LINE FIGURES AND INCOMPLETENESS
TANBIGLE INCOMPLETENESS SERIES
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ABSTRACT. We use finite lists of closed line segments to obtain Π^0_2 independence results at the level of the Kruskal and Extended Kruskal theorems. Also see the Addendum.

There are by now a large number of examples of Tangible Incompleteness from PA (Peano Arithmetic), with the original examples being Goodstein Sequences (discovered in [Go44], Incompleteness established in [KP82]), and the Paris/Harrington Ramsey Theorem (discovered and established in [PH77]). There have been far fewer beyond what is usually referred to as Impredicativity, with associated proof theoretic ordinal Γ_0, starting with

i. My finite form of Kruskal's theorem (KT) and variants with labels and structure. These have a much higher associated proof theoretic ordinal starting at $\Theta(\Omega^\omega)(0)$. Original [Fr81].
ii. My Extended Kruskal's theorem (EKT), with a yet higher associated proof theoretic ordinal $\Theta(\Omega_\omega)(0)$. Original [Fr82].
iii. My finite form of the Graph Minor Theorem (GMT), also with $\theta(\Omega_\omega)(0)$. but as a lower bound only, with [KR18] establishing a somewhat higher upper bound). [FRS87].

In every new substantial result in Intangible Incompleteness, there are some new interesting and desirable features, although some other features present in other such results may be lacking in the new ones. For here, the connections with vividly clear elementary plane geometry are a plus. Others are strong on numerics. It is the general expectation that at least with regards to PA, the opportunities for new Intangible Incompleteness are vast, having seen only a fairly short term development by most mathematical standards. We expect more or less the same for these higher levels referred to above, at the level of at most iterated or transfinitely iterated inductive definitions, but probably at a more limited pace.

The idea of making lists of objects and finding mappings sending parts of the list into other parts of the list is inspired by my work in [Fr01]. There I look at sufficiently long sequences drawn from a fixed finite alphabet, and ask for $i < j$ such that $x_i,\ldots,x_{2i}$ is a subsequence of $x_j,\ldots,x_{2j}$. The longest sequence with no such $i < j$, using k letters, is called n(k). n(1) = 1, n(2) = 11, and n(3) is ridiculously large, well into Ackermann numbers. n(3) > $A_{7198}(158386)$ is proved there with the help of computer investigations by Randall Dougherty. That is the 7,198th Ackermann function at 159,386. Subsequently I remember claiming that n(4) is bigger than iterating $A_k(k)$ a lot starting at 1. How many times? $A_5(5)$ times. That all n(k) exists is just beyond multi recursion, and descent recursion through $<\omega^{\omega\hat{}\omega}$. The full theorem for all finite sets is provable in $I\Sigma_3$ but not in $I\Sigma_2$. So there are a myriad of huge phase transitions between positive integers here.

A natural step in this research is to change the $x_i,\ldots,x_{2i}$ and $x_j,\ldots,x_{2j}$. For example, to $x_i,\ldots,x_{3i}$ and $x_j,\ldots,x_{3j}$. How does that effect early values of n(k)? Of course, there is the obvious possibility of using asymmetric numbers here, something I have never even considered. When is it even true, and when does it get quantitatively easy, using $x_i,\ldots,x_{f(i)}$ and $x_j,\ldots,x_{g(j)}$?

An orthogonal step is to ask for $i_1 < \ldots < i_m$ such that $(x_{i\_1},\ldots x_{2i\_n}),(x_{i\_2},\ldots,x_{2i\_2}),\ldots,(x_{i\_m},\ldots,x_{2i\_m})$ forms a chain each being a subsequence of the next? One does not have to be using intervals j,...,2j as we are, here.

Now bear in mind that this is only using long sequences drawn from a fixed finite set. Here we want to talk about using long sequences drawn from geometric objects in the plane. My work from the 80's involved long finite sequences drawn from finite trees and finite graphs, and I just compared a single term with a single later term. But we want the terms of the sequences to be objects simpler than trees.

By a segment we will mean a closed line segment in the plane with distinct endpoints. The basic object we are interested in here is that of a finite list of segments, $S_1,...,S_n$. We look at a finite union V of segments as a topological subspace of $\Re^2$. We have the usual notions of connected and simply connected.

V will always refer to a nonempty finite union of segments. It is convenient to remove crossings in V and redundancies in V. The former is where we cross like in an "X". Redundancies occur when two segments have a segment in common. There is a standard procedure to remove both of these nuisances, leading to a unique reduction RED(V).

THEOREM 1. (Well known). V is connected and simply connected if and only if RED(V) becomes a rooted tree when some endpoint in RED(V) is designated as the root, in the sense of graph theory, if and only if RED(V) becomes a rooted tree when any endpoint in RED(V) is designated as the root, in the sense of graph theory. (Full precise explanation of this is well known).

Note that if V is merely simply connected then we analogously have a forest.

Now there is a particularly simple SEQUENTIAL approach to all of this that gets straight to the heart of the matter for this approach to Incompleteness. We now write V for finite sequences $S_1,...,S_n$ of segments. We say that V is outward if and only if

$$\text{for all } 1 \le i \le n,\ |S_i \cap (S_1 \cup ... \cup S_{i-1})| \le 1$$

THEOREM 2. (Well known). V is simply connected if $V = S_1 \cup ... \cup S_n$ for some outward list $S_1,...,S_n$.

We now come to a finite form of Kruskal's Theorem (no labels).

THEOREM 3. Let $k \geq 1$. In every sufficiently long outward list of segments $S_1,\ldots,S_n$, there exists $k \leq i < j \leq n$ such that $S_i \cup \ldots \cup S_{2i}$ is continuously embeddable into $S_j \cup \ldots \cup S_{2j}$.

Proof: Restate this purely combinatorially so we can assume that it is false and apply Weak Konig's Lemma. We obtain an infinite sequence $S_1,S_2,\ldots$ of segments such that for no $k \leq i < j$ do we have $S_i \cup \ldots \cup S_{2i}$ continuously embeddable into $S_j \cup \ldots \cup S_{2j}$. But now look at the infinite sequence $S_k \cup \ldots \cup S_{2k}$, $S_{k+1} \cup \ldots \cup S_{2k+2}$, ... These are forests, and hence by Kruskal's theorem for forests, one is continuously embeddable into a later one, which is a contradiction. QED

THEOREM 4. Theorem 3 implies my finite form of Kruskal's Theorem (no labels) over RCA_0. Theorem 3 is provably equivalent to no elementary recursive descending sequences through every initial segment of $\theta(\Omega^\omega)(0)$ and equivalently, Π^1_2-BI_0.

Proof: It is very convenient to use a known result that Kruskal's theorem and its finite forms (no labels) are equivalent to those with uniform valence (uniform splitting) and finitely many labels with label preservation. We only need two labels for this purpose. Let $k,r \geq 1$. Let $T_1,\ldots,T_n$ be a sufficiently long sequence of finite trees of uniform valence $\leq r$, where $|T_i| \leq i$. We want to prove that there exists $k \leq i < j \leq n$ such that T_i is inf preserving embeddable into T_j. Also one of my forms has $|T_i| = i$ rather than $|T_i| \leq i$, which is more convenient here. Note that $T_1,\ldots,T_{k-1}$ have no role here just like when we construct the S's, $S_1,\ldots,S_{k-1}$ also play no role.

If we could just be constructing $S_k \cup \ldots \cup S_{2k}$, $S_{2k+1} \cup \ldots \cup S_{4k+2}$, and so forth, we would have no interference, and just make these exactly T_k, T_{k+1}, ... So we want to make sure that these unions, if they are anywhere near being close together, let alone overlap, cannot have the inf preserving embedding. The idea is to use uniform valence r+1 and the second label, to enforce "no close together". Also there is no reason to be wedded to the same k so we can start putting this extra information in comfortably.

We are deriving the finite form of Kruskal's Theorem with no labels, but we are using UNSTRUCTURED trees. However, is known to be as strong as using STRUCTURED trees. QED

We now come to trivalent graphs. A trivalent graph is an undirected graph with no loops, every vertex has out degree at most 3.

THEOREM 5. In any infinite sequence of finite trivalent graphs, one is continuously embeddable into a later one as topological spaces.

This is an immediate consequence of the Graph Minor Theorem because for trivalent graphs, minor inclusion is the same as continuously embeddable. QED

In [FRS87], Paul Seymour claimed in print that the proof EKT (my extended Kruskal's theorem) from GMT can be refined to give a proof of EKT from the Trivalent Graph Theorem. Recall that my EKT corresponds to Π^1_1-CA_0 or finitely iterated induction definitions. I am in contact with him about this.

We continue to use sequences of segments, $S_1,...,S_n$. Only now we use trivalent. V is trivalent if and only if no four segments from $S_1,...,S_n$ have exactly one point in common.

THEOREM 6. Let $k \geq 1$. In every sufficiently long trivalent list of segments $S_1,...,S_n$, there exists $k \leq i < j \leq n$ such that $S_i \cup ... \cup S_{2i}$ is continuously embeddable into $S_j \cup ... \cup S_{2j}$.

To imitate the proof that this is strong, we should have some hierarchy here to make a similar no interference argument. In [FRS87] there was the hierarchy of tree width. More tree width climbs up the ID_n. There should be an analogous thing for trivalent graphs. In any case this no interference argument should be a technical detail that is easily handled. So we would have

THEOREM 7. Theorem 6 is provably equivalent no elementary recursive descending sequences through $\Theta(\Omega_\omega)(0)$ and equivalently, 1-Con(Π^1_1-CA_0).

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NOTE: There is another manuscript from the 1980's, also handwritten notes concerning Kruskal's theorem. I don't see a date on it, and my copy has two sections, sections 2 and 3. I think I might have planned a section 1 at the time. The first section is entitled "Some proofs of Kruskal's theorem and its restrictions", 53 pages. The second section is entitled "Some ordinal calculations for Kruskal's theorem and its restrictions", 110 pages. I think that Andreas Weiermann has a copy of these two items.

ADDENDUM: AN alternative to the finite forms of Kruskal type Theorems and trivalent graph theorem discussed above using line segments, we can instead proceed purely combinatorially.

THEOREM 7. Let n >> k ≥ 1, and T = (T,<) be a sufficiently large rooted tree with vertices 1,...,n. There exists k ≤ i,j ≤ n such that T|{i,...,2i} is inf preserving embeddable into T|{j,...,2j}.

It is also convenient to work with labels, which removes the need for this inert k.

THEOREM 8. Let n >> k ≥ 1, and T = (T,<) be a sufficiently large rooted tree with {1,...,k} labeled vertices 1,...,n. There exists i,j ≤ n such that T|{i,...,2i} is inf and label preserving embeddable into T|{j,...,2j}.

We can add the gap condition to Theorem 8. Also we need to investigate

THEOREM 9. Let n >> k ≥ 1, and T = (T,<) be a sufficiently large trivalent graph with vertices 1,...,n. There exists k ≤ i,j ≤ n such that T|{i,...,2i} is homeomorphically embeddable into T|{j,...,2j}.

I wrote continuously embeddable above but as we are talking about one-one maps, better to use homeomorphically embeddable. In addition, presumably there is a version like this:

THEOREM 10. Let n >> k ≥ 1, and T = (T,<) be a sufficiently large trivalent graph with {1,...,k} labeled vertices 1,...,n. There exists k ≤ i,j ≤ n such that T|{i,...,2i} is homeomorphically embeddable into T|{j,...,2j}.

Is there such a thing as the trivalent graph theorem with k labels with a "gap condition"? What exactly does that say and how strong is it?

Incidentally, I think the trivalent graph theorem is often called the subcubic graph theorem.