TANGIBLE INCOMPLETENESS INTERIM REPORT

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ABSTRACT. Tangible Incompleteness concerns Incompleteness in the realm of mathematically transparent discrete and finite mathematics that has continuously occupied the author since just before his Ph.D. in 1967. This Interim Report covers the state of the art MAXIMALITY and SET COMPLEMENTATION, which are parts 2,3 of the forthcoming book Tangible Incompleteness. Part 1 is Boolean Relation Theory, which appears in essentially final form on my website https://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/ The development here of Maximality is in the most basic fundamental combinatorial directions. We plan to incorporate richer mathematical contexts in the future. The Incompleteness under Maximality is at the level of the SRP hierarchy (stationary Ramsey property). The Incompleteness under Set Complementation takes us from SRP to HUGE to I3 (strictly between I3 and I2). HUGE and I3/I2 represent Extremely Large Cardinals rather than merely Large Cardinals.

TANGIBLE INCOMPLETENESS - book

2. MAXIMALITY

2.1. INTRODUCTION

Maximality is a theme that cuts across virtually all areas of mathematics. We use maximality in some particularly elementary countable contexts. A very general relevant formulation that does not use countability reads as follows.

S is a square in E if and only if S is some $A^2 \subseteq E$. A maximal square in E is a square in E that is not a proper subset of any square in E.

GENERAL MAXIMAL SQUARES. Every set contains a maximal square.

Note that if there are no ordered pairs in E then $\emptyset$ is the unique square in E.

THEOREM 2.1.1. (ZF) General Maximal Squares is equivalent to the Axiom of Choice.

Of course, for this work on Tangible Incompleteness, we want to stay far away from such set theoretic generalities and necessary uses of the axiom of choice. Instead we aim for tangible discrete and finite mathematics, and so we work in the countable.

COUNTABLE MAXIMAL SQUARES THEOREM. Every countable set contains a maximal square.

These countable versions do not require the axiom of choice and in fact, when formulated for reverse mathematics, are equivalent to ACA_0 over RCA_0.

So far, we remain very far from any Incompleteness. However, we have discovered that if we impose very innocent looking symmetry properties on maximal squares (and other related maximal objects), then deep metamathematical phenomena arise, including Tangible Incompleteness. These symmetry properties are presented as various kinds of "stability". See section 2.4 and more comprehensively, section 2.6.

Our Tangible Incompleteness is driven by three ingredients: maximality, symmetry (called stability here), and order equivalence.

We cannot emphasize too much the fundamental character of order equivalence for tuples of rational numbers:

ORDER EQUIVELNCE COMBINATORIALLY. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

ORDER EQUIVALENCE GEOMETRICALLY. $x,y \in Q^k$ are order equivalent if and only if there is a strictly increasing bijection $h:Q \rightarrow Q$ which sends x to y under the coordinate action.

For various reasons, we work in the subspaces $Q[-n,n]^k$ of the Q^k, where $Q[-n,n] = Q \cap [-n,n]$, and so we use order equivalence on $Q[-n,n]^k$. This induces the crucial notion of order invariant subsets of $Q[-n,n]^k$: $E \subseteq Q[-n,n]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[-n,n]^k$, $x \in E \leftrightarrow y \in E$.

There are only finitely many order invariant subsets of any given $Q[-n,n]^k$. This finite number depends on k and not on n. In section 2.7, we work in the integers for explicitly finite Tangible Incompleteness, using the $\{-nt,...,nt\}^k$, t >> k,n. We actually use t >> k,n,r to accommodate the parameter r if its is used.

Our Tangible Incompleteness uses ten alternative settings, as can be seen from the title of section 2.5. These various settings are well motivated by very elementary mathematics, and all have their conceptual advantages and disadvantages for various audiences. Some settings are obviously more general than others, but less is more and more is less when it comes to Incompleteness. Less generality is simpler and more generality is stronger.

Here are our audience expectations. See section 2.5 for definitions.

1. Maximal squares in order invariant subsets of $Q[-n,n]^k$. For the experienced mathematician looking for the simplest and most familiar Tangible Incompleteness that we have. This setting drives this Introduction as well as sections 2.3, 2.4, with our Lead Tangible Incompleteness, called Negatively Stable Maximal Squares.
2. Maximal sides, maximal r-cubes, maximal r-sides, in order invariant subsets of $Q[-n,n]^k$. For the experienced mathematician right after absorbing the Squares development.
3. Maximal cliques, maximal r-cliques, in order invariant graphs and order invariant r-graphs. For the experienced mathematician interested in or familiar with or willing to tolerate basic graph theory.
4. Maximal emulators, maximal duplicators, maximal r-emulators, maximal r-duplicators of subsets of $Q[-n,n]^k$. For Mathematically Gifted Youth. Emulators are more convenient than duplicators, and we are preparing materials on emulators for the leading Mathematically Gifted Youth Programs.

For simplicity, we continue our discussion here with maximal squares.

Here is the general shape of our lead Intangible Incompleteness.

EVERY ORDER INVARIANT SUBSET OF $Q[-n,n]^k$ CONTAINS A <u>*GOOD*</u> MAXIMAL SQUARE

DEFINITION (section 2.4). $S \subseteq Q[-n,n]^k$ is stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, $x \in S \leftrightarrow y \in S$.

A stable maximal square is an example of a good maximal square. See section 2.4.

STABLE MAXIMAL SQUARE THEOREM. SMST. Every order invariant subset of $Q[-n,n]^k$ contains a <u>*STABLE*</u> maximal square.

Although this Theorem is provable well within ZFC, not even close to Incompleteness, the only proof I know is more infinitary in nature than would be expected. We have not had a chance to work on a reversal here.

Now we naturally strengthen stability by adding lower parameters.

DEFINITION (section 2.4). $S \subseteq Q[-n,n]^k$ is negatively stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in Q[-n,n]^k$ are obtained from x,y by replacing zero or more x_i,y_i by $p,p < 0$, then $x' \in S \leftrightarrow y' \in S$.

Thus a negatively stable maximal square is an even better maximal square than a merely stable maximal square.

EXAMPLE OF SUCH LOWERING: Use $Q[-8,8]^6$, and order equivalent $(1,4,2,4,8,6)$, $(2,4,3,4,6,5) \in Q[-8,08]^6$. Replace third coordinates by -3/2, and fourth coordinates by -8, obtaining (1,4,-3/2,-8,8,6), (2,3,-3/2,-8,6,5).

NEGATIVELY STABLE MAXIMAL SQUARES. NSMS. Every order invariant subset of $Q[-n,n]^k$ contains a <u>*NEGATIVELY STABLE*</u> maximal square.

We have chosen NSMS as the first Tangible Incompleteness that appears in this manuscript. It is the Lead Tangible Incompleteness.

We show that NSMS is independent of ZFC, under commonly believed assumptions about the coherence of abstract set theory.

NOTE: Here and elsewhere we use "a sentence φ is independent of T" to mean "φ is neither provable nor refutable in T". We caution the reader that this terminology has not yet become standard. Some authors take "independent" to merely mean "unprovable".

We now give a very careful statement of the key metamathematical facts about NSMS. We begin with the core fact about NSMS from which all other facts can be derived using general methods from mathematical logic that have nothing to do with NSMS in particular.

METAMATHEMATICAL PROPERTIES OF NSMS
all established in EFA

CORE NSMS STATUS. WKL_0 proves NSMS $\leftrightarrow$ Con(SRP).

PPROVAILITY FROM LARGE CARDINALS. SRP^+ proves NSMS.

UNPROVABILITY IN ZFC. NSMS is not provable in ZFC if and only if ZFC is consistent.

UNPROVABILITY IN SRP. NSMS is not provable in SRP if and only if SRP is consistent.

GENERAL UNPROVABILITY. NSMS is not provable in any consistent set of theorems of SRP that proves RCA_0 in set theoretic form.

INDEPENDENT OF ZFC. NSMS is independent of ZFC if and only if SRP does not refute its own consistency.

INDEPENDENT OF SRP. NSMS is independent of SRP if and only if SRP does not refute its own consistency.

We also know that there are small integers n,k for which NSMS is unprovable in ZFC (assuming ZFC is consistent). This is also the case for the other Tangible Incompleteness in section 2.6. I will be making a major effort to determine the n,k where we have unprovability in ZFC.

We now take up the Tangibility of NSMS. NSMS is obviously infinitary in that it uses quantification over infinite objects. Actually it uses only a single existential quantifier over a countably infinite object, namely the maximal square.

However, there is a way of looking at NSMS that reveals its fundamentally finitary character just from its form. As a nice undergraduate logic exercise, it is clear that NSMS asserts that for certain finite data X, there is an associated sentence φ[X] in first order predicate calculus with <,= and a predicate symbol P for the maximal square (whose arity is the dimension k), such that the following holds. NSMS is equivalent to asserting that every such φ[X] is satisfiable (i.e., has a model).

Now according to the Gödel Completeness Theorem, NSMS is equivalent to asserting that each φ[X] is consistent in the first order predicate calculus with <,=,P. This equivalence is provable in WKL_0. All of the components of this argument are provable in RCA_0 except for the "consistency implies satisfiability" part of the Gödel Completeness Theorem. We have glossed over some subtleties here, which are covered in section 2.7.

Thus we say that NSMS is implicitly Π^0_1 over WKL_0 via the Gödel Completeness Theorem. We also show that NSMS is what we call WKL_0 falsifiable in section 2.7. This says that "if NSMS is false then NSMS is provably false in WKL0", and this implication is provable in EFA.

Falsifiability is a highly desirable property of an Incompleteness that is crucial from the point of view of general science. For we want our theories or hypotheses to be falsifiable through the results of experiments - such results are viewed as incontrovertible.

In section 2.8, we present the same Incompleteness in a way that does not involve quantification over infinite objects. This is done through a systematic principled weakening of the notion of maximality.

In section 2.9, instead of using the $Q[-n,n]^k$ with the distinguished points 0,1,...,n, we use the $\{-nt,...,nt\}^k$ with the distinguished points 0,t,...,nt, where t >> k,n (t >> k,n,r if the parameter r is used). The development is almost identical. The usual notion of maximality is appropriate for stability, but too strong in this finite setting for the other four, stronger, notions of stability. Instead we use weak maximality.

The Incompleteness in section 2.9 is explicitly Π^0_3 because of the t >> k,n,r. With the a priori estimate t > (8knr)!!, the Incompleteness becomes explicitly Π^0_1.

2.2. PRELIMINARIES

DEFINITION 2.2.1. N, Z^+, Q are, respectively, the set of nonnegative integers, the set of positive integers, and the set of rationals. We use i,j,k,n,m,r,s,t with or without subscripts for positive integers, unless otherwise indicated. We use p,q with or without subscripts for rational numbers, unless otherwise indicated. We use A,B,C,D,E,K,S,T,U,V,W with or without subscripts for sets unless otherwise indicated. We work with the closed rational intervals $Q[-n,n] = Q \cap [-n,n]$. min(x), max(x) is the least, greatest coordinate of $x \in Q^k$, respectively. $x_1...x_n$ is the concatenation of the tuples $x_1,...,x_n$.

DEFINITION 2.2.2. Let $x,y \in Q^k$. x,y are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. Let $B \subseteq Q$. x,y are B order equivalent if and only if for all $b \in B$, (x,b),(y,b) are order equivalent.

DEFINITION 2.2.3. $E \subseteq Q[-n,n]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[-n,n]^k$, $x \in E \leftrightarrow y \in E$. $E \subseteq Q[-n,n]^k$ is B order invariant if and only if for all B order equivalent $x,y \in Q[-n,n]^k$, $x \in E \leftrightarrow y \in E$.

DEFINITION 2.2.4. $H:Q[-n,n]^s \rightarrow Q]-n,n]^t$ is order invariant (B order invariant) if and only if it is order invariant (B order invariant) as a subset of $Q[-n,n]^{s+t}$. We make this definition also with Q[-n,n] replaced by Q.

DEFINITION 2.2.5. A quasi well ordering is a relation R (set of ordered pairs) which is reflexive, transitive, connected, and where every nonempty subset of its field has an R mminimal element. Thus if we factor out by the equivalence relation $x R y \wedge y R x$, then we get a reflexive well ordering.

Note that order equivalence on $Q[-n,n]^k$ is an equivalence relation on $Q[-n,n]^k$ with finitely many cosets. The number is bounded by a double exponential in k. There are finitely many order invariant $E \subseteq Q[-n,n]^k$. The number is bounded by a triple exponential in k.

THEOREM 2.2.1. $E \subseteq Q[-n,n]^k$ is order invariant if and only if $E = \{x \in Q[-n,n]^k\text{: } \varphi\}$ where φ is a Boolean combination of inequalities $x_i < x_j$, $1 \leq i,j \leq k$. $E \subseteq Q[-n,n]^k$ is B order invariant if and only if $E = \{x \in Q[-n,n]^k: \varphi\}$ where φ is a Boolean combination of inequalities $x_i < x_j$, $x_i < b$, $b < x_i$, $1 \leq i,j \leq k$, $b \in B$.

We most commonly work with order invariance and N order invariance.

DEFINITION 2.2.6. $S \subseteq Q^k$ is order theoretic if and only if there exists finite $B \subseteq Q$ such that S is B order invariant.

THEOREM 2.2.2. $S \subseteq Q^k$ is order theoretic if and only if $S = \{x \in Q^k: \varphi\}$ where φ is a Boolean combination of inequalities $x_i < x_j$, $x_i < b$, $b < x_i$, $1 \leq i,j \leq k$, where $b \in Q$. The b's used are called the parameters.

Order theoretic will be used in section 3.

2.3. SQUARES IN $Q[-n,n]^k$

We use ten friendly contexts for our Tangible Incompleteness. In this section we focus on the one of the ten that is arguably the most immediately mathematically transparent. The remaining nine contexts are treated in section 2.5.

DEFINITION 2.3.1. A square in $E \subseteq Q[-n,n]^k$ is a set $A^2 \subseteq E$. A maximal square in $E \subseteq Q[-n,n]^k$ is a square in $E \subseteq Q[-n,n]^k$ which is not a proper subset of any square in $E \subseteq Q[-n,n]^k$.

Here is some background information about maximal squares.

THEOREM 2.3.1. (RCA_0) Every order invariant $E \subseteq Q[-n,n]^k$ has a recursive maximal square.

THEOREM 2.3.2. The following are equivalent over RCA_0.
i. Every $E \subseteq Q[-n,n]^k$ contains a maximal square.
ii. Every order invariant $E \subseteq Q[-n,n]^k$ contains a maximal square containing any given square in $E \subseteq Q[-n,n]^k$.
iii. ACA_0.

2.4. STABLE AND NEGATIVELY STABLE SUBSETS OF $Q[-n,n]^k$

DEFINITION 2.4.1. $S \subseteq Q[-n,n]^k$ is stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, $x \in S \leftrightarrow y \in S$.

STABLE MAXIMAL SQUARES. SMS. (WKL_0 + Con(PA)) Every order invariant subset of $Q[-n,n]^k$ has a stable maximal square.

The right endpoint n appearing in x,y causes difficulties in proving SMS. An easier result, provable in RCA_0, is to only use x,y in $\{0,...,n-1\}^k$ in the definition of stable, avoiding the right endpoint.

STABLE MAXIMAL SQUARES TEMPLATE. SMST. Fix k,n. Every order invariant subset of $Q[-n,n]^k$ has a stable maximal square.

Thus SMST consists of one statement for each choice of k,n.

CONJECTURE. Every sentence in SMST is provably equivalent, over WKL_0, to Con($I\Sigma_n$) or 1 = 0, for some n effectively obtained from the sentence by low computational complexity.

We now move to a parametric form of stability.

DEFINITION 2.4.2. $S \subseteq Q[-n,n]^k$ is negatively stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in Q[-n,n]^k$ is obtained from x,y by replacing zero or more x_i,y_i by $p,p < 0$, then $x' \in S \leftrightarrow y' \in S$.

Note that all x',y' so arising are order equivalent. Of course they may have fractional coordinates (all negative).

EXAMPLE OF SUCH NEGATIVE PARAMETERIZATION: Use $Q[-8,8]^6$, (1,4,2,4,8,6), (2,4,3,4,6,5). Replace third coordinates by -3/2, and fourth coordinates by -7, obtaining (1,4,-3/2,-7,8,6), (2,4,-3/2,-7,6,5).

NEGATIVELY STABLE MAXIMAL SQUARES. NSMS. Every order invariant subset of $Q[-n,n]^k$ has a negatively stable maximal square.

THEOREM 2.4.1. NSMS is provably equivalent to Con(SRP) over WKL_0 where the implication to Con(SRP) is provable in RCA_0.

NSMS is our Lead Tangible Incompleteness, as discussed in the Introduction.

NEGATIVELY STABLE MAXIMAL SQUARES TEMPLATE. NSMST. Fix k,n. Every order invariant subset of $Q[-n,n]^k$ has a negatively stable maximal square.

Thus NSMST consists of one statement for each choice of k,n.

THEOREM 2.4.2. Every sentence in NSMST is provable in SRP. In fact every sentence in NSMST is provable in WKL_0 + Con(SRP[n]), for some n effectively determinable in the sentence. For every n we can effectively find a sentence in NSMT which implies Con(SRP[n]) over RCA_0.

CONJECTURE. NSMST under $A \leq B \leftrightarrow WKL_0$ proves $B \rightarrow A$, forms a recursive quasi well ordering of low level computational complexity.

2.5. SQUARES, SIDES, CLIQUES, EMULATORS, DUPLICATORS, r-CUBES, r-SIDES, r-CLIQUES, r-EMULATORS, r-DUPLICATORS in $Q[-n,n]^k$

In section 2.3 we focused on Squares. Here we incorporate all ten contexts into the discussion, all ten listed in the title of this section.

Squares, sides, cliques, emulators, duplicators are 2-squares, 2-sides, 2-cliques, 2-emulators, 2-duplicators, respectively. Therefore we need only be concerned with r-cubes, r-sides, r-cliques, r-emulators, r-duplicators. We show that the Tangible Incompleteness is already realized with squares, sides, cliques, emulators, duplicators.

DEFINITION 2.5.1. An r-cube in $E \subseteq Q[-n,n]^k$ is a set $A^r \subseteq E$. An r-side in $E \subseteq Q[-n,n]^{kr}$ is a set $A \subseteq Q[-n,n]^k$ such that $A^r \subseteq E$.

DEFINITION 2.5.2. An r-graph (as in hypergraph) on V is an H = (V,E), where V is the set of vertices and $E \subseteq V^r$ is the set of r-edges. It is required that $x \in E \rightarrow (x_1 \neq \ldots \neq x_r \wedge$ every permutation of x lies in E). G is an order invariant r-graph on $Q[-n,n]^k$ if and only if $V = Q[-n,n]^k$ and $E \subseteq Q[-n,n]^{rk}$ is order invariant.

DEFINITION 2.5.3. An r-clique in an r-graph $H = (V,E)$ is an $S \subseteq V$ such that for all distinct $x_1,\ldots,x_r \in S$, $(x_1,\ldots,x_s) \in E$.

DEFINITION 2.5.4. An r-emulator of $E \subseteq Q[-n,n]^k$ is a set $S \subseteq Q[-n,n]^k$ such that for every $x_1,\ldots,x_r \in S$ there exists $y_1,\ldots,y_r \in E$ such that $x_1\ldots x_r$ is order equivalent to $y_1\ldots y_r$. An r-duplicator of $S \subseteq Q[-n,n]^k$ is a set $S \subseteq Q[-n,n]^k$ such that for every $x_1,\ldots,x_r \in S$ there exists $y_1,\ldots,y_r \in E$ such that $x_1\ldots x_r$ is order equivalent to $y_1\ldots y_r$, and vice versa.

Here are the five crucial categories of sets of subsets of the various $Q[-n,n]^k$ that we work with. We establish easy relationships between these five categories. This will establish various easy relationships between the Tangible Incompleteness formulated in terms of the various categories.

DEFINITION 2.5.5. Fix k,n,r. The k,n,r-sets are the sets of subsets of $Q[-n,n]^k$ of the following five forms:
C1. The set of r-cubes in an order invariant $E \subseteq Q[-n,n]^k$.
C2. The set of r-sides in an order invariant $E \subseteq Q[-n,n]^{kr}$.
C3. the set of r-cliques in an order invariant r-graph on $Q[-n,n]^k$.
C4. The set of r-emulators of subsets of $Q[-n,n]^k$.
C5. The set of r-duplicators of subsets of $Q[-n,n]^k$.

In C4, C5, the use of all subsets of $Q[-n,n]^k$ is for simplicity. We shall see that every $E \subseteq Q[-n,n]^k$ has the same r-emulators (r-duplicators) as some finite subset of E, so that we are really working with finite subsets of $Q[-n,n]^k$ in C4,C5.

It is important to note that order invariance does not appear in the statements of C4,C5. But order equivalence is hidden in the definition of r-emulators and r-duplicators. On the other hand, order invariance and order equivalence do not appear in the definition of r-cubes, r-sides, r-cliques.

DEFINITION 2.5.6. Let K be a set of sets. A maximal element of K is an $S \in K$ which is not a proper subset of any $S' \in K$.

We apply this definition of maximal to C1-C5.

Here is some background information about k,n,r-sets.

THEOREM 2.5.1. (RCA_0) Every k,n,r-set has a recursive maximal element.

THEOREM 2.5.2. (RCA_0) The following are equivalent.
i. Every k,n,r-set of the forms C1,C2,C3 with "order invariant" removed, has a maximal element.
ii. ACA_0.
For C1, we can fix any n ≥ 1, r ≥ 2, and k = jr, j ≥ 2. For C2, we can fix any n ≥ 1 and any k,r ≥ 2. For C3 we can fix any n ≥ 1 and k,r ≥ 2.

THEOREM 2.5.3. (RCA_0) The Following are equivalent.
i. Every (order invariant) $E \subseteq Q[-n,n]^k$ has a maximal r-cube containing any given r-cube in E. Also for order invariant E and any fixed r ≥ 2.
ii. Every (order invariant) $E \subseteq Q[-n,n]^k$ has a maximal r-side containing any given r-side in E.
iii. Every (order invariant) graph on $Q[-n,n]^k$ has a maximal r-clique containing any given r-clique in G. Also for order invariant G and any fixed r ≥ 2.
iv. Every (finite) $E \subseteq Q[-n,n]^k$ has a maximal r-emulator containing any given r-emulator of E. Also for any fixed r ≥ 2.
v. Every (finite) $E \subseteq Q[-n,n]^k$ has a maximal r-duplicator containing any given r-duplicator of E. Also for any fixed r ≥ 2.
vi. ACA_0.
In i, we can fix any n ≥ 1, r ≥ 2, and any k = jr, j ≥ 2. In ii, we can fix n ≥ 1 and k,r ≥ 2. In iii, we can fix any k,r ≥ 2. In iv,v we can fix any r ≥ 2. QED

In section 2.6 we prove all of our stability statements for all k,n,r-sets with five forms of stability. It is clear that it suffices to show this for C2,C5 (i.e., for r-sides and r-duplicators) with full stability. The proof for C2 (r-sides) needs only minor tweaking to handle C5 (r-duplicators) along the lines of Theorem 2.5.3.

For the reversals, we show that the statements in section 2.6 for k,n,2-sets (squares, sides, cliques, emulators, duplicators) and the last four forms of stability are strong. In particular, at the SRP level with negatively and fully stable, at the MAH level with fully min stable, at the WZC level with negatively min stable. It suffices to show this for C3,C4 (cliques, emulators) are strong. We don't want to carry out two reversals, so we work with a restricted form of C3. We need to show that C3 is covered by restricted C3. We sketch this development here.

DEFINITION 2.5.7. A restricted order invariant graph on $Q[-n,n]^k$ is an order invariant graph on $Q[-n,n]^k$ such that the following holds. For all $x,y \in Q[-n,n]^k$, if $\min(x) > \max(y) \vee \max(x) < \min(y)$, then x,y are adjacent.

THEOREM 2.5.4. For all restricted order invariant graphs G on $Q[-n,n]^k$ there exists $x_1,\ldots,x_m \in Q[-n,n]^k$ such that the following holds. The emulators of $x_1,\ldots,x_m \in Q[-n,n]^k$ are exactly the same as the cliques in G. The maximal emulators of $x_1,\ldots,x_m \in Q[-n,n]^k$ are exactly the same as the maximal cliques in G.

Proof: First enumerate the elements of $Q[-n,n]^k$ up to order equivalence, and for each such x, let x* be order equivalent to x, and arrange that for two distinct x*, all coordinates of the first are less than all coordinates of the second, or vice versa. Now enumerate the edges (x,y) in G up to order equivalence of 2k-tuples. For each such pair we assign $(x,y)^* = (z,w) \in Q[-n,n]^{2k}$, where (x,y) and (z,w) are order equivalent, and also that for distinct (x,y)*, (x',y')*, all coordinates in the first are less than all coordinate in the second, or vice versa.

Let $x_1,\ldots,x_m$ list all k-tuples used here. The emulators of $\{x_1,\ldots,x_m\} \in Q[-n,n]^k$ are exactly the same as the cliques in G. QED

When we actually work up the reversals elsewhere, and soon, we need only reverse the statements in section 2.6 for the restricted order invariant graphs on the $Q[-n,n]^k$.

2.6. FROM STABLE TO FULLY STABLE MAXIMAL $S \subseteq Q[-n,n]^k$

DEFINITION 2.6.1. $S \subseteq Q[-n,n]^k$ is stable if and only if for all order equivalent $x,y \in \{0,\ldots,n\}^k$, $x \in S \leftrightarrow y \in S$. $S \subseteq Q[-n,n]^k$ is negatively stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in Q[-n,n]^k$ are obtained from x,y by replacing zero or more x_i,y_i by p,p < 0, then $x' \in S \leftrightarrow y' \in S$. $S \subseteq Q[-n,n]^k$ is fully stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in Q[-n,n]^k$ are obtained from x,y by replacing zero or more x_i,y_i by p,p < min(xy), then $x' \in S \leftrightarrow y' \in S$.

DEFINITION 2.6.2. $S \subseteq Q[-n,n]^k$ is negatively min stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$ with the same min, if $x',y' \in Q[-n,n]^k$ are obtained from x,y by replacing zero or more of the x_i,y_i by p,p < 0, then $x' \in S \leftrightarrow y' \in S$. $S \subseteq Q[-n,n]^k$ is fully min stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$ with the same min, if $x',y' \in Q[-n,n]^k$ are obtained from x,y by replacing zero or more x_i,y_i by p,p < min(x), then $x' \in S \leftrightarrow y' \in S$.

Note that the x',y' so arising in these Definitions are of course order equivalent.

Thus we have stable, negatively stable, fully stable, negatively min stable, fully min stable. Note that each of these implies stable and each are implied by fully stable.

STABLE MAXIMALITY. SM. (WKL_0 + Con(PA)) Every k,n,r-set has a stable maximal element.

NEGATIVELY STABLE MAXIMALITY. NSM. Every k,n,r-set has a negatively stable maximal element.

FULLY STABLE MAXIMALITY. FSM. Every k,n,r-set has a fully stable maximal element.

NEGATIVELY MIN STABLE MAXIMALITY. NMSM. Every k,n,r-set has a negatively min stable maximal element.

FULLY MIN STABLE MAXIMALITY. FMSM. Every k,n,r-set has a fully min stable maximal element.

THEOREM 2.6.1. NSM, FSM are each provably equivalent to Con(SRP) over WKL_0 where the implication to Con(SRP) is provable in RCA_0. NMSM is provably equivalent to Con(WZC) over WKL_0 where the implication to Con(WZC) is provable in RCA_0. FMSM is provably equivalent to Con(MAH) over WKL_0 where the implication to Con(MAH) is provable in RCA_0. These results hold even if we use any one of the five components of these four statements NSM, FSM, NMSM, FWSM corresponding to C1,C2,C3,C4,C5, and fix r = 2 (squares, sides, cliques, emulators, duplicators).

CONSOLIDATED STABILITY MAXIMALITY TEMPLATE. CSMT. Given k,n,r. Every k,n,r-set has a (stable, negatively stable, fully stable, negatively min stable, fully min stable) maximal element.

Thus CSMT consists of 25 = 5 × 5 statements for each k,n,r, using the five forms of stability and the five kinds of k,n,r-sets, C1,C2,C3,C4,C5 of Definition 2.5.5.

CONJECTURE. CSMT under $A \leq B \leftrightarrow WKL_0$ proves $B \rightarrow A$, forms a recursive well ordering with low level computational complexity. What is the ordinal?

2.7. EXPLICIT, IMPLICIT FINITENESS AND FALSIFIABILITY

The Incompleteness presented in section 2.6 is radically different than earlier mathematical incompleteness in many ways, and in particular in terms of their logical form. These are Σ^1_1 sentences. Note that the mathematical incompleteness in Boolean Relation Theory are also Σ^1_1.

Σ^1_1 does not reflect deeper hidden aspects of the Tangibility of these statements from section 2.6 that independent of ZFC. The following gives such an indication of some hidden Tangibility.

THEOREM 2.7.1. For each of the statements in section 2.6, we can require that the maximal object be arithmetic. These strengthened statements are provably equivalent to the originals over ACA.

We can take this further as follows.

THEOREM 2.7.2. For each of the statements in section 2.6, we can require that the maximal object be a Δ^0_2 set. These strengthened statements are provably equivalent to the originals over ACA_0. These strengthened statements are seen to be Σ^0_4 statements by a quantifier calculation.

Thus these statements become arithmetical when so strengthened. But there is more hidden Tangibility to these statements from section 2.6. They are what we call implicitly Π^0_1 in the following sense.

DEFINITION 2.7.1. Let T be a first order theory whose language includes 0,S,+,•,< (on a sort for the nonnegative integers), which is recursively axiomatized and proves PFA. φ is implicitly Π^0_1 over T if and only if there is a Π^0_1 sentence ψ such that T proves $\varphi \leftrightarrow \psi$.

There are two ways of showing that the statements in section 2.6 are implicitly Π^0_1 over WKL_0. One way is to build a recursive 0,1 tree, for each statement, and show, in RCA_0, that from any maximal object given by the statement we can recover an infinite path, and from each infinite path we can recover a maximal object given by the statement. This establishes the that statement is provably equivalent to the 0,1 tree being infinite, over WKL_0 - a Π^0_1 statement. A second way is to bury tree constructions somehow in the use of the Gödel Completeness Theorem, ultimately showing that the statements in section 2.6 are provably equivalent, over WKL_0, to consistency statements. We will present both ways in the book, but here we will only present the second way.

The usual Gödel Completeness Theorem for countable models is given in two parts. Soundness asserts that if a set of sentences T (in first order predicate calculus with equality) has a countable model then T is consistent (in first order predicate calculus with equality). Completeness asserts that if a set of sentences T is consistent then it has a countable model. The natural theory in which to formulate this most general formulation for countable models is ACA since implicit in the statement is the existence of the satisfaction relation for any countable structure. It is also clear that Gödel Completeness is provable in ACA.

THEOREM 2.7.3. The statements in section 2.6 are implicitly Π^0_1 over ACA.

Proof: Let P be one of the statements in section 2.6 with parameters k,n,r. (If r is not used it is considered to be 2). We now conveniently axiomatize $S \subseteq Q[-n,n]^k$ being the required maximal object. We use <,=,-n,0,...,n and S as a k-ary predicate symbol. We assert that < is a dense linear ordering with endpoints -n,n and -n < 0 < ... < n. This takes on the form ∀∀∀∃ using -n,n,<,=. Membership in the relevant k,n,r-set takes the form ∀...∀ using S,<,=. The maximality asserts that no new k-tuple can be adjoined to S and still stay in the relevant k,n,r-set. This takes the form ∀...∀∃...∃ using S,<,=. The stability also takes the form ∀...∀ and uses the constants 0,...,n. Let T be the this finitely axiomatized theory. Now suppose the maximal object $S \subseteq Q[-n,n]^k$ for the statement exists. Then (Q[-n,n],<,S,-n,0,...,n) satisfies T. Now suppose we have a countable model (D,<',S',-n',0',...,n') of T. Then we can make an isomorphism onto (Q[-n,n],<,S*,-n,0,...,n). Isomorphisms preserve sentences and so T holds in (Q[-n,n],<,S*,-n,0,...,n). It follows that S* is a maximal object required by the statement on $Q[-n,n]^k$.

So we have shown in ACA that the statement from section 2.6 with parameters k,n,r is equivalent to the countable satisfiability of T. By the Gödel Completeness Theorem in ACA, this is equivalent to the consistency of T in first order predicate calculus with equality, which is explicitly Π^0_1 in the parameters k,n,r. Now the statement quantifies over k,n,r, and therefore the statement is provably equivalent to the Π^0_1 sentence obtained by quantifying the consistency statement over k,n,r, resulting in the desired Π^0_1 sentence. QED

THEOREM 2.7.4. The statements in section 2.6 are implicitly Π^0_1 over WKL_0.

Proof: We adapt the proof of Theorem 2.7.3. First of all, note that the theories T built in the proof of Theorem 2.7.3 consist entirely of sentences in ∀...∀∃...∃, and so there is no longer any issue on how to formulate satisfaction. However, in order to do separation with respect to satisfaction, we are going to need ACA0. In fact, there are issues here with soundness, even with ACA_0, because proofs may have arbitrary formulas. So some sort of cut elimination is needed even to carry out the proof of Theorem 2.7.4 in ACA_0 instead of ACA. Actually it is clearer and cleaner to use Herbrand's theorem. And when we do this below, we see that a lot can be done in RCA_0, except that at one point we need WKL_0.

We now work in WKL_0. First Skolemize the ∀...∀∃...∃ axioms for the maximal object by introducing function symbols in the obvious way. Let T' be this Skolemization. We can pass from a maximal object for the statement to a model of T and then to a countable model of T', all within RCA0. And from a model of T', we already have a countable model of T and then a maximal object for the statement as before, again all in RCA_0. Now we apply Herbrand's Theorem. T' is countably satisfiable if and only if every finite set of substitution instances is satisfiable with a term model, and that is provable in WKL_0. (The forward direction in RCA_0 and the reverse direction uses WKL_0). QED

We now take up falsifiability, a very strong motivator for seeking implicitly or explicitly Π^0_1 sentences independent of ZFC.

DEFINITION 2.7.2. φ is T falsifiable if and only if T proves "¬φ → (T proves ¬φ)". I.e., T proves "if φ is false then φ is provably false".

THEOREM 2.7.7. (RCA_0) Let T be a recursively axiomatized theory that proves PFA. Every sentence implicitly Π^0_1 over T is T falsifiable. Furthermore assume T is finitely axiomatized. Every T falsifiable sentence is implicitly Π^0_1 over T augmented with induction for all formulas. In fact, induction only for formulas of quantifier complexity at most the maximum of T,φ is needed.

Proof: Let T be as given. Let φ be implicitly Π^0_1 over T. Let ψ be Π^0_1 where T proves φ ↔ ψ. Arguing s T, assume ¬φ. Then ¬ψ, and so T proves ¬ψ. Now T sees that T proves φ ↔ ψ. Therefore T sees that T proves ¬φ.

Now suppose φ is T falsifiable. We claim φ ↔ Con(T+φ) is provable in T with induction. To see this, argue in T with induction. We see if φ is false then φ is refutable in T. Hence ¬φ → ¬Con(T+φ). Now suppose ¬Con(T+φ). By cut elimination, we get a refutation of φ in T with a proof of quantifier complexity at most that of the maximum of the quantifier complexities of T,φ. We then perform an induction in T to derive ¬φ. QED

COROLLARY 2.7.8. The statements in section 2.6 are all WKL_0 falsifiable.

Proof: By Theorem 2.7.5, the statements are implicitly Π^0_1 over WKL_0. By Theorem 2.7.6, the statements are all WKL_0 falsifiable. QED

Falsifiability is a highly desirable property of an Incompleteness from the point of view of general scientific practice. A scientific hypothesis is generally accepted as meaningful if and only if it can be refuted, at least in principle, by an experiment. Such refutation by experimentation is a case of falsifiability since the results of a properly conducted experiment are supposed to be incontrovertible.

For issues of Tangibility, the results of the next two sections are more satisfying than the results of this section. There we give explicitly Π^0_1 forms that preserve all or virtually all of the fundamental naturalness of the original statements from section 2.6.

2.8. STABILITY IN FINITE WEAKLY MAXIMAL S ⊆ $Q[-n,n]^k$

We have found a way to weaken the notion of maximal. We can always find finite weakly maximal sets. This keeps us in the realm of finite sets of rational vectors.

We now use Definition 2.2.4.

DEFINITION 2.8.1. Let S ⊆ $Q[-n,n]^k$. The extendings of S are the images of the N order invariant $H:Q[-n,n]^{k^2} \to Q[-n,n]^k$ on S^k that properly contain S.

THEOREM 2.8.1. (EFA) Every k,n,r-set includes some finite element without including any of its extendings.

STABLE WEAK MAXIMALITY. SWM. (WKL_0 + Con(PA)) Every k,n,r-set includes some stable finite element without including any of its extendings.

NEGATIVELY STABLE WEAK MAXIMALITY. NSWM. Every k,n,r-set includes some negatively stable finite element without including any of its extendings.

FULLY STABLE WEAK MAXIMALITY. FSWM. Every k,n,r-set includes some fully stable finite element without including any of its extendings.

NEGATIVELY MIN STABLE WEAK MAXIMALITY. NMSWM. Every k,n,r-set includes some negatively min stable finite element without including any of its extendings.

FULLY MIN STABLE WEAK MAXIMALITY. FMSWM. Every k,n,r-set includes some fully min stable finite element without including any of its extendings.

These five statements are explicitly Π^0_2. However, we can place an exponential bound on the cardinality of the maximal set and apply quantifier elimination for (Q,<) to put them in explicitly Π^0_1 form. We can alternatively avoid using quantifier elimination by directly bounding the magnitude of the numerators and denominators involved.

THEOREM 2.8.2. NSWM, FSWM are each provably equivalent to Con(SRP) over WKL_0 where the implication to Con(SRP) is provable in RCA_0. NMSWM is provably equivalent to Con(WZC) over WKL_0 where the implication to Con(WZC) is provable in RCA_0. FMSWM is provably equivalent to Con(MAH) over WKL_0 where the implication to Con(MAH) is provable in RCA_0. These results hold even if we use any of the five components of these four statements M, NMSWM, FWSWM corresponding to C1,C2,C3,C4,C5, and fix r = 2 (squares, sides, cliques, emulators, duplicators).

2.9. STABILITY IN WEAKLY MAXIMAL S ⊆ $\{-nt,...,nt\}^k$

Since only finite sets of rational vectors are involved in section 2.8, it is fairly easy to recast section 2.8 using integer vectors instead of rational vectors. More specifically, we move from the $Q[-n,n]^k$ to the $\{-nt,...,nt\}^k$, where t >> k,n (t >> k,n,r if the parameter r is involved). The 0,...,n in Q[-n,n] are replaced by the 0,t,...,nt in {-nt,...,nt}. The Tangible Incompleteness here are obviously explicitly Π^0_3 (because of the >>), and with easy a priori estimates replacing the >>, become explicitly Π^0_1.

DEFINITION 2.9.1. Fix k,n,t,r. The k,n,t,r-sets are the sets of subsets of $\{-nt,...,nt]^k$ of the following five forms:
D1. The set of r-cubes in an order invariant E ⊆ $\{-nt,...,nt\}^k$.
D2. The set of r-sides in an order invariant E ⊆ $\{-nt,...,nt\}^{kr}$.
D3. the set of r-cliques in an order invariant r-graph on $\{-nt,...,nt\}^k$.
D4. The set of r-emulators of subsets of $\{-nt,...,nt\}^k$.
D5. The set of r-duplicators of subsets of $\{-nt,...,nt\}^k$.

DEFINITION 2.9.2. $S \subseteq \{-nt,...,nt\}^k$ is stable if and only if for all order equivalent $x,y \in \{0,t,...,nt\}^k$, $x \in S \leftrightarrow y \in S$. $S \subseteq \{-nt,...,nt\}^k$ negatively stable if and only if for all order equivalent $x,y \in \{0,t,...,nt\}^k$, if $x',y' \in \{-nt,...,nt\}^k$ are obtained from x,y by replacing zero or more x_i,y_i by $p,p < 0$, then $x' \in S \leftrightarrow y' \in S$. $S \subseteq \{-nt,...,nt\}^k$ is fully stable if and only if for all order equivalent x,y $\in \{0,t,...,nt\}^k$, if $x',y' \in \{-nt,...,nt]^k$ are obtained from x,y by replacing zero or more x_i,y_i by $p,p < \min(xy)$, then $x' \in S \leftrightarrow y' \in S$.

DEFINITION 2.9.3. $S \subseteq \{-nt,...,nt]^k$ is negatively min stable if and only if for all order equivalent $x,y \in \{0,t,...,nt\}^k$ with the same min, if $x',y' \in \{-nt,...,nt]^k$ are obtained from x,y by replacing zero or more of the x_i,y_i by $p,p < 0$, then $x' \in S \leftrightarrow y' \in S$. $S \subseteq \{-nt,...,nt]^k$ is fully min stable if and only if for all order equivalent $x,y \in \{0,t,...,nt\}^k$ with the same min, if $x',y' \in \{-nt,...,nt]^k$ are obtained from x,y by replacing zero or more x_i,y_i by $p,p < \min(x)$, then $x' \in S \leftrightarrow y' \in S$.

DEFINITION 2.9.4. Let $S \subseteq \{-nt,...,nt\}^k$. The extendings of S are the images of the tN order invariant $H:\{-nt,...,nt\}^{k^2} \to \{-nt,...,nt\}^k$ on S^k that properly contain S.

THEOREM 2.9.1. (EFA) Every k,n,t,r-set includes some finite element without including its proper supersets. I.e., every k,n,t,r-set has a maximal element.

STABLE WEAK MAXIMALITY/Z. SWM/Z. (WKL_0 + Con(PA)) Let t >> k,n,r. Every k,n,t,r-set includes some stable finite element without including any of its extendings.

NEGATIVELY STABLE WEAK MAXIMALITY/Z. NSWM/Z. Let t >> k,n,r. Every k,n,t,r-set includes some negatively stable finite element without including any of its extendings.

FULLY STABLE WEAK MAXIMALITY/Z. FSWM/Z. Let t >> k,n,r. Every k,n,t,r-set includes some fully stable finite element without including any of its extendings.

NEGATIVELY MIN STABLE WEAK MAXIMALITY/Z. NMSWM/Z. Let t >> k,n,r. Every k,n,t,r-set includes some min stable finite element without including any of its extendings.

FULLY MIN STABLE WEAK MAXIMALITY/Z. FMSWM/Z. Let t >> k,n,r. Every k,n,t,r-set includes some stable finite element without including any of its extendings.

These five statements are explicitly Π^0_3. However, we can replace t >> k,n,r by an exponential bound, thereby putting them in explicitly Π^0_1 form.

THEOREM 2.9.2. NSWM/Z, FSWM/Z are both provably equivalent to Con(SRP) over WKL_0 where the implication to Con(SRP) is provable in EFA. NMSWM/Z is provably equivalent to Con(MAH) over EFA. NMSWM/Z is provably equivalent to Con(ZC) over EFA. These results hold even if we use any of the five components of these four statements NSWM/Z, FSWM/Z, NMSWM/Z, FWSWM/Z corresponding to D1,D2,D3,D4,D5, and fix r = 2 (squares, sides, cliques, emulators, duplicators).

2.10. EMULATORS IN $Q[-1,1]^2$ AND MATHEMATICALLY GIFTED YOUTH

See [Fr20]. We restate the special case of emulators and negative stability.

NEGATIVELY STABLE MAXIMALITY/EMULATORS. NSM/EM. Every subset of $Q[-n,n]^k$ has a negatively stable maximal emulator.

With arbitrary k,n this is equivalent to Con(SRP) over WKL_0, and we cannot expect to have Mathematically Gifted Youth really engage with NSM/EM.

However, consider the special case of n = 1 and k = 2, with $Q[-1,1]^2$. In this case, the main definitions take on the following simplified forms which we use for the Mathematically Gifted Youth.

S is an emulator of $E \subseteq Q[-1,1]^2$ if and only if $S \subseteq Q[-1,1]^2$, and for all $x,y \in S$ there exists $z,w \in E$ such that xy is order equivalent to zw.

S is a maximal emulator of $E \subseteq Q[-1,1]^2$ if and only if S is an emulator of $E \subseteq Q[-1,1]^2$ which does not remain an emulator if a new pair is inserted into S.

$S \subseteq Q[-1,1]^2$ is stable if and only if $(0,0) \in S \leftrightarrow (1,1) \in S$.

$S \subseteq Q[-1,1]^2$ is negatively stable if and only if

i. $(0,0) \in S \leftrightarrow (1,1) \in S$.
ii. For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$.
iii. For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.

MAIN STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, has an algorithmic negatively stable maximal emulator.

The proof of this Main Student Theorem is given in detail in [Fr20] for the students to work through. It is extremely elementary and is straightforwardly conducted in RCA_0. There are loads of details to be checked.

The "easiest" proof of Main Student Theorem is by a transfinite recursion of length a large cardinal, building a big maximal emulator of $E \subseteq Q[-1,1]^2$ as a subset $S \subseteq \lambda$ of the large cardinal λ. The large cardinal combinatorics of λ is applied very easily to get a set of indiscernibles, of which we only use the first two. Then we pull the development down into the countable and therefore into $Q[-1,1]^2$ with 1 as the second indiscernible.

There is nothing in this highly abstract and sophisticated transfinite construction that suggests any kind of algorithmic computability. Hence this only gives the Main Student Theorem without "algorithmic".

With some care using constructible set methods, we can get away with using ω_1 instead of λ so we are squarely within ZFC. More effort shows that we can get away with Π^i_2-CA_0, a subsystem of Z_2.

OPEN PROBLEM. Does every $E \subseteq Q[-1,1]^2$ have an algorithmic negatively stable maximal emulator? What about subsets of $Q[-1,1]^2$ of cardinality 4?

Our fancy proof actually proves the following in Z_2.

THEOREM. Every $E \subseteq Q[-n,n]^2$ has a fully stable maximal r-emulator.

There is real hope that this stronger Theorem can be refuted with algorithmic, and reversed to $Con(Z_2)$.

2.11. FORMAL SYSTEMS USED

EFA Exponential function arithmetic. Based on 0, successor, addition, multiplication, exponentiation and bounded induction. Same as $I\Sigma_0$(exp), [HP93], p. 37, 405.

RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].

WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].

ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics. [Si99,09].

Z_2 Second order arithmetic as a two sorted first order theory. [Si99,09].

WZ(C) Weak Zermelo set theory. Uses bounded comprehension instead of full comprehension.

Z(C) Zermelo set theory (with the axiom of choice). This is the same as ZF(C) without the axiom scheme of replacement.

ZF(C)\P ZF(C) without the power set axiom. [Ka94]

ZF(C) Zermelo Frankel set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Ka94].

MAH[k] ZFC + $(\exists\lambda)(\lambda$ is strongly k-Mahlo), for fixed k. Defined by induction on k. λ is strongly 0-Mahlo if and only if λ is a strong limit cardinal. λ is strongly j-Mahlo if and only if every closed unbounded subset of λ contains a strongly (j-1)-Mahlo cardinal.

MAH ZFC + $(\exists\lambda)(\lambda$ is strongly k-Mahlo), as a scheme in k.

MAH+ ZFC + (∀k)(∃λ)(λ is a strongly k-Mahlo cardinal).

SRP[k] ZFC + (∃λ)(λ has the k-SRP), for fixed k. In every partition of the unordered k-tuples from λ into two pieces, there is a stationery subset of λ all of whose unordered k-tuples lie in the same piece.

SRP ZFC + (∃λ)(λ has the k-SRP), as a scheme in k.

SRP^+ ZFC + (∀k)(∃λ)(λ has the k-SRP).

2.12. REFERENCES

[Co63,64] P.J. Cohen, The independence of the continuum hypothesis. Proc. Nat. Acad. Sci. U.S.A. 50 (1963), 1143-1148; 51, 105-110.

[Fr01] H. Friedman, Subtle cardinals and linear orderings, Annals of Pure and Applied Logic, Volume 107, Issues 1-3, 15 January 2001, 1-34.

[Fr11] H. Friedman, My Forty Years on His Shoulders, in: Kurt Gödel and the Foundations of Mathematics, Horizons of Truth, Ed. Matthias Baaz, Christos H. Papadimitrou, Hilary W. Putnam, Dana S. Scott, Charles L. Harper, Jr., Cambridge University Press, 2011, p. 399 - 432.

[Fr20] Tangible Incompleteness for Gifted Youth: Emulation Theory, July 16, 2020, 53 pages. https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/ #111.

[Go31] K. Gödel, On formally undecidable propositions of Principia mathematica and related systems I, in: [Go86-03], Vol. 1 (145-195), 1931.

[Go40] K. Gödel, The consistency of the axiom of choice and the generalized continuum hypothesis with the axioms of set theory, Annals of mathematics studies, vol. 3 (Princeton University Press), in:

[Go86-03] K. Gödel, Collected works, Vol. I-V, Oxford: Oxford University Press (S. Feferman et al, editors).

[Gr62] O.A. Gross, Preferential arrangements, American Mathematical Monthly 69 (1962), p. 4-8.

[HP93] P. Hajek, P. Pudlak, Metamathematics of First-Order Arithmetic, Perspectives in Mathematical Logic, Springer, 1993.

[Ka94] A. Kanamori, The Higher Infinite, Perspectives in Mathematical Logic, Springer, 1994.

[Ra30] F.P. Ramsey, On a Problem of Formal Logic, Proc. London Math. Soc. 39, 264-286.

[Si99,09] S. Simpson, Subsystems of Second Order Arithmetic, Springer Verlag, 1999. Second edition, ASL, 2009.

[Slxx] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, 1964-, https://oeis.org/A000670.

3. SET COMPLEMENTATION

3.1. PRELIMINARIES

DEFINITION 3.1.1. N, Z^+, Q are, respectively, the set of nonnegative integers, the set of positive integers, and the set of rationals. We use i,j,k,n,m,r,s,t with or without subscripts for positive integers, unless otherwise indicated. We use p,q with or without subscripts for rational numbers, unless otherwise indicated. We use A,B,C,D,E,K,S,T,U,V,W with or without subscripts for sets unless otherwise indicated. We work with the closed rational intervals Q[-n,n] = Q ∩ [-n,n]. min(x), max(x) is the least, greatest coordinate of x ∈ Q^k, respectively. $x_1...x_n$ is the concatenation of the tuples $x_1,...,x_n$.

DEFINITION 3.1.2. A* is the set of all finite sequences from A (empty sequence allowed). f::A → B if and only if f is a function with dom(f) ⊆ A ∧ rng(f) ⊆ B. Let h::Q → Q. Extend h coordinatewise to h::Q^k → Q^k. Extend further to h::POW(Q*) → POW(Q*) via the action of h on elements of Q*. The upper shift, ush. maps Q into Q by p if p < 0; p+1 otherwise. $S^{\leq}$ = {x ∈ S: x_1 ≤ ... ≤ x_k}. S|<p = S ∩ $\{-\infty,p)^k$.

DEFINITION 3.1.3. Let R ⊆ Q^{2k}. When we write R[S] or $R_<$[S] we are committing to S ⊆ Q^k. R[S] = {y ∈ Q^k: (∃x ∈ S)((x,y) ∈ R)}. $R_<$[S] = {y ∈ Q^k: (∃x ∈ S)((x,y) ∈ R ∧ max(x) < max(y))}. Let A,B,C be subsets of Q^n,Q^m,Q^r, respectively. A complements B on C if and only if n = m = r ∧ (∀x ∈ C)(x ∈ A ↔ x ∉ B). S# is the least E^k containing S ∪ $\{0\}^k$. fld(S) is the set of all coordinates of elements of S.

3.2. UPPER SHIFT, Θ COMPLEMENTATION IN Q^k

We begin with our favorite Incompleteness at the level of SRP in this section 3.2.

UPPER SHIFT COMPLEMENTATION/1. USC/1. For all order invariant R ⊆ Q^{2k}, some S = S#\$R_<$[S] ⊇ ush(S).

In preparation for SUSC below, we restate this as follows.

UPPER SHIFT COMPLEMENTATION/2. USC/2. For all order invariant R ⊆ Q^{2k}, some S ⊇ ush(S) complements $R_<$[S] on S#.

We can weaken this as follows, in preparation for SUSC.

UPPER SHIFT COMPLEMENTATION/3. USC/3. For all order invariant R ⊆ Q^{2k}, some S ⊇ ush(S) complements $R_<$[S] on $S\#^{\leq}$.

THEOREM 3.2.1. USC/1-3 are provably equivalent to Con(SRP) over ACA_0 where RCA_0 suffices for the forward implication.

We now go much further, into the HUGE cardinal hierarchy. To do this, we must strengthen the containment relation.

DEFINITION 3.2.1. Let $S,T \subseteq Q^k$. $S \supseteq^* T$ if and only every $T|\leq n$ is $\{x \in S: S(y_{11},\ldots,y_{1k}) \wedge \ldots \wedge S(y_{m1},\ldots,y_{mk})\}$, where the y's are among $x_1,\ldots,x_k$ and constants from Q.

STRONG UPPER SHIFT COMPLEMENTATION. SUSC. For all order invariant $R \subseteq Q^{2k}$, some $S \supseteq^*$ ush(S) complements $R_<[S]$ on $S\#^{\leq}$.

THEOREM 3.2.2. SUSC is provably equivalent to Con(HUGE) over WKL_0, and implies Con(HUGE) over RCA_0.

Note that ush:Q → Q is a rational piecewise linear function (with finitely many pieces).

PIECEWISE LINEAR TEMPLATE. PLT. Let f:Q → Q be rational piecewise linear. USC with ush replaced by f. SUSC with ush replaced by f.

CONJECTURE. Every instance of PLT is refutable in RCA_0 or provable in SRP^+. with ⊇, and $HUGE^+$ with ⊇*. PLT under A ≤ B ↔ WKL_0 proves B → A, forms a recursive quasi well ordering. What ordinal?

Below we will be using functions that are not quite piecewise linear. They are allowed to involve N.

DEFINITION 3.2.2. f::Q → Q is partial N piecewise linear if and only if the graph of f is quantifier free definable in (Q,N,<,+).

We now use a particular carefully constructed partial function Θ::Q → Q which is obviously N piecewise linear, and of course much better than that.

DEFINITION 3.2.3. Θ::Q → Q is defined by p if $p < -1$; p/2 if $-1 \leq p < 0$; p + 1 if $p \in N$. We extend to Θ::Q* → Q* and Θ:POW(Q*) → POW(Q*) as we did for ush.

STRONG Θ COMPLEMENTATION. SΘC. For all order invariant $R \subseteq Q^{2k}$, some $S \supseteq^* \Theta(S)$ complements $R_<[S]$ on $Q^{k_{\leq}}$.

THEOREM 3.2.3. WKL_0 proves Con(I2) → SΘC → Con(I3). SΘC is provable in I2 but not in I3, assuming I3 is consistent.

N PIECEWISE LINEAR TEMPLATE. NPLT. Let f::Q → Q be N partial rational piecewise linear. SUSC with ush replaced by f.

CONJECTURE. Every instance of NPLT is refutable in RCA_0 or provable in I2. NPLT under A ≤ B ↔ WKL_0 proves B → A, forms a recursive quasi well ordering.

3.3. FORMAL SYSTEMS USED

See Section 2.9 for any systems not covered below.

λ is n-huge if and only if there is a nontrivial elementary embedding j:V(λ) → V(μ). where λ = j...j(κ) with k j's and κ is the critical point of j.

HUGE is ZFC + $\{$there exists k-huge λ$\}_k$

HUGE+ is ZFC + (∀k)(∃λ)(λ is k-huge)

I3 is ZFC + there exists a nontrivial elementary embedding from some V(λ) into V(λ).

I2 is NBG + AxC + there exists a nontrivial elementary embedding from V into some transitive class M such that V(λ) ⊆ M, where λ is the least fixed point of j above its critical point.

I1 is ZFC + there exists a nontrivial elementary embedding from some V(λ+1) into V(λ+1).