). Let one of the four forms be given. Let T be an extension of ZFC (VBC) such that T proves that some specific algorithm correctly identifies the true sentences in the given form. If T is consistent with the large cardinal axiom heretofore associated with the given form, then it proves slightly weaker forms of that large cardinal axiom. If T is consistent with, e.g., two inaccessible cardinals, then T proves the existence of one inaccessible cardinal.

We can obtain analogous results for other important forms on On and <On, where we do not yet have any good bi-directional transfer with N. We mention one case of this involving the subtle cardinal hierarchy (see [4] and [5]).

THEOREM 5.5. “On is n-subtle for all $n \in N$” suffices to establish the decidability of the set of true sentences in M(On,k,p,E,A), where E is the class of all infinite cardinals (or alternatively the class of all ordinal powers of 2), and is necessary in the sense of Theorem 5.4. If the class of all infinite successor cardinals is used, then it is analogously necessary and sufficient to use “On is n-Mahlo for all $n \in N$.”

Finally, we discuss the forms of section 1.

THEOREM 5.6. The set of valid formulas of each of the forms M(<On,wr,k,p,A),W(<On,wr,k,p,A),I(<On,wr,k,p,A), M(On,wr,k,p,A),W(On,wr,k,p,A),I(On,wr,k,p,A) are provably recursive using an appropriate large cardinal axiom. In the case of M this is n-subtlety; for W this is n-almost ineffability (see [4],[5]); for I this is n-ineffability. In fact, specific natural decision procedures will be provided which, provably using the associated large cardinal axiom, work, and run in exponential time.

We also obtain the analogous results to Theorem 5.2 and 5.3. And we have the corresponding open question, with the analogous result to Theorem 5.4.

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