SIMPLIFIED AXIOMS FOR CLASS THEORY

by
Harvey M. Friedman
Distinguished University Professor
of Mathematics, Philosophy, Computer Science
Emeritus
Ohio State University
Columbus, Ohio
https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/
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ABSTRACT. The usual axiomatization of the theory of classes is the system NBG of Von Neumann, Bernays, and Gödel. NBG is well known to be finitely axiomatized, which for many audiences is greatly preferable to infinite axiomatizations with logical schemes. However, the known finite axiomatizations do not reflect the kind of set theoretic constructions so familiar in ordinary mathematical and set theoretic practice. Rather they have the unmistakable appearance of axioms specifically designed to facilitate the logician's proof of the finite axiomatizability of NBG. Here we offer a finite axiomatization which does reflect familiar mathematical and set theoretic constructions. This opens up many rich opportunities for improvements and insufficiency results.

1. STANDARD NBG AXIOMATIZATION

NBG is a theory of classes credited to Von Neumann, Bernays, and Gödel. There are a few slightly different standard versions of NBG. We use one with the language $\in,=,M$, where M is a unary predicate symbol read "x is a set". Everything is a class, but some classes are sets.

The symbol set for the language of class theory, LCT, is $\in,=,M,\neg,\wedge,\vee,\rightarrow,\leftrightarrow,\exists,\forall,(,),v_1,v_2,\ldots$. To adhere to the idea that formulas are finite strings in a finite alphabet, raise the subscripts on the variables from subscripts to center line, and use base 2 expansions. Thus the official finite symbol set for LCT is $\in,=,M,\neg,\wedge,\vee,\rightarrow,\leftrightarrow,\exists,\forall,(,),v,0,1$. However, we will completely ignore this detail.

The formulas of LCT are inductively defined as follows.

a. For $i,j \geq 1$, $v_i = v_j$, $v_i \in v_j$, and $M(v_i)$ are formulas.
b. If φ,ψ are formulas then $(\neg\varphi),(\varphi \wedge \psi),(\varphi \wedge \psi),(\varphi \rightarrow \psi),(\varphi \leftrightarrow \psi)$ are formulas.
c. If φ is a formula and $i \geq 1$ then $(\exists v_i)(\varphi),(\forall v_i)(\varphi)$ are formulas.

We need the notion of a limited formula in LCT. These are the formulas (in LCT) where all quantification is over sets only. The limited formulas are inductively defined as follows.

a. For $i,j \geq 1$, $v_i = v_j$, $v_i \in v_j$, and $M(v_i)$ are limited formulas.
b. If φ,ψ are formulas then $(\neg\varphi),(\varphi \wedge \psi),(\varphi \wedge \psi),(\varphi \rightarrow \psi),(\varphi \leftrightarrow \psi)$ are limited formulas.
c. If φ is a formula and $i \geq 1$ then $(\exists v_i)(M(v_i) \wedge \varphi),(\forall v_i)(M(v_i) \rightarrow \varphi)$ are limited formulas.

Finally we use relativized quantifiers as an abbreviation. This works as follows. $(\exists v_i \in v_j)(\varphi)$ abbreviates $(\exists v_i)(v_i \in v_j \wedge \varphi)$. $(\forall v_i \in v_j)(\varphi)$ abbreviates $(\forall v_i)(v_i \in v_j \rightarrow \varphi)$. $(\exists v_i|M)(\varphi)$ abbreviates $(\exists v_i)(M(v_i) \wedge \varphi)$. $(\forall v_i|M)(\varphi)$ abbreviates $(\forall v_i)(M(v_i) \rightarrow \varphi)$.

Here is a standard infinite axiomatization.

Below we give the English description of each axiom (scheme), where we use "class" for arbitrary element, and "set" for classes with M.

1. SETS. The sets are the elements. $M(v_1) \leftrightarrow (\exists v_2)(v_1 \in v_2))$.

2. EXTENSIONALITY. If v_1, v_2 have the same elements then $v_1 = v_2$. $(\forall v_3)(v_3 \in v_1 \leftrightarrow v_3 \in v_2) \rightarrow v_1 = v_2$.
3. PAIRING. For sets v_1, v_2, $\{v_1, v_2\}$ is a set. $M(v_1) \wedge M(v_2) \rightarrow (\exists v_3|M)(\forall v_4)(v_4 \in v_3 \leftrightarrow v_4 = v_1 \vee v_4 = v_2))$.
4. UNION. For sets v_1, $\cup v_1$ is a set. $M(v_1) \rightarrow (\exists v_2|M)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\exists v_4 \in v_1)(v_3 \in v_4)))$.
5. POWER SET. If v_1 is a set then $\{v_2: v_2 \subseteq v_1\}$ is a set. $M(v_1) \rightarrow (\exists v_2|M)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\forall v_4)(v_4 \in v_3 \rightarrow v_4 \in v_1)))$.
6. INFINITY. There is a nonempty set v_1 with no $\in$ maximal element. $(\exists v_1|M)(\exists v_2)(v_2 \in v_1) \wedge (\forall v_2 \in v_1)(\exists v_3 \in v_1)(v_2 \in v_3))$.
7. CLASS FOUNDATION. Every nonempty class v_1 has an $\in$ minimal element v_2. $v_2 \in v_1 \rightarrow (\exists v_2 \in v_1)(\forall v_3 \in v_1)(\neg v_3 \in v_2))$.
8. COLLECTION SCHEME. If every v_1 in a set v_2 is related to some set v_3 by a limited formula then there exists a set v_4 such that every v_1 in v_2 is related to some v_3 in v_4 by that limited formula. $M(v_2) \wedge (\forall v_1 \in v_2)(\exists v_3|M)(\varphi) \rightarrow (\exists v_4|M)(\forall v_1 \in v_2)(\exists v_3 \in v_4)(\varphi))$, where φ is limited and v_4 is not in φ.
9. LIMITED COMPREHENSION SCHEME (LCS). There is a class v_1 consisting of all sets v_2 obeying a limited formula in LCT. $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (M(v_2) \wedge \varphi))$, where φ is a limited formula in LCT and v_1 is not in φ.

This axiomatization avoids the use of certain annoying details in some other axiomatizations. In particular, the use of $\emptyset, \cup, \{\ \}$ in Infinity, and the use of ! (uniqueness quantifier) in Replacement. The latter is annoying because $(\exists! v_i)(\varphi)$ expanded out requires some care (clash of variables). Of course, variable clashing in a more direct form has to be handled in Collection and Limited Comprehension. Here we handle this by simply banning v_4 in φ and v_1 in φ, respectively, which is simpler than bringing in free variables.

Note that we have used one of the many standard versions of Infinity. It is the one standard version that seems simplest to state in primitive notation. It is well known that we do not have to be really careful about how we formulate Infinity in this context. All reasonable formulations of Infinity seem to be equivalent in the presence of the other axioms. However, it is not known just how to formulate this idea rigorously.

In particular, we can think of Infinity as calling for a single set that is some sort of universe for finite set theory. From this point of view it is natural to use a strong form that asserts the existence of a transitive set A closed under the unordered pair, union (unary), and power set operations. This form of Infinity is well known to be derivable from the other axioms.

Gödel proved that we obtain a logically equivalent system if we use the ostensibly weaker Set Foundation instead of Class Foundation. I.e., we require that v_1 be a set in axiom 7.

2. NEW AXIOMS FOR NBG

In contrast to NBG, our NBG* is presented with 15 simple axioms, using a number of abbreviations that greatly improve readability. The language of NBG* is the same as the language LCT of NBG.

Let $i \geq 3$.

$v_i = \{v_1,v_2\}$ for $(\forall v_{i+1})(v_{i+1} \in v_i \leftrightarrow v_{i+1} = v_1 \vee v_{i+1} = v_2)$.

$v_i = \cup v_1$ for $(\forall v_{i+1})(v_{i+1} \in v_i \leftrightarrow (\exists v_{i+2} \in v_1)(v_2 \in v_{i+2}))$.

$v_i = v_1 \cap v_2$ for $(\forall v_{i+1})(v_{i+1} \in v_i \leftrightarrow v_{i+1} \in v_1 \wedge v_{i+1} \in v_2)$.

$v_i \subseteq v_1$ for $(\forall v_{i+1})(v_{i+1} \in v_i \rightarrow v_{i+1} \in v_1)$.

$v_2 \subseteq \cup v_1$ for $(\forall v_3 \in v_2)(\exists v_4 \in v_1)(v_3 \in v_4)$.

$v_2 \subseteq \cup v_3$ for $(\forall v_1 \in v_2)(\exists v_4 \in v_3)(v_1 \in v_4)$.

Here are the 15 axioms of NBG*. It starts with all of the axioms of NBG other than the Replacement and Limited Comprehension Schemes, without any change. It is followed by 8 new axioms.

1. SETS.
2. EXTENSIONALITY.
3. PAIRING.
4. UNION.
5. POWER SET.
6. INFINITY.
7. FOUNDATION.

A. BOOLEAN. The classes are closed under complement and pairwise union. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\neg v_3 \in v_1 \wedge M(v_3))) \wedge (\exists v_3)(v_3 = v_1 \cup v_2)$.
B. CLASS UNION. The union of every class v_1 is a class. $(\exists v_3)(v_3 = \cup v_1)$.
C. LOCAL PAIRING. $\{\{v_1,v_2\}: v_1 \in v_3 \wedge v_2 \in v_4\}$ is a class. $(\exists v_5)(\forall v_6)(v_6 \in v_5 \leftrightarrow (\exists v_1 \in v_3)(\exists v_2 \in v_4)(v_6 = \{v_1,v_2\}))$.
D. LOCAL UNION. $\{\cup v_1: v_1 \in v_2\}$ is a class. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow (\exists v_1 \in v_2)(v_4 = \cup v_1))$.
E. LOCAL INTERSECTION. $\{v_1 \cap v_2: v_1 \in v_3 \wedge v_2 \in v_4\}$ is a class. $(\exists v_5)(\forall v_6)(v_6 \in v_5 \leftrightarrow (\exists v_1 \in v_3)(\exists v_2 \in v_4)(v_6 = v_1 \cap v_2))$.
F. UNION MEMBERSHIP. $\{v_1: \cup v_1 \in v_2\}$ is a class. $(\exists v_3)(\forall v_1)(v_1 \in v_3 \leftrightarrow (\exists v_5 \in v_2)(v_5 = \cup v_1))$.
G. EPSILON CLASS. $\{\{v_1,v_2\}: v_1 \in v_2\}$ is a class. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow (\exists v_1)(\exists v_2)(v_4 = \{v_1,v_2\} \wedge v_1 \in v_2))$.
H. CAPTURE. For all classes v_1 and sets $v_2 \subseteq \cup v_1$, there exists a set $v_3 \subseteq v_1$ such that $v_2 \subseteq \cup v_3$. $(\forall v_1)(\forall v_2)(M(v_2) \wedge v_2 \subseteq \cup v_1 \rightarrow (\exists v_3)(M(v_3) \wedge v_3 \subseteq v_1 \wedge v_2 \subseteq \cup v_3))$.

LEMMA 2.1. NBG* is logically provable in NBG.

Proof: Derive each of axioms A-H in NBG as follows.

Boolean. Obvious by LCS.

Class Union. Obvious by LCS.

Local Pairing. By pairing, all of the relevant $\{v_1,v_2\}$ are sets. Apply LCS.

Local Union. By Union, all of the relevant ∪v1 are sets. Apply LCS.

Local Intersection. By LCS, all of the relevant $v_1 \cap v_2$ are sets. Apply LCS.

Union Inclusion. Suppose $\cup v_1 \subseteq v_2$. Then every element of v1 is a subset of v_2. Hence every element of v_1 lies in the power set of v_2 which is a set (Power Set). So v_1 is a subset of the power set of v_2. Hence v_1 lies in the power set of the power set of v_2, and so the relevant v_1 all lie in a single set. Now apply LCS.

Epsilon Class. By pairing, all of the relevant $\{v_1,v_2\}$ are sets. Apply LCS.

Capture. Fix class v_1. Fix set $v_2 \subseteq \cup v_1$. I.e., every element of v_2 is an element of an element of v_1. For each $v_4 \in v_2$ let $f(v_4)$ be the least ordinal α such that v_4 is an element of an element of $v_1 \cap V(\alpha)$. By Replacement, let β be greater than all $f(v_4)$. Set $v_3 = v_1 \cap V(\beta)$.

QED

3. EQUIVALENCE WITH NBG

We first prove the Limited Comprehension Scheme from NBG*. This is the essence of the proof. We then prove Collection using axiom H.

DEFINITION 3.1 For sets x, $\wp(x)$ is the set of all subsets of x.

UNIVERSAL SET. The class V of all sets exists.

Proof: Let x be a set by Infinity. By Boolean, there is a complement $C(x)$. By Boolean, $x \cup C(x)$ is a class, and consists of all sets. QED

LEMMA 3.1. Every subclass of a set is a set. In Local Pairing, if v_3,v_4 are sets then v_5 is a set. In Local Union if v_2 is a set then v_3 is a set. In Local Intersection if v_3,v_4 are sets then v_5 is a set. In Union Membership if v_2 is a set then v_3 is a set.

Proof: Every subclass of a set is an element of its power set, and therefore is a set. For Local Pairing note that v_5 consists of subsets of $v_1 \cup v_2$, and so $v_5 \subseteq \wp(v_1 \cup v_2)$, and hence a subclass of a set. For Local Union, note that every element of every $\cup v_1$, $v_1 \in v_2$, is an element of an element of an element of v_2. Hence every $\cup v_1$, $v_1 \in v_2$, is a subclass of $\cup v_2$. Hence every $\cup v_1$, $v_1 \in v_2$, is an element of $\wp\cup v_2$. Hence v_3 is a subclass of $\wp\cup v_2$ and therefore a set. For

Local Intersection note that v_5 consists of subsets of elements of v_3, or subsets of $\bigcup v_3$, or elements of $\wp\bigcup v_3$, and hence v_5 is a subclass of $\wp\bigcup v_3$, and therefore a set. In Union Membership, Assume $\bigcup v_1 \in v_2$. Then every element of $\bigcup v_1$ is an element of $\bigcup v_2$, and so $\bigcup v_1 \subseteq \bigcup v_2$. Now every element of v_1 is a subset of $\bigcup v_1$, and therefore $v_1 \subseteq \wp\bigcup v_2 \wedge v_1 \in \wp\wp\bigcup v_2$. Therefore v_3 is a subclass of $\wp\wp\bigcup v_2$, and therefore is a set. QED

EMPTY SET. $\emptyset$ is a set.

Proof: $\emptyset$ is a class by Boolean, taking the complement of V. Since $\emptyset$ is a subclass of the set given by Infinity, $\emptyset$ is a set. QED

LOCAL SINGLETON. $\{\{x\}: x \in y\}$ is a class. If y is a set then $\{\{x\}: x \in y\}$ is a set.

Proof: Suppose $y \neq V$, and let $a \in V\backslash y$. By Local Pairing, let $A = \{\{x,z\}: x,z \in y\}$, $B = \{\{x,a\}: x \in y\}$. Apply Local Intersection to A,B to obtain class $C = \{\{x\}: x \in y\}$ or $\{\{x\}: x \in y\} \cup \{\emptyset\}$. By Boolean, $\{\{x\}: x \in y\}$ is a class. Now we have $\{\{x\}: x \in V\backslash\{\emptyset\}\}$ is a class, and so $\{\{x\}: x \in V\}$ is a class by Boolean. Suppose y is a set. Then $\{\{x\}: x \in y\}$ is a subclass of $\wp y$, and hence a set. QED

LOCAL PAIRWISE UNION. $\{x_1 \cup x_2: x_1 \in x_3 \wedge x_2 \in x_4\}$ is a class. If x_3,x_4 are sets then $\{x_1 \cup x_2: x_1 \in x_3 \wedge x_2 \in x_4\}$ is a set.

Proof: $A = \{\{x_1,x_2\}: x_1 \in x_3 \wedge x_2 \in x_4\}$ is a class by Local Pairing. $\{\bigcup y: y \in A\} = \{x_1 \cup x_2: x_1 \in x_3 \wedge x_2 \in x_4\}$ is a class by Local Union. Suppose x_1,x_2 are sets. $A = \{\{x_1,x_2\}: x_1 \in x_3 \wedge x_2 \in x_4\}$ is a set by Lemma 3.1. $\{\bigcup y: y \in A\} = \{x_1 \cup x_2: x_1 \in x_3 \wedge x_2 \in x_4\}$ is a set by Lemma 3.1. QED

LOCAL MULTIPLE UNION. $\{x_1 \cup \ldots \cup x_m: x_1 \in y_1 \cup \ldots x_m \in y_m\}$ is a class. If $y_1,\ldots,y_m$ are sets then $\{x_1 \cup \ldots \cup x_m: x_1 \in y_1 \cup \ldots \cup x_m \in y_m\}$ is a set.

Proof: By Local Singleton, for $1 \leq i \leq m$, $A_i = \{\{x_i\}: x_i \in y_i\}$ is a class. Apply Local Pairwise Union to A_1,A_2, and take that and apply local pairwise Union to that and A_3, and so forth, through A_m. Now suppose $y_1,\ldots,y_m$ are sets. $A = \{x_1 \cup \ldots \cup x_m: x_1 \in y_1 \wedge \ldots \wedge x_m \in y_m\} \subseteq \cup(y_1 \cup \ldots \cup y_m)$, and so A is a subclass of a set, and therefore is a set by Lemma 3.1. QED

SUPERSET CLASS. Let c be a set. $\{x: c \subseteq x\}$ is a class.

Proof: Note that $\{x: c \subseteq x\} = \{c \cup x: x \in V\}$ by Boolean. Apply Local Pairwise Union to $\{c\}$ and V to obtain the class of all unions of elements of $\{c\}$ and V, which is $\{c \subseteq x: x \in V\}$. QED

DEFINITION 3.2. For classes x,y, $x \cup y = \{z: z \in x \vee z \in y\}$. $x \cap y = \{z: z \in x \wedge z \in y\}$. $C(x) = \{y: M(y) \wedge \neg y \in x\}$. $V = \{x: M(x)\}$. For sets x,y, $\{x\} = \{y: y = x\}$ and $\{x,y\} = \{z: z = x \vee z = y\}$.

LEMMA 3.2. For classes x,y, $x \cup y$, $x \cap y$, $C(x)$, V are classes. For sets x,y, $x \cup y$, $x \cap y$, $\{x\}$, $\{x,y\}$ are sets.

Proof: The first claim is by Boolean. Let x,y be sets. By Pairing, $\{x\},\{x,y\}$, is a set. By Union, $\cup(\{x,y\}) = x \cup y$ is a set. $x \cap y$ is a subclass of x, and therefore a set. QED

LEMMA 3.3. For sets a,x,y, $\{a,x\} = \{a,y\} \rightarrow x = y$.

Proof: Let $\{a,x\} = \{a,y\}$. Then $x = a \vee x = y$. Suppose $x = a$. Now $y = a \vee y = x$. Hence $x = a = y = x$. QED

LEMMA 3.4. Let $n \geq 1$ be an external integer. Let $a_1,\ldots,a_{2n}$ be distinct sets and $x_1,\ldots,x_n,y_1,\ldots,y_n$ be sets. Suppose $\{\{a_1,x_1\},\{a_2,x_1\},\ldots,\{a_{2n-1},x_n\},a_{2n},x_n\}\} = \{\{a_1,y_1\},\{a_2,y_1\},\ldots,\{a_{2n-1},y_n\},\{a_{2n},y_n\}\}$. Then $x_1 = y_1 \wedge \ldots \wedge x_n = y_n$.

Proof: Look at $\{a_{2i-1},x_i\},\{a_{2i},x_i\}$. These are distinct by Lemma 3.3, and are respectively equal to two distinct terms in the right side, $\{a_j,y_k\},\{a_p,y_q\}$. The complication is that we are dealing with unordered pairs rather than ordered pairs. We have

i. $\{a_{2i-1},x_i\} = \{a_j,y_k\}$.
ii. $\{a_{2i},x_i\} = \{a_p,y_q\}$.
iii. $j \neq p$.
iv. k is the ceiling of j/2,
v. q is the ceiling of p/2.

case 1. $x_i \neq a_{2i-1},a_{2i},a_j,a_p$. Then $x_i = y_k = y_q$. Hence $a_{2i-1} = a_j \wedge a_{2i} = a_p$. By iv,v, $k = i = q$. So $x_i = y_i$.

case 2. $x_i = a_{2i-1}$. Then $a_{2i-1} = x_i = a_j = y_k$. By iv, $k = i$. Hence $x_i = y_i$.

case 3. $x_i = a_{2i}$. Then $a_{2i} = x_i = a_p = y_q$. By v, $q = i$. Hence $x_i = y_s$.

case 4. $x_i = a_j$. By i, $a_{2i-1} = y_k$. Also by ii, $x_i = a_p \vee x_i = y_q$. By iii, $x_i = a_j = y_q$. By ii, $\{a_{2i},a_j\} = \{a_p,a_j\}$. Hence $a_{2i} = a_p$. By v, $q = i$. So $x_i = y_q = y_i$.

case 5. $x_i = a_p$. BY ii, $a_{2i} = y_q$. Also by i, $x_i = a_j \vee x_i = y_k$. By iii, $x_i = a_p = y_k$. By i, $\{a_{2i-1},a_p\} = \{a_j,a_p\}$. Hence $a_{2i-1} = a_j$. By iv, $k = i$. So $x_i = y_k = y_i$.

Since $1 \leq i \leq n$ is arbitrary we have $x_1 = y_1 \wedge \ldots \wedge x_n = y_n$. QED

LEMMA 3.5. Let $a,b \in V$ be distinct, and u be a class.
i. $\{\{a,b,x\}: x \in u\}$ is a class.
ii. $\{\{\{a,x\},\{b,y\}\}: x,y \in u\}$.
iii. $\{\{\{a,x\},\{b,x\}\}: x \in u\backslash\{a,b\}\}$ is a class.
iv. $\{\{\{a,x\},\{b,x\}\}: x \in u\}$ is a class.

Proof: $A = \{\{x\}: x \in u\backslash\{a,b\}\}$ by Local Singleton. Apply Local Pairwise Union to $A,\{a,b\}$ to obtain $B = \{\{a,b,x\}: x \in u\backslash\{a,b\}\}$. Apply Local Pairing to $\{\{a,x\}: x \in u\backslash\{a,b\}\}$, $\{\{b,y\}: y \in u\backslash\{a,b\}\}$ to obtain $C = \{\{\{a,x\},\{b,y\}\}: x,y \in u\backslash\{a,b\}\}$. Apply Union Membership to obtain $D = \{v: \bigcup v \in B\}$. Then $C \cap D$ must be the elements of C where $x = y$, obtaining the class $E = \{\{\{a,x\},\{b,x\}\}: x \in u\backslash\{a,b\}\}$. Note that

$\{\{\{a,x\},\{b,x\}\}: x \in u\}$ is $\{\{\{a,x\},\{b,x\}\}: x \in u\backslash\{a,b\}\}$ union some finite set. Hence iv. QED

LEMMA 3.6. Let $a_1,...,a_n \in V$ be distinct and u be a class. $\{\{\{a_1,x_1\},\{a_2,x_1\},...,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\}: x_1,...,x_n \in u\}$ is a class.

Proof: For each $1 \leq i \leq n$, $A_i = \{\{\{a_{2i-1},x_i\},\{a_{2i},x_i\}\}: x_i \in u\}$ is a class by Lemma 3.5iv. Now apply Local Multiple Union to $A_1,...,A_n$. QED

LEMMA 3.7. Let $a,b,c,d \in V$ be distinct and u be a class. $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x,y \in u\}$ is a class.

Proof: Use Local Pairwise Union on $\{\{a,x\},\{b,x\}: x \in u\}$, $\{\{c,y\},\{d,y\}: y \in u\}$, both of which are classes by Lemma 3.5iv. QED

LEMMA 3.8. Let $a,b,c,d \in V$ be distinct and u be a class. The following are classes.

i. $\{\{x,y\}: x,y \in u\}$.

ii. $\{\{x,y\}: x,y \in u\backslash\{a,b,c,d\}\}$.

iii. $\{\{x,y\}: x,y \in u\backslash\{a,b,c,d\} \wedge x \in y\}$.

iv. $\{\{a,b,c,d,x,y\}: x,y \in u\backslash\{a,b,c,d\} \wedge x \in y\}$.

v. $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x,y \in u\backslash\{a,b,c,d\}\}$.

vi. $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x,y \in u\backslash\{a,b,c,d\} \wedge x \in y\}$.

Proof: Apply Local Pairing to u,u, to obtain i. For ii, apply Local Pairing to $u\backslash\{a,b,c,d\},u\backslash\{a,b,c,d\}$. Let $A = \{\{x,y\}: x,y \in u\backslash\{a,b,c,d\}\}$. Let B be the Epsilon Class $\{\{x,y\}: x \in y\}$. Then $A \cap B$ is the class of all α such that α is an $\{x,y\}$ with $x,y \in u\backslash\{a,b,c,d\}$ and α is an $\{x',y'\}$ with $x' \in y'$. We claim that this the class C of all α such that α is an $\{x,y\}$ such that $x,y \in \alpha\backslash\{a,b,c,d\}$ and $x \in y$. Let α be an $\{x,y\}$ with $x,y \in u\backslash\{a,b,c,d\}$ and an $\{x',y'\}$ with $x' \in y'$. Let $\alpha = \{x,y\} = \{x',y'\}$, where $x,y \in u\backslash\{a,b,c,d\}$ and $x' \in y'$. If $\{x,y\} = \{x',y'\}$ then the claim is established. Otherwise set $x = y' \wedge y = x'$, again establishing the claim. The converse is immediate.

Apply Local Pairwise Union to $\{\{a,b,c,d\}\}$ and C to obtain the class $D = \{\{a,b,c,d,x,y\}: x,y \in u\backslash\{a,b,c,d\} \wedge x \in y\}$, establishing iv. Now apply Local Pairwise Union to $\{\{\{a,x\},\{b,x\}\}: x,y \in u\backslash\{a,b,c,d\}\}$ and $\{\{\{c,y\},\{d,y\}\}: x,y \in u\backslash\{a,b,c,d\}\}$ to obtain the class $E = \{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x,y \in u\backslash\{a,b,c,d\}\}$. This establishes v. For vi, apply Union Membership to obtain the class $F = \{z: \bigcup z \in D\}$, and set $G = E \cap F$. Thus $G = \{z \in E: \bigcup z \in D\}$.

We claim that $z \in G$ if and only if z is some $\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}$ with $x,y \in u\backslash\{a,b,c,d\} \wedge x \in y$. To see this first let $z \in G$. Then $z \in E \wedge \bigcup z \in D$. Let $z = \{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}$, $x,y \in u\backslash\{a,b,c,d\}$. Then $\{a,b,c,d,x,y\} \in D$. Then $x \in y \vee y \in x$. Thus z is some $\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}$ with $x,y \in u\backslash\{a,b,c,d\} \wedge x \in y$. Conversely, let $w = \{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}$ with $x,y \in u\backslash\{a,b,c,d\} \wedge x \in y$. Then $w \in E$ and $\bigcup z = \{a,b,c,d,x,y\}$ which lies in D. QED

LEMMA 3.9. Let $a,b,c,d \in V$ be distinct and u be a class.
i. $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x \in u\backslash\{a,b,c,d\} \wedge y \in \{a,b,c,d\} \wedge x \in y\}$ is a class.
ii. $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x \in \{a,b,c,d\} \wedge y \in u\backslash\{a,b,c,d\} \wedge x \in y\}$ is a class.

Proof: For i, note that the aggregate is a union of four aggregates obtained by setting $x = a,b,c,d$. They all are handled the same way. We will use α for any of a,b,c,d. So we are looking at $\{\{\{a,\alpha\},\{b,\alpha\},\{c,y\},\{d,y\}\}: y \in u\backslash\{a,b,c,d\} \wedge \alpha \in y\}$. By Local Pairwise Union it suffices to obtain $\{\{\{\{c,y\},\{d,y\}\}: y \in V\backslash\{a,b,c,d\} \wedge \alpha \in y\}$. This is an obvious intersection, so it suffices to obtain $\{\{\{c,y\},\{d,y\}\}: y \in u\backslash\{a,b,c,d\}\}$ and $\{\{\{c,y\},\{d,y\}\}: \alpha \in y\}$. For the former, use Local Pairing for $\{\{c,y\}: y \in V\backslash\{a,b,c,d\}\}$, $\{\{d,y\}: y \in V\backslash\{a,b,c,d\}\}$. The former is by Local Pairing applied to $\{c\},V\backslash\{a,b,c,d|$, and the latter is by Local Pairing applied to $\{d\}$, $V\backslash\{a,b,c,d\}$. Finally we obtain $\{\{\{c,y\},\{d,y\}\}: \alpha \in y\}$ as follows. First $\{y: \alpha \in y\} = \{y: \{\alpha\} \subseteq y\}$ is a class $\gamma(\alpha)$ by Superset Class. Then we obtain $\{\{\{\alpha,c\},\{d,y\}\}: \alpha \in y\}$ as $\{\{\{c,y\},\{d,y\}\}: y \in \alpha(c)\}$

in the same way that we obtained $\{\{\{\{c,y\},\{d,y\}\}: y \in u\backslash\{a,b,c,d\}\}$. ii is handled analogously. QED

LEMMA 3.10. Let $a,b,c,d \in V$ be distinct. $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x \in y\}$ is a class.

Proof: $\{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x \in y\} = \{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x,y \in V\backslash\{a,b,c,d\} \wedge x \in y\} \cup \{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x \in \{a,b,c,d\} \wedge y \in V\backslash\{a,b,c,d\} \wedge x \in y\} \cup \{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x \in V\backslash\{a,b,c,d\} \wedge y \in \{a,b,c,d\} \wedge x \in y\} \cup \{\{\{a,x\},\{b,x\},\{c,y\},\{d,y\}\}: x,y \in \{a,b,c,d\} \wedge x \in y\}$. We have shown that the first three terms are classes, and we now observe that the fourth term is finite. QED

Lemma 3.10 is used in a crucial way when we handle atomic formulas $v_i \in v_j$. The next Lemma is used when we handle $\exists v_i$.

LEMMA 3.11. Let $a_1,\ldots,a_{2n} \in V$ be distinct, where $n \geq 1$ is external. Let $A \subseteq \{\{\{a_1,x_1\},\{a_2,x_1\},\ldots,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\}: x_1,\ldots,x_n \in V\}$. Let B be the reduction of A by removing all terms $\{a_{2i-1},x_i\},\{a_{2i},x_i\}$ from all of the elements of A. Let C be the expansion of B obtained by adding a single pair $\{a_{2i-1},y\},\{a_{2i},y\}$, for each $y \in V$, to every element of B. C is the same as the class of all $\{\{a_1,x_1\},\{a_2,x_1\},\ldots,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\}$ such that $(\exists x_i)(\{\{a_1,x_1\},\{a_2,x_1\}\ldots,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\} \in A)$.

Proof: Let $D = \{\{\{a_{2i-1},x_i\},\{a_{2i},x_i\}\}: x_i \in V\}$, which is a class by Lemma 3.5iv. Now B results from A by taking out the elements of D from each element of A (or equivalently intersecting each with the complement of D), and hence B is obtained by Local Intersection applied to A,B. Now C is obtained from B by adjoining every element of B with the elements of D. Hence C is obtained by Local Pairwise Union applied to B,D.

Now suppose $\{\{a_1,x_1\},\{a_2,x_1\},\ldots,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\} \in C$. Then $\{\{a_1,x_1\},\{a_2,x_1\},\ldots,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\}\backslash\{\{a_{2i-1},x_i\},\{a_{2i},x_i\}\} \in B$. By the definition of B, this element of B must have come from an element of A by removing some pair $\{a_{2i-1},x_i\},\{a_{2i},x_i\}$. Hence $(\exists x_i \in V)((a_1,x_1\},\{a_2,x_1\},\ldots,\{a_{2n-}$

$_1,x_n\},\{a_{2n},x_n\}\} \in A$. Conversely, suppose $(\exists x_i)((a_1,x_1\},\{a_2,x_1\},...,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\} \in A$. Let $((a_1,x_1\},\{a_2,x_1\},...,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\} \in A$. Then $((a_1,x_1\},\{a_2,x_1\},...,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\}\backslash\{\{a_{2i-1},x_i\},\{a_{2i},x_i\}\} \in B$. Hence $((a_1,x_1\},\{a_2,x_1\},...,\{a_{2n-1},x_n\},\{a_{2n},x_n\}\} \in C$. QED

Recall that we are proving LCS from NBG*. It is expositionally convenient, though not necessary, to use specific simple integer like sets for the first listed elements of unordered pairs.

DEFINITION 3.3. For external integers $i \geq 0$ we inductively define i* as follows. $0^* = \emptyset$, $i+1^* = \{i^*\}$.

LEMMA 3.12. For distinct external integers $i,j \geq 0$, i^* is a set $\wedge$ $i^* \neq j^*$.

Proof: By induction on external $i \geq 0$, i^* is a set. Now assume $i^* = j^*$ and write $\{...\{\emptyset\}...\} = (...\{\emptyset\}...\}$, where there are i pairs of braces on the left and j pairs of braces on the right. We can successively eliminate these braces getting down to $\emptyset$ on one side and some $\{...\{\emptyset\}...\}$ on the other. This is a contradiction. QED

We won't be using any other properties of the i*.

DEFINITION 3.4. K is the external set of all limited formulas in $\in,=,M,\neg,\wedge,\exists,(,),v_1,v_2,...$. For $\varphi \in K$, $S(\varphi)$ is the external set of all $\alpha = (a_1,...,a_n;b_1,...,b_m)$ such that $a_1,...,a_n,b_1,...,b_m$ are without repetition, with $n \geq 1$, which includes all subscripts of variables in φ. ($S(\varphi)$ is the set of all signatures for φ). An α-assignment consists of classes $x_{b1},...,x_{bm}$, where we think of these classes as assigning to the variables $v_{b1},...,v_{bm}$, respectively.

NOTE: α-assignments do not assign to variables $v_{a1},...,v_{an}$. Only to the variables $v_{b1},...,v_{bm}$.

DEFINITION 3.5. Let $\varphi \in K$ and $\alpha \in S(\varphi)$. φ is α-secured if and only if for all α-assignments $x_{b1},...,x_{bm}$, the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a2}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\}$ such that φ

holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}$, exists. φ is secured if and only if for all $\alpha \in S(\varphi)$, φ is α-secure.

Note that for fixed $\varphi \in K$ and fixed $\alpha \in S(\varphi)$, the notions of α-assignment and α-secured are internal. The notion "φ is secure", even for fixed $\varphi \in K$, is external. This is because the $\alpha \in S(\varphi)$ can be of arbitrary long finite length. However, $\varphi \in K$ consists of countably many internal statements, namely the internal statements "φ is α-secured" for the various $\alpha \in S(\varphi)$.

We now show by external induction on formulas $\varphi \in K$ that φ is secured. I.e., each instance "φ is α-secured" is provable in NBG*.

LEMMA 3.13. $v_i \in v_j$ is secured.

Proof: Let $\alpha = (a_1,...,a_n;b_1,...,b_m) \in S(v_i \in v_j)$. We prove that $v_i \in v_j$ is α-secured. Let an α-assignment $x_{b1},...,x_{bm}$ be given. We need to show that the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in x_j$, exists.

case 1. $i,j \in \{a_1,...,a_n\}$. Let $i = a_p \wedge j = a_q$. By Lemma 3.10, the class A of all $\{\{2p-1^*,x_i\},2p^*,x_i\},\{2q-1^*,x_j\},2q^*,x_j\}\}$ such that $x_i \in x_j$, exists. Now $B = \{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\backslash\{\{2p-1^*,x_i\},\{2p^*,x_i\},\{2q-1^*,x_j\},2q^*,x_j\}\}: x_1,...,x_n\backslash x_i,x_j \in V\}$ exists by Lemma 3.6. Finally apply Local Pairwise Union to A,B.

case 2. $i \in \{a_1,...,a_n\} \wedge j \in \{b_1,...,b_m\}$. Let $i = a_p$. Then x_j has been fixed by the α-assignment. By Lemma 3.5iv, the class C of all $\{\{2p-1^*,x_i\},2p^*,x_i\}\}$ such that $x_i \in x_j$, exists. By Lemma 3.6, the class $C = \{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\backslash\{\{2p-1^*,x_i\},\{2p^*,x_i\}\}: x_{a1},...,x_{an}\backslash x_i \in V\}$ exists by Lemma 3.6. Apply Local Pairwise Union to C,D.

case 3. $i \in \{b_1,...,b_m\} \wedge j \in \{a_1,...,a_n\}$. Let $j = a_p$. Then x_i has been fixed by the class assignment. By Superset Class, let $S = \{x_j: \{x_i\} \subseteq x_j\} = \{x_j: x_i \in x_j\}$. By Lemma

3.5iv, the class E of all $\{\{2p-1^*,x_j\},\{2p^*,x_j\}\}$ such that $x_j \in S$, exists. By Lemma 3.6, the class F = $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\backslash\{\{2p-1^*,x_j\},\{2p^*,x_j\}\}: x_{a1},...,x_{an}\backslash x_j \in V\}$ exists by Lemma 3.6. Apply Local Pairwise Union to E,F.

case 4. $i,j \in \{b_1,...,b_m\}$. Then x_i,x_j have been fixed. If $x_i \in x_j$ then the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in x_j$, exists by Lemma 3.6. If $x_i \notin x_j$, then the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in x_j$, is $\emptyset$.

QED

LEMMA 3.14. Let $M(v_i) \in K$. $M(v_i)$ is secured.

Proof: Let $\alpha = \{a_1,...,a_n,b_1,...,b_m\} \in S(M(v_i))$. We prove that $M(v_i)$ is α-secured. Let an α-assignment $x_{b1},...,x_{bm}$ be given. We need to show that the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in V$, exists.

case 1. $i \in \{a_1,...,a_n\}$. The class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in V$, exists by Lemma 3.6.

case 2. $i \in \{b_1,...,b_m\}$. Then x_i has been fixed by the class assignment. If x_i is a set, then $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in V$, exists by Lemma 3.6. If x_i is not a set then $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $x_i \in V$, is $\emptyset$.

QED

LEMMA 3.15. Let $\varphi \in K$ be secured. Then $(\exists v_i)(M(v_i) \wedge \varphi) \in K$ is secured.

Proof: Let $\alpha = (a_1,...,a_n;b_1,...,b_m) \in S((\exists v_i)(M(v_i) \wedge \varphi))$. Let an α-assignment $x_{b1},...,x_{bm}$ be given. We need to show that the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm})\}$, exists.

case 1. $i \in \{a_1,...,a_n\} \wedge v_i$ not in φ. Let $i = a_p$. Let α' be α with i deleted. Then $\alpha' \in S(\varphi)$ and $x_{b1},...,x_{bm}$ is also an α'-assignment. By the induction hypothesis let A be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\backslash\{\{2p-1^*,x_i\},\{2p^*,x_i\}\}$ such that φ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}\backslash x_i$. Then $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\backslash\{\{2p-1^*,x_i\},\{2p^*,x_i\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}\backslash x_i$, exists. Let B be the class of all $\{\{2p-1^*,x_i\},\{2p^*,x_i\}\}$ such that $x_i \in V$. Apply Local Pairwise Union to A,B to obtain the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm})\}$.

case 2. $i \in \{a_1,...,a_n\} \wedge v_i$ is in φ. Let $i = a_p$. By the induction hypothesis let A be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that φ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}$. By Lemma 3.11, the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $(\exists x_i)(\varphi$ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm})$, exists. But this is the same as the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}$.

case 3. $i \in \{b_1,...,b_m\} \wedge v_i$ is not in φ. Let $i = b_q$. Let α' be α with i deleted. Then $\alpha' \in S(\varphi)$ and $x_{b1},...,x_{bm}\backslash x_{bq}$ is an α'-assignment. By the induction hypothesis let A be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that φ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}\backslash x_i$. Then A be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},...,x_{an},x_{b1},...,x_{bm}$.

case 4. $i \in \{b_1,...,b_m\} \wedge v_i$ is in φ. Let $i = b_q$. Let $\alpha' \in S(\varphi)$ be $(a_1,...,a_n,b_q;b_1,...,b_m\backslash bq)$ and we use the α'-assignment $x_{b1},...,x_{bm}\backslash x_i$. By the induction hypothesis, let A be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\},\{2n+1^*,x_{bq}\},\{2n+2^*,x_{bq}\}\}$ such that φ holds at $x_{a1},...,x_{an},x_{bq},x_{b1},...,x_{bm}/x_i$. As in case 2, using α', let B be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},...,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\},\{2n+1^*,x_i\},\{2n+2^*,x_i\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},...,x_{an},x_i,x_{b1},...,x_{bm}/x_i$. By Local Pairwise

Intersection, let C be the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},\ldots,x_{an},x_i,x_{b1},\ldots,x_{bm}/x_i$. Obviously $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},\ldots,x_{an},x_i,x_{b1},\ldots,x_{bm}/x_i$ if and only if $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},\ldots,x_{an},x_{b1},\ldots,x_{bm}$. Hence C is the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}\}$ such that $(\exists v_i)(M(v_i) \wedge \varphi)$ holds at $x_{a1},\ldots,x_{an},x_{b1},\ldots,x_{bm}$.

QED

LEMMA 3.16. Suppose $\varphi \in K$ is secured. Then $\neg\varphi \in K$ is secured.

Proof: Let $\varphi \in K$ be secured. Let $\alpha = (a_1,\ldots,a_n;b_1,\ldots,b_m) \in S(\neg\varphi)$. Let $x_{b1},\ldots,x_{bm}$ be an α-assignment. Since $\neg\varphi$ is secured, the class A of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that φ holds at $x_{a1},\ldots,a_n,x_{b1},\ldots,x_{bm}$, exists. By Lemma 3.6, the class B of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ exists. Hence by Boolean applied to A,B, the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $\neg\varphi$ holds at $x_{a1},\ldots,x_{an},x_{b1},\ldots,x_{bm}$, exists. QED

LEMMA 3.17. Suppose $\varphi,\psi \in K$ are secured. Then $\varphi \wedge \psi$ is secured.

Proof: Let $\varphi,\psi \in K$ be secured. Let $\alpha = (a_1,\ldots,a_n;b_1,\ldots,b_m) \in S(\varphi \wedge \psi)$. Let $x_{b1},\ldots,x_{bm}$ be an α-assignment. Then $\alpha \in S(\varphi),S(\psi)$. By the induction hypothesis, The class A of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that φ holds at $x_{a1},\ldots,x_{an},x_{b1},\ldots,x_{bm}$, and the class B of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that ψ holds at $x_{a1},\ldots,x_{an},x_{b1},\ldots,x_{bm}\}$ are classes. Therefore by Boolean, the class of all $\{\{1^*,x_{a1}\},\{2^*,x_{a1}\},\ldots,\{2n-1^*,x_{an}\},\{2n^*,x_{an}\}\}$ such that $\varphi \wedge \psi$ holds at $x_{a1},\ldots,x_{an},x_{b1},\ldots,x_{bm}$, exists. QED

LEMMA 3.18. Every formula in K is secured.

Proof: By Lemmas 3.14 - 3.17. QED

LEMMA 3.19. Let φ be a limited formula. There exists $\varphi' \in K$ such that $\varphi \leftrightarrow \varphi'$ is logically provable from Extensionality. If v_1 is not in φ then we can require that v_1 is not in φ'.

Proof: We construct such φ' by induction on the given limited formula φ.
1. φ is $v_i \in v_j$. φ' is $v_i \in v_j$.
2. φ is $M(v_i)$. φ' is $M(v_i)$.
3. φ is $v_i = v_j$. φ' is $\neg(\exists v_k)(v_k \in v_i \wedge \neg v_k \in v_j) \wedge \neg(\exists v_k)(v_k \in v_j \rightarrow v_k \in v_i)$, where $k = \max(i,j)+1$.
4. φ is $\neg\psi$, $\psi \wedge \rho$. φ' is $\neg\psi'$, $\psi' \wedge \rho'$.
5. φ is $\psi \vee \rho$, $\psi \rightarrow \rho$, $\psi \leftrightarrow \rho$. φ' is $\neg(\neg\psi' \wedge \neg\rho')$, $\neg(\psi' \wedge \neg\rho')$, $\neg(\psi' \wedge \neg\rho') \wedge \neg(\psi' \wedge \neg\rho')$.
6. φ is $(\exists v_i)(M(v_i) \wedge \psi)$. φ' is $(\exists v_i)(M(v_i) \wedge \psi')$
10. φ is $(\forall v_i)(M(v_i) \rightarrow \psi)$. φ' is $\neg(\exists v_i)(M(v_i) \wedge \neg\psi')$.

Extensionality is used only for 3. It is clear in 1-10 that if v_1 does not appear in φ then v_1 does not appear in φ'. QED

LEMMA 3.20. NBG* proves Limited Comprehension Scheme. In particular, NBG* proves every instance of LCS.

Proof: We prove $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (M(v_2) \wedge \varphi))$ in NBG*, where φ is a limited formula without v_1. By Lemma 3.19, let $\psi \in K$ be such that $\varphi \leftrightarrow \psi$ is logically provable from Extensionality, where v_1 is not in ψ. It suffices to prove

*) $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (M(v_2) \wedge \psi))$

in NBG*. By Lemma 3.18, ψ is secured. Let $\alpha \in S(\psi)$ be $(2;b_1,\ldots,b_m)$ where $v_{b1},\ldots,v_{bm}$ lists all of the variables in ψ without repetition other than v_2. Then ψ is α-secured. With the aim of proving *), let $x_{b1},\ldots,x_{bm}$ be an α-assignment. Since ψ is α-secured, the class A of all $\{\{1^*,x_2\},\{2^*,x_2\}\}$ such that ψ holds at $x_2,x_{b1},\ldots,x_{bm}$, exists. By Class Union, $\cup A$ exists, and it is the class of all x_2 such that ψ holds of $x_2,x_{b1},\ldots,x_{bm}$ together with $1^*,2^*$. By Pairing, the set B of all $x_2 \in \{1^*,2^*\}$ such that ψ fails of $x_2,x_{b1},\ldots,x_{bm}$ exists. Now apply Boolean to A,B to obtain the class C of all x_2 such that ψ holds of $x_2, x_{b1}, \ldots, x_{bm}$ exists. QED

THEOREM 3.21. NBG* and NBG are logically provably equivalent.

Proof: It remains to derive the Collection Scheme from NBG*. Let A be a set where $(\forall x \in A)(\exists y \in V)(\varphi(x,y))$, φ limited without w. We don't show the side parameters in φ. Let B be the class of all $\{x,\{x,y\}\} = \langle x,y\rangle$ such that for some $x \in A$, we have $\varphi(x,y)$. Note that B exists by an application of LCS. It is clear that for each $x \in A$ there exists $z \in B$ such that $x \in z$. In particular, $A \subseteq \cup B$. Let C be a subset of B such that $A \subseteq \cup C$. Then for all $x \in A$, there exists $\{x,\{x,y\}\} \in C$, and for any of the $[x,\{x,y\}\} \in C$, we have $\varphi(x,y)$. Note that $\cup\cup C$ includes as elements those y such that for some $x \in A$, $\varphi(x,y)$. Thus for all $x \in A$ there exists $y \in \cup\cup C$ such that $\varphi(x,y)$. QED

4. SOME ADDITIONAL RESULTS

We have chosen to use the Collection Scheme rather than the Replacement Scheme in the axiomatization of NBG that we presented in section 1. It is well known that using Replacement instead of Collection is equivalent For NBG. Our NBG* uses Capture in a form akin to Collection - without the use of the uniqueness hypothesis used in Replacement. However, we can weaken Capture to Capture/Unique where we use $x \subseteq! \cup y$ if and only if for all $z \in x$ there exists unique $w \in y$ with $x \in w$.

We now take up the variants where we do not use Foundation. NBG without Foundation has some foundational significance because Foundation is probably the most rarely used of all of the axioms of NBG in actual mathematical practice. It is well known that Replacement is strictly weaker than Collection in NBG if we omit Foundation.

THEOREM 4.1. NBG without Foundation, as formalized by 1-6,8,9 of section 1, is logically equivalent to NBG* without Foundation, as formalized by 1-6 + A-H.

Proof: The derivation of LCS in section 3 from NBG* did not use Foundation. The derivation of Collection in section 3 from Capture did not use Foundation either. QED

THEOREM 4.2. NBG without Foundation, as formalized by 1-6,8 and Replacement, is logically equivalent to NBG* without Foundation, as formalized by 1-6 + A-G with Capture/Unique.

Proof: The derivation of LCS in section 3 from NBG* did not use Foundation or Replacement or Collection or Capture or Capture/Unique. The derivation of Replacement can be analogously made with Capture/Unique and does not use Foundation. QED

There are also many variants of Infinity. The choice of any remotely reasonable Infinity in NBG and in NBG* does not affect the logical equivalence, both in the main result, Theorem 3.21, and in these variants Theorems 4.1, 4.2. In fact, the reasonable choice of Infinity does not have to match in NBG and in NBG*. The reason is that first of all, the derivation of LCS in section 3 does not use Infinity except in one spot where it is used only to derive the existence of a set. (Note that the existence of a class, which is the existence of something, is guaranteed as a principle of pure logic.) And Infinity is not used to prove Collection or Replacement from Capture or Capture/Unique.

We now take up a more substantial variant. Foundation and Replacement and Collection are easily the most rarely used axioms of class theory (or set theory), and by a very large margin. So a kind of minimal set theoretic system for doing mathematics which is remarkably accommodating is as follows. This is a set theory and there are no proper classes. The language is ∈,=. There is no M. This language is written LST.

Z, A VERSION OF ZERMELO SET THEORY

1. EXTENSIONALITY. If v_1,v_2 have the same elements then $v_1 = v_2$. $(\forall v_3)(v_3 \in v_1 \leftrightarrow v_3 \in v_2) \rightarrow v_1 = v_2$.
2. PAIRING. For v_1,v_2, $\{v_1,v_2\}$ exists. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow v_4 = v_1 \vee v_4 = v_2))$.
3. UNION. For v_1, $\cup v_1$ exists. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\exists v_4 \in v_1)(v_3 \in v_4)))$.

4. POWER SET. For any v_1, $\{v_2: v_2 \subseteq v_1\}$ is a set. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\forall v_4)(v_4 \in v_3 \rightarrow v_4 \in v_1)))$.
5. INFINITY. There is a set containing $\emptyset$ and closed under $x \cup \{y\}$. $(\exists v_1)(\exists v_2)((v_2 \in v_1 \wedge v_2 = \emptyset) \wedge (\forall v_3)(\forall v_4)(\exists v_5)(v_5 \in v_1 \wedge v_5 = v_3 \cup \{v_4\}))$.
6. SEPARATION. There exist v_1 consisting of all $v_2 \in v_3$ obeying a formula in LST. $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (v_2 \in v_3 \wedge \varphi))$, where φ is a formula in LST and v_1 is not in φ.

Here it does make a difference what version we take for Infinity. Without Replacement, they are not generally equivalent. So it is natural to use a strongest formulation, in some sense that we don't really know how to characterize. We picked the obvious one that gives us the set $V(\omega)$, as the inclusion least set with the property presented in Infinity above.

This version of Z (Zermelo set theory) is well known not to be finitely axiomatizable. In fact, Z proves the consistency of any one of its finite fragments.

WZ, A VERSION OF WEAK ZERMELO SET THEORY

1. EXTENSIONALITY. If v_1,v_2 have the same elements then $v_1 = v_2$. $(\forall v_3)(v_3 \in v_1 \leftrightarrow v_3 \in v_2) \rightarrow v_1 = v_2$.
2. PAIRING. For v_1,v_2, $\{v_1,v_2\}$ exists. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow v_4 = v_1 \vee v_4 = v_2))$.
3. UNION. For v_1, $\cup v_1$ exists. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\exists v_4 \in v_1)(v_3 \in v_4)))$.
4. POWER SET. For any v_1, $\{v_2: v_2 \subseteq v_1\}$ is a set. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\forall v_4)(v_4 \in v_3 \rightarrow v_4 \in v_1)))$.
5. INFINITY. There is a set containing $\emptyset$ and closed under $x \cup \{y\}$. $(\exists v_1)(\exists v_2)((v_2 \in v_1 \wedge v_2 = \emptyset) \wedge (\forall v_3)(\forall v_4)(\exists v_5)(v_5 \in v_1 \wedge v_5 = v_3 \cup \{v_4\}))$.
6'. WEAK SEPARATION. There exist v_1 consisting of all $v_2 \in v_3$ obeying a restricted formula in LST. $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (v_2 \in v_3 \wedge \varphi^{v4}))$, where φ is a formula in LST without v_1,v_4.

Here for formulas φ in LST, let φ^{vi} be φ with all quantifiers restricted to "$\in v_i$".

WZ is well known to be finitely axiomatizable. We now give a finite axiomatization.

WZ*

1. EXTENSIONALITY. If v_1,v_2 have the same elements then $v_1 = v_2$. $(\forall v_3)(v_3 \in v_1 \leftrightarrow v_3 \in v_2) \rightarrow v_1 = v_2$.
2. PAIRING. For v_1,v_2, $\{v_1,v_2\}$ exists. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow v_4 = v_1 \vee v_4 = v_2))$.
3. UNION. For v_1, $\cup v_1$ exists. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\exists v_4 \in v_1)(v_3 \in v_4)))$.
4. POWER SET. For any v_1, $\{v_2: v_2 \subseteq v_1\}$ is a set. $(\exists v_2)(\forall v_3)(v_3 \in v_2 \leftrightarrow (\forall v_4)(v_4 \in v_3 \rightarrow v_4 \in v_1)))$.
5. INFINITY. There is a set containing $\emptyset$ and closed under $x \cup \{y\}$. $(\exists v_1)(\exists v_2)((v_2 \in v_1 \wedge v_2 = \emptyset) \wedge (\forall v_3)(\forall v_4)(\exists v_5)(v_5 \in v_1 \wedge v_5 = v_3 \cup \{v_4\}))$.
A. DIFFERENCE. The set of all elements of v_1 that are not in v_2 exists. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow (v_4 \in v_1 \wedge \neg v_4 \in v_2))$.
B. LOCAL PAIRING. $\{\{v_1,v_2\}: v_1 \in v_3 \wedge v_2 \in v_4\}$ is a set. $(\exists v_5)(\forall v_6)(v_6 \in v_5 \leftrightarrow (\exists v_1 \in v_3)(\exists v_2 \in v_4)(v_6 = \{v_1,v_2\}))$.
C. LOCAL UNION. $\{\cup v_1: v_1 \in v_2\}$ is a set. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow (\exists v_1 \in v_2)(v_4 = \cup v_1))$.
D. LOCAL INTERSECTION. $\{v_1 \cap v_2: v_1 \in v_3 \wedge v_2 \in v_4\}$ is a set. $(\exists v_5)(\forall v_6)(v_6 \in v_5 \leftrightarrow (\exists v_1 \in v_3)(\exists v_2 \in v_4)(v_6 = v_1 \cap v_2))$.
E. UNION MEMBERSHIP. $\{v_1: \cup v_1 \in v_2\}$ is a set. $(\exists v_3)(\forall v_1)(v_1 \in v_3 \leftrightarrow (\exists v_5 \in v_2)(v_5 = \cup v_1))$.
F. EPSILON CLASS. $\{\{v_1,v_2\}: v_1 \in v_2 \wedge v1,v2 \in v3\}$ is a set. $(\exists v_3)(\forall v_4)(v_4 \in v_3 \leftrightarrow (\exists v_1)(\exists v_2)(v_4 = \{v_1,v_2\} \wedge v_1 \in v_2))$.

THEOREM 4.3. WZ and WZ* are logically provably equivalent.

Proof: This very closely follows the equivalence for NBG and NBG*. QED