TANGIBLE INCOMPLETENESS SERIES
LECTURE 4, ZOOM,
BABY EMULATION THEORY
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I have reworked some of the details of Baby Emulation Theory in the previous Gent Lecture Notes Number 3. So let's start the treatment of Baby Emulation Theory systematically from scratch.

Baby Emulation Theory lives in $Q[-1,1]^2$. Here $Q[-1,1]$ is $Q \cap [-1,1]$. We only use these four notions.

1. Order equivalence of $x,y \in Q^4$.
2. $S \subseteq Q[-1,1]^2$ is an emulator of $E \subseteq Q[-1,1]^2$.
3. $S \subseteq Q[-1,1]^2$ is a maximal emulator of $E \subseteq Q[-1,1]^2$.
4. The upper shift of $S \subseteq Q[-1,1]^2$.

Gifted Youth can be walked through definition 1 by examples in dimensions 2,3,4:

DEFINITION 1. $x,y \in Q^4$ are order equivalent if and only if $(\forall i,j)(1 \leq i,j \leq 4 \rightarrow (x_i < x_j \leftrightarrow y_i < y_j))$.

We are older and can handle this definition:

DEFINITION 2. $x,y \in Q^k$ are order equivalent if and only if $(\forall i,j)(1 \leq i,j \leq k \rightarrow (x_i < x_j \leftrightarrow y_i < y_j))$.

In Baby Emulation Theory, we only use order equivalence for $x,y \in Q[-1,1]^4$.

Let ot(k) be the number of cosets of order equivalence on Q^k. There is a lot of work on ot(k) in the combinatorics community. If you are interested, look up, say, "preferential arrangements", "races with ties", "fubini numbers". Also the site https://oeis.org/A000670 from Sloane's online encyclopedia of integer sequences site. We let ot(k) be the number of cosets of order equivalence on Qk. We call the cosets the order types of k-tuples. ot(1) = 1, ot(2) = 3, ot(3) = 13, ot(4) = 75. From the site:

1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 230283190977853, 5315654681981355, 130370767029135901, 3385534663256845323, 92801587319328411133, 2677687796244384203115

We now come to the crucial definition.

DEFINITION 3. An emulator of $E \subseteq \in Q[-1,1]^2$ is an $S \subseteq Q[-1,1]^2$ such that the concatenation of any two elements of S is order equivalent to the concatenation of some two elements of E.

I.e., any two elements of S look like some two elements of E.

Note that there appears to be two uses of infinite sets in the definition of emulator. Actually there is only one real one, S, the emulator. But E, the set being emulated, isn't really infinite in light of the following.

THEOREM 1. Every $E \subseteq Q[-1,1]^2$ has exactly the same emulators as some $E' \subseteq E$ with at most 150 elements.

Proof: The number of cosets under order equivalence on Q^4 is exactly ot(4) = 75. Let $W_1,\ldots,W_r$, $0 \leq r \leq 75$, be an enumeration without repetition of the cosets of the various $x \bullet y$, $x,y \in E$. Now choose $z_1,\ldots,z_{2r} \in E$ such that each $z_{2i-1} \bullet z_{2i} \in W_i \cap E$. Then $z_1,\ldots,z_{2r}$ has the same emulsators as E. QED

There are lots of opportunities for reducing the number 150 in Theorem 1. The most obvious is to show that we need only consider certain order types of 4-tuples for being an emulation of $x_1,...,x_m \in Q[-1,1]^2$. In particular, we exploit x•y OE z•w ↔ y•x OE w•z, where OE is "order equivalent".

In particular, we need only use order types (p,q,r,s) where p ≤ r. And then we should be able to reduce this somewhat further, which we haven't really investigated. Here is the list using coordinates from 1,2,3,4. Here we list all 75 except only half of the permutations, the ones with p ≤ r. We put an x against those without p ≤ r. So the total number of order types of 4-tuples with p ≤ r is 12 + 7 + 7 + 7 + 4 + 3 + 3 + 1 = 12 + 21 + 11 = 44.

1234
1243
1324
1342
1423
1432
2134
2143
2341
2431
3142
3241

1123
1132
1213
1312
1231
1321
2113 x
3112 x
2131
3121 x
2311 x
3211 x

2213 x
2231
2123
2321
2312 x
2132
1223

3221 x
1232
3212 x
1322
3122 x

3312 x
3321 x
3132
3231
3123 x
3213 x
1332
2331
1323
2313 x
1233
2133

1122
1212
1221
2112 x
2121
2211 x

1112
1121
1211
2111 x

2221
2212 x
2122
1222

1111

THEOREM 2. Every $E \subseteq Q[-1,1]^2$ has exactly the same emulators as some $E' \subseteq E$ with at most 88 elements.

I think that some clever computer algorithms should help in reducing the number 88. I am quite curious to know what the least number is that can be used.

DEFINITION 4. A maximal emulator of $E \subseteq Q[-1,1]^2$, $m \geq 0$, is an emulator of $E \subseteq Q[-1,1]^2$ which is not a proper subset of any emulator of $E \subseteq Q[-1,1]^2$.

BABY MAXIMAL EMULATION. BME. Every $E \subseteq Q[-1,1]^2$ has a maximal emulator. The maximal emulator can be taken to be elementary recursive.

Proof: Let $E \subseteq Q[-1,1]^2$, which we assume is nonempty. By Theorem 2, let $x_1,...,x_{88} \in E$ have the same emulators as E. If you are greedy you know how to construct this. Let $y_1,y_2,...$ be an enumeration without repetition of all of $Q[-1,1]^2$. Then collect some of the y's into a set $S \subseteq Q[-1,1]^2$ as follows. Suppose we have put some of the $y_1,...,y_i$, $i \geq 0$, in S, so that we so far have an emulator of $\{x_1,...,x_{88}\} \subseteq Q[-1,1]^2$. Put y_{i+1} in S if and only if we still have an emulator of $E \subseteq Q[-1,1]^2$. Otherwise reject y_{i+1}. and move on to y_{i+2}. Clearly this is effective, assuming the $y_1,y_2,...$ is effective, because you have merely to see if certain finite sets are emulators of fixed $E \subseteq Q[-1,1]^2$. That does require that, at least superficially, we have to look to see if $S \cup \{y_{i+1}\}$ is an emulator of $\{x_1,...,x_{88}\} \subseteq Q[-1,1]^2$ by looking at all pairs (z,y_{i+1}), $z \in S$ so far, and checking to see if it is order equivalent to some $x_i \bullet x_j$. QED

There is a related computational complexity issue. Suppose E is given by a finite list from $Q[-1,1]^2$. We look for a maximal emulator S_E, and we want to understand the computational complexity of $\{(E,z): z \in S_E\}$. This moves somewhere into low level PTIME. So this does seem to be of possible interest for computational complexity.

We now come to the last definition needed for BME.

DEFINITION 5. The upper shift $ush:Q^2 \to Q^2$ is defined as follows. ush(x) is the result of adding 1 to all nonnegative coordinates of x. The upper shift $ush:\wp(Q^2) \to \wp(Q^2)$ is given by $ush(S) = \{ush(x): x \in X\}$.

We now observe BABY INVARIANT MAXIMAL EMULATION. BIME. Every subset of $Q[-1,1]^2$ has a maximal emulator containing its upper shift intersect $Q[-1,1]^2$.

There are some weak things about BIME. Note that in order for both x,ush(x) to lie in the ambient space $Q[-1,1]^2$, we must have max(x) = 0. Hence as far as BIME is concerned, only 0 gets shifted by 1 (from 0 to 1), and negative coordinates get left alone.

For the other weak thing about BIME, first note that the condition "$S \supseteq ush(S) \cap Q[-1,1]^2$" is equivalent to the following:

#) $(0,0) \in S \rightarrow (1,1) \in S$
For $p < 0$, $(p,0) \in S \rightarrow (p,1) \in S$
For $p < 0$, $(0,p) \in S \rightarrow (1,p) \in S$

However, this falls quite short of the following stronger condition:

##) $(0,0) \in S \leftrightarrow (1,1) \in S$
For $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$
For $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$

The conditions that we are placing on S are best thought of in the context of general invariance which we now present.

DEFINITION 6. Let X be a set (ambient space) and R be a set of ordered pairs (binary relation). We do not assume that $R \subseteq X^2$. $S \subseteq X$ is R invariant if and only if for all x R y with $x,y \in X$, we have $x \in S \rightarrow y \in S$.

Notice the role of the ambient space X in Definition 5. There is an important stronger form of invariance sometimes called complete invariance. Here we use the word "stable".

DEFINITION 7. Let X be a set (ambient space) and R be a set of ordered pairs (binary relation). We do not assume that $R \subseteq X^2$. $S \subseteq X$ is R stable if and only if for all x R y with $x,y \in X$, we have $x \in S \leftrightarrow y \in S$.

Thus in invariance, membership is preserved, whereas in stability, truth value of membership is preserved.

In Baby Emulation Theory we actually use stability with respect to functions. With functions treated as sets of ordered pairs, this is a special case of stability with respect to a relation. We spell this out now.

DEFINITION 8. Let X be a set (ambient space) and f be a function (from anywhere to anywhere). $S \subseteq X$ is f invariant if and only if for all $x,f(x) \in X$, we have $x \in S \rightarrow f(x) \in S$.

DEFINITION 9. Let X be a set (ambient space) and f be a function (from anywhere to anywhere). $S \subseteq X$ is f stable if and only if for all $x,f(x) \in X$, we have $x \in S \leftrightarrow f(x) \in S$.

We now restate BIME and also present the naturally stronger BSME (baby stable maximal emulation).

BABY INVARIANT MAXIMAL EMULATION. BIME. Every subset of $Q[-1,1]^2$ has a ush invariant maximal emulator.

BABY STABLE MAXIMAL EMULATION. BSME. Every subset of $Q[-1,1]^2$ has a ush stable maximal emulator.

For Gifted High School we have notes first walking them through this:

EASY BABY EMULATION. Every subset of $Q[-1,1]^2$ with at most two elements has a ush stable maximal emulator. The construction can be made computable.

And then seriously walk through more of this, which has some real substance - although purely elementary:

ADVANCED BABY EMULATION. Every subset of $Q[-1,1]^2$ with at most three elements has a ush stable maximal emulator. The construction can be made computable.

This is proved by a painstaking organization and analysis of the finitely many (although large) number of cases involved. See Downloadable Manuscripts, #111.

However, using this approach becomes rather daunting for four element subsets of $Q[-1,1]^2$, and daunting beyond daunting for five element subsets of $Q[-1,1]^2$. Recall that Theorem 2 talks of ≤ 88 element subsets of $Q[-1,1]^2$.

We will prove full BSME using a transfinite recursion of length $\omega_1 \times 3$. This is a bit beyond Z_2 or ZFC\P. But it doesn't provide a computable ush stable maximal emulator.

QUESTION 1. Does every subset of $Q[-1,1]^2$ have a recursive ush stable maximal emulator? How about four element subsets?

I have my doubts that I can get a reversal going for BSME. So we have the obviously related question:

QUESTION 2. Is BSME provable in Z_2? Provable in ACA_0? Provable in RCA_0?

BSME calls for a generally infinite object - the ush stable maximal emulator - and so at least ostensibly, leaves much to be desired in terms of Tangibility. However, the essence of the matter is totally finite combinatorics. This is because BSME really calls for a countable models of sentences in predicate calculus and we can apply the Gödel Completeness Theorem.

THEOREM 3. For each instance of BSME there is a very effectively constructed sentence in first order predicate calculus such that the BSME instance is easily seen to be equivalent to the existence of a countable model of that sentence. This equivalence proof can be carried out in RCA_0. Thus each instance of BSME is provably equivalent, over WKL_0, to a Π^0_1 sentence via Gödel's Completeness Theorem. The implication here to the Π^0_1 sentence can be carried out in RCA_0, whereas the converse here from the Π^0_1 sentence is carried out in WKL0. Also this applies to BSME itself. Hence the ush stable maximal emulator in BSME can be taken to be recursive in the Turing jump.

Proof: Let $x_1,...,x_m \in Q[-1,1]^2$. The language is $<,P,-1,0,1$, where $<,P$ are binary relation symbols. The axioms are
i. $<$ is a dense linear ordering with two endpoints -1,1, left and right.
ii. Every pair from P, concatenated, is order equivalent to some x_i,x_j, concatenated.
iii. If $P \cup \{(x,y)\}$ has ii then $P(x,y)$.
iv. $P(0,0) \leftrightarrow P(1,1)$.
v. $(\forall x < 0)(P(0,x) \leftrightarrow P(1,x))$.
vi. $(\forall x < 0)(P(x,0) \leftrightarrow P(x,1))$.

Note that i is $\forall\forall\exists$, ii is $\forall\forall\forall\forall$, and iii is $\forall\forall\exists\exists\exists\exists$.

Let $M = (A,<,P,-1,0,1)$ be a model of these axioms. Let $h:A \rightarrow Q[-1,1]$ be any order preserving bijection mapping the -

1,0,1 of the model to -1,0,1. Then h(P) is a ush stable maximal emulator of $x_1,...,x_m$. QED

We leave you with some BSME examples.

Obviously, ∅ is vacuously an emulator of any sequence, and is a maximal emulator of the empty sequence.

EX1. {(0,0)}. The emulators are ∅ and singletons {(p,p)}, -1 ≤ p ≤ 1. The maximal emulators are these singletons. {(-1,-1)} is ush stable.

EX2. {(0,1)}. The emulators are ∅ and singletons {(p,q)}, -1 ≤ p < q ≤ 1. The maximal emulators are these singletons. {(-1,-1/2)} is ush stable.

EX3. {(0,0),(1,1)}. The emulators are the subsets of {(p,p): -1 ≤ p ≤ 1}. Exactly one is maximal, {(p,p): -1 ≤ p ≤ 1}. This set is ush stable.

EX4. {(0,0),(0,1)}. The emulators are the sets that are contained in some {p} × Q[p,1], -1 ≤ p < 1. The maximal emulators are the sets {p} × Q[p,1], -1 ≤ p ≤ 1. {-1} × Q[-1,1] is ush stable.

EX5. {(0,2/5),(1/5,3/5),(2/5,4/5),(3/5,1)}. The emulators are the graphs of strictly increasing partial f:Q[0,1) → Q(0,1], where each defined f(x) > x. There are continuumly many maximal emulators. Let f be such that 0 ∉ rng(f) and f(1/2) = 1. Then f is ush stable.

EX6. E = {(1/6,1/4),(1/7,1/3),(0,1/5),(1/2,1)}. The idea behind E is that no coordinate of an element of E is a coordinate of any other element of E, and the pairs of elements are in general position relative to that restriction. Then S is an emulator of E ⊆ $Q[-1,1]^2$ if and only if S ⊆ $Q[-1,1]^{2<}$ and no coordinate of an element of S is a coordinate of any other element of S. Let S be a maximal emulator of E which contains (0,1). Then S is ush stable.

EX7. {(p,q) ∈ $Q[-1,1]^2$: -1 ≤ p < 1/2 < q ≤ 1}. We claim that S is an emulator of E ⊆ $Q[-1,1]^2$ if and only if S ⊆ $Q[-1,1]^{2<}$ and the first coordinate of every element of S is less than the second coordinate of every element of S.

Let S be a maximal emulator of E. Obviously S is nonempty. Let α be the sup of the first terms of the pairs in S and β be the inf of the second terms of the pairs in S. Clearly $\alpha \leq \beta$. If $\alpha < \beta$ then $S \cup \{(p,q)\}$ is an emulator of E where $\alpha < p < q < \beta$, violating the maximality of S. Hence $\alpha = \beta$.

The maximal emulators are the sets S =
i. $\{(p,q) \in Q[0,1]^2: -1 \leq p \leq \alpha < q \leq 1\}$, where α is a real number in $[-1,1)$.
ii. $\{(p,q) \in Q[0,1]^2: -1 \leq p < \alpha \leq q \leq 1\}$, where α is a real number in $(-1,1]$.

If $\alpha < 0$ then these sets are ush stable.