HIGHER TANGIBLE INCOMPLETENESS
$Q[-1,1]^k$ EMULATION PROOF
TANGIBLE INCOMPLETENESS SERIES
GENT LECTURE NOTES NUMBER 5

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with Addendum on Reverse Mathematical aspects

In Lecture 4, we introduced Baby Emulation Theory, which is Emulation Theory on $Q[-1,1]^2$. We stated the following theorem without proof.

BABY STABLE MAXIMAL EMULATION. BSME. Every subset of $Q[-1,1]^2$ has a ush stable maximal emulator.

BRIEF REVIEW: S is an emulator of $E \subseteq Q[-1,1]^2$ if and only if the concatenation of any two elements of S is order equivalent to the concatenation of some two elements of E. ush is the upper shift which adds 1 to all nonnegative coordinates. Because the space Q[-1,1] is so limited, ush can only change a 0 to a 1 and stay in the space. Stability is invariance with if and only if, sometimes called complete invariance. We can a priori restrict to $E \subseteq Q[-1,1]^2$ with at most 88 elements for the front end. The back end is generally infinite. We have seen that BSME is implicitly Π^0_1 via Gödel's Completeness Theorem.

GIFTED HIGH SCHOOL: I prove BSME for them where $E \subseteq Q[-1,1]^2$ has at most 3 elements. The proof is very effective, producing ush stable maximal emulators of low computational complexity. I don't know if this can be done effectively for $|E| \leq 4$. Perhaps we shouldn't work on this because that is Gifted High School research territory.

Today I am going to prove the full BSME using transfinite recursion of length $\omega_1 + \omega_1$. Very crudely, this puts BSME as provable in Z_3. NOTE: SEE ADDENDUM.

PROVACATIVE CONJECTURE. BSME can be proved in RCA_0.

A proof in RCA_0 definitely seems to need some new theory, as a case study appears totally overwhelming. I'm presenting some new theory here, and it works for all of BSME, but it is highly non effective and uses a bit more than Z_2.

However, assuming we stick with the interval Q[-1,1] where ush is so weak, there are much stronger versions of BSME. After we prove plain old BSME, we will take up these stronger versions, with more or less the same proof, but with somewhat more transfinite recursion. We like the idea of starting with BSME because I think it rather exciting to be doing transfinite recursions on $\omega_1 + \omega_1$ simply to control ordered pairs of rational numbers with only < around.

There is some elementary L theory from set theory that is good to separate off and bring it in at the right time in the proof.

SOME RELEVANT TRANSFINITE RECURSION THEORY

The following is a particular application of standard techniques in set theory going back to Gödel.

THEOREM 1. There are countable powers of ω, $\lambda_0 < \lambda_1$ such that the following holds. For all $\alpha < \lambda_0$, $\varphi(\alpha)$ holds in $L(\lambda_0 + \lambda_0)$ if and only if $\varphi(\alpha)$ holds in $L(\lambda_1 + \lambda_1)$, where φ is first order without parameters. This is provable in a weak fragment of third order arithmetic.

Proof: Start with $(L(\omega_1 + \omega_1),\in)$. This structure satisfies V = L + "there is a unique ordinal γ such that the ordinals $\gamma+\delta$, $\delta < \gamma$, comprise all of the ordinals $\geq \gamma$". Let $(A,\in)$ be a countable elementary substructure of $(L(\omega1 + \omega1),\in)$. Since $(A,\in)$ satisfies extensionality, it is isomorphic to some $(A',\in)$, where A' is countable and transitive. Since $(A',\in)$ satisfies the same sentences as $(L(\omega_1 + \omega_1),\in)$, we see that A' must be $L(\lambda_0 + \lambda_0)$ for some countable power of ω, λ_0. Since the isomorphism is the identity on λ_0 and maps ω_1 to λ_0, we have our conclusion for λ_0 and ω_1. Now we just have to show we can replace ω_1 here with another countable power of ω, $\lambda_1 > \lambda_0$. We can take another countable elementary substructure of $(L(\omega1 + \omega1),\in)$, this time making sure that λ_0+1 is contained in it. Then collapse to some $L(\lambda_1 + \lambda_1)$ as before. QED

Theorem 1 is the core set theoretic component of what we need to prove BSME. The 0,1 in Q[-1,1] correspond to λ_0,λ_1 in the proof below. But in say dimension 3 and the interval $Q[-2,2]^3$, we encounter the need for this and more: we need $\lambda_0 < \lambda_1 < \lambda_2 < \omega_1$ corresponding to 0,1,2, where for all $\alpha < \lambda_0$, $\varphi(\alpha,\lambda_0)$ holds in $L(\lambda_1+\lambda_1)$ if and only if $\varphi(\alpha,\lambda_1)$ holds in $L(\lambda_2+\lambda_2)$.

In fact, even the weaker $\lambda_0 < \lambda_1 < \lambda_2 < \omega_1$ with for all $\alpha < \lambda_0$, $\varphi(\alpha,\lambda_0)$ holds in $L(\lambda_1)$ if and only if $\varphi(\alpha,\lambda_1)$ holds in $L(\lambda_2)$ is a problem. It cannot be proved in ZFC even with the existence of an indescribable cardinal.

We now apply this to first order transfinite recursion on pairs of countable ordinals. Of course, the most common such is the lexicographic ordering, but we will want condition ii below, and also minimize order types of the initial segments α^2.

DEFINITION 1. $(\alpha,\beta) <^* (\gamma,\delta)$ if and only if
i. $\max(\alpha,\beta) < \max(\gamma,\delta)$; or
ii. $\max(\alpha,\beta) = \max(\gamma,\delta) \wedge \min(\alpha,\beta) < \min(\gamma,\delta)$; or
iii. $\alpha < \beta \wedge \alpha = \delta \wedge \beta = \gamma$.
Let $H(\alpha)$ be the order type of the set $\alpha2$.

THEOREM 2. $<^*$ is well ordered. $H(0) = 0$, H is strictly increasing, H is continuous.

i. $H(\alpha+1) = H(\alpha) + \alpha + 1$.
ii. $H(\alpha) \leq \alpha^2$.
iii. If la is a double power of ω then $H(\lambda) = \lambda \wedge H(\lambda+1) = \lambda + \lambda + 1$.

We look at the following form of recursion on ω_1^2.

DEFINITION 2. $R \subseteq \omega_1^4$ is controlled if and only if $R(\alpha,\beta,\gamma,\delta) \leftrightarrow L(\max(\alpha,\beta,\gamma,\delta)+\omega)$ satisfies $\varphi(\alpha,\beta,\gamma,\delta)$, where φ is first order with no parameters.

DEFINITION 3. Controlled <* recursion on ω_1. Let $R \subseteq \omega_1^4$ be controlled. Define $RCN(R) \subseteq \omega_1^2$ uniquely by $(\alpha,\beta) \in RCN(R)$ if and only if $(\forall\gamma,\delta)((\gamma,\delta) <^* (\alpha,\beta) \wedge (\gamma,\delta) \in RCN(R) \rightarrow R(\alpha,\beta,\gamma,\delta)))$.

THEOREM 3. Let $R \subseteq \omega_1^4$ be controlled. There exists double powers of ω, $\lambda_0 < \lambda_1 < \omega_1$ such that $(\forall\alpha < \lambda_0)(((\alpha,\lambda_0) \in RCN(R) \leftrightarrow (\alpha,\lambda_1) \in RCN(R)) \wedge ((\lambda_0,\alpha) \in RCN(R) \leftrightarrow (\lambda_1,\alpha) \in RCN(R)))$.

Proof: We apply Theorem 1. It suffices to prove that for controlled $R \subseteq \omega_1^4$ there are φ,ψ such that the following holds. For countable double powers γ of ω and $\alpha < \gamma$, $(\alpha,\gamma) \in RCN(R) \leftrightarrow \varphi(\alpha)$ holds in $L(\gamma+\gamma)$, and $(\gamma,\alpha) \in RCN(R) \leftrightarrow \psi(\alpha)$ holds in $L(\gamma+\gamma)$. For countable double powers γ of ω, it is easy to see how to uniformly define RCN(R) at $\alpha,\beta < \gamma$, in $L(\gamma)$. Then we can clearly define RCN(R) at $\alpha < \gamma$ and γ uniformly in $L(\gamma+\gamma)$. QED

PROOF OF BSME

BABY STABLE MAXIMAL EMULATION. BSME. Every subset of $Q[-1,1]^2$ has a ush stable maximal emulator.

Fix $E \subseteq Q[-1,1]^2$. The only role that E plays is to provide us with a quantifier free formula $\varphi(p,q,r,s)$ in < only, without parameters, so that the $S \subseteq Q[-1,1]^2$ we are looking for must satisfy $(\forall p,q,r,s)((p,q),(r,s) \in S \rightarrow \varphi(p,q,r,s))$. In fact, maximally satisfy this, and also be ush stable.

But there aren't any ordinals here let alone a transfinite recursion on pairs of ordinals as in Theorem 3. So the first thing we do is to bring ordinals into the picture. In particular, we form the lexicographic product $\omega_1 \times Q[0,1)$. Thus $(\alpha,p) <' (\beta,q) \leftrightarrow \alpha < \beta \vee (\alpha = \beta \wedge p < q)$. This linear ordering is dense with the left endpoint (0,0), but has no right endpoint.

We are going to build the desired maximal emulator by transfinite recursion of the kind in Theorem 3. Right now of course, <' is not well ordered, but let's pretend it is for just a second. How do we build a maximal emulator by transfinite recursion on pairs? By GREED. If we have a well ordering of the pairs, then greed says that we put a pair in S if and only if it will still be an emulator after we put it in. Otherwise we pass it up and move on to the next pair along the well ordering. Greedy emulators are obviously maximal emulators, and even much better than a typical maximal emulator. Being an emulator is defined as above using φ. Here φ must use <'.

But we can't use <' to perform the transfinite recursion because <' is ill founded. Of course the ill foundedness is coming from the second lexicographic factor, Q[0,1). So we fix any effectively given enumeration without repetition $0 = p_0,p_1,...$ of Q[0,1). We now work with <* given by $(\alpha,p) <^* (\beta,q) \leftrightarrow \alpha < \beta \vee (\alpha = \beta \wedge (\exists i<j)(p = p_i \wedge q = p_j))$. Note that <* is a well ordering.

We now use <* to make the greedy construction of the maximal emulator $S \subseteq \omega_1 \times Q[0,1)$ identifying $(\omega_1 \times Q[0,1),<^*)$ with $(\omega_1,<)$ by the obvious unique isomorphism. But what has become of the crucial 4-ary relation R that we always use for the kind of transfinite recursion on pairs we are doing? It is as before for emulation, using <', where we go from ω onto Q[0,1). I.e., from i to p_i. Thus the 4-ary relation R is easily controlled as in Definition 2 above.

So we now have a maximal emulator $S = RCN(R) \subseteq \omega_1 \times Q[0,1)$ and limits $(\lambda_0,0),(\lambda_1,0)$ in <*, such that for all $(\alpha,p) <^* (\lambda_0,0)$, $((\alpha,p),(\lambda_0,0)) \in S \leftrightarrow ((\alpha,p),(\lambda_1,0)) \in S$, and $((\lambda_0,0),(\alpha,p)) \in S \leftrightarrow ((\lambda_1,0),(\alpha,p)) \in S$. It is important to note that the limits in <* are exactly the pairs other than (0,0), whose second coordinate 0, and also $(\alpha,p) <^* (\lambda_0,0) \leftrightarrow \alpha < \lambda_0$.

$(\omega_1 \times Q[0,1),<')$ is a dense linear ordering and we can take the initial segment ending with the right endpoint $(\lambda_1,0)$, to form $(\omega_1 \times Q[0,1),<')|\leq'(\lambda_1,0)$ which is dense with both endpoints present. Note that S is a maximal emulator here using the original $\varphi(p,q,r,s)$. If we consider the ush to be just like the real ush, only here $(\lambda_0,0)$ is considered zero and $(\lambda_1,0)$ is considered one, then we have obtained an uncountable version of BSME exactly, with this S. Any countable elementary substructure gives us the original countable version in the rationals.

Now what about the promised stronger forms of BSME?

DEFINITION 4. S is an r-emulator of $E \subseteq Q[-1,1]^k$ if and only if $S \subseteq Q[-1,1]^k$, where the concatenation of any r elements of S is order equivalent to the concatenation of some r elements of E.

STABLE MAXIMAL EMULATION/Q[-1,1]. BSME/Q[-1,1]. Every subset of $Q[-1,1]^k$ has a ush stable maximal r-emulator.

Thus we are using order equivalence for kr-tuples. The use of r-emulators instead of emulators = 2-emulators does not change the proof at all. The r comes in only in the greedy definition of an r-emulator, instead of just an emulator. In the greedy construction, we need to preserve being an r-emulator, not just being an emulator.

There is a difference in the proofs when we move from dimension $k = 2$ to dimension $k \geq 3$. We use a k dimensional form of Theorem 3. Here the recursion involved in the greedy construction used to be of length λ to handle pairs $(\alpha,p),(\beta,p)$, $\alpha,\beta < \lambda$, λ a double power of ω, but then to handle pairs $(\lambda,0),(\alpha,p)$, $\alpha < \lambda$, we need length roughly $\lambda+\alpha$. So we get to length $\lambda+\lambda$ to handle all pairs $(\lambda,0),(\alpha,p),(p,\alpha)$, $\alpha < \lambda$. So for 3 dimensions, the same line of reasoning leads to a transfinite recursion of length $\lambda \times 3$. And for k dimensions, length $\lambda \times k$.

So in this way, for full BSME/Q[-1,1], we will use transfinite recursion of length $\omega_1 \times \omega$.

ADENDUM
REVERSE MATHEMATICAL ASPECTS

added after Lecture
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So I have given a proof here of BSME (first with k = r = 2 and then for k,r ≥ 2), and the question is where does this proof reside? An obvious answer is of course in Z_3 = third order arithmetic. But that seems like great overkill. It is obviously much much closer to Z_2. If the proof just used transfinite recursion on ω_1 in nice form, then probably it could be relatively easily formalized in say WKL_0 + Con(Z_2). But there is more transfinite recursion going on than that. Namely we wrote "using transfinite recursion on $\omega_1 + \omega_1$" and "using transfinite recursion on $\omega_1 \times k$" and "using transfinite recursion on $\omega_1 \times \omega$".

But what kind of RM framework would make proper sense of this assertion? Something much more informative than Z_3. One could look for fragments of Z_3 and that is in fact probably quite a reasonable idea here. I don't think there has been much work on fragments of Z_3 mainly because there is not so much mathematics that is profitably formulated in the language of Z_3 rather than the language of Z_2. I haven't thought much about the appropriate weak forms of comprehension and other principles in the Z_3 fragment context. But here we seem to be naturally using ω_1 to prove a theorem in discrete mathematics.

We can also envision using fragments of set theory for this purpose. We could have a Reverse Set Theory where the main systems are fragments of $ZF\backslash\wp$. Then one can directly talk of the $L(\alpha)$'s, α countable.

But when we come to "transfinite recursion on ω_1" and "transfinite recursion on $\omega_1 \times k$", there is a problem in using a set theoretic base theory with too much replacement. For with replacement one gets immediate all of this transfinite recursion just from the existence of ω1. So a plausible interesting base theory for this might be the following.

1. extensionality.
2. pairing.
3. union.
4. Δ_0-separation
5. infinity

6. ω-collection

Then, for example, $\exists L(\omega_1)$ doesn't imply $\exists L(\omega_1 + \omega_1)$ doesn't imply $\exists L(\omega_1 \times 3)$, and so forth. Or one can more abstractly define what we mean by "transfinite recursion on $\omega_1 \times k$", for various $k \geq 1$.

We now come to a way of avoiding this issue entirely, but it is not completely satisfactory. We can focus on the application(s) only of transfinite recursion on the $\omega_1 \times k$, like Theorem 3 (and adaptions to higher dimensions). And these applications are entirely COUNTABLE. They clearly fall totally within the purview of my ordinary RM.

This leads to an interesting area of RM that has had limited exploration. The effective base theory would be my ATR_0, particularly after the main preliminary result that of course just having transfinite recursions along any well ordering is equivalent to ATR_0 over my RCA_0 base theory. But from the above developments, there is clearly of form of transfinite recursion that is very fundamental, namely transfinite recursion with respect to order invariant relations R. This is equivalent to parameterless ATR_0 over RCA_0. Parameterless ATR_0 doesn't appear much in RM as far as I know.

I haven't thought it through, but it would seem that there is probably a very satisfying graph theoretic or hypergraph theoretic interpretation of transfinite recursion, where the condition of order invariance is natural and fundamental. This might profitably bring parameterless ATR_0 into RM.

In any case, it seems like there is an area of order invariant or countable transfinite recursion, in the countable. I wrote

https://cpb-us-w2.wpmucdn.com/u.osu.edu/dist/1/1952/files/2014/01/MetaComp100701-1h9ey86.pdf
https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3133675

and not sure it has been followed up properly. It has the following result:

THEOREM 4.5. The following are provably equivalent over

RCA_0.
i) ATR_0;
ii) For any two countable metric spaces, there is a continuous embedding from one into the other;
iii) For any two sets of rationals, there is a continuous embedding from one into the other;
iv) For any two compact well ordered sets of rationals, there is a continuous embedding from one into the other;
v) For any two countable well orderings with greatest elements, there is an order continuous embedding from one into the other.
vi) For any two countable well orderings, there is an order continuous embedding from one into the other.

The developments in this talk suggest a big realm of statements concerning the relationship between various transfinite recursions of different length and how they cohere. The strengths involved would go from maybe just ATR0 to large cardinals. And there would also be paramterless versions, based on how elemental the front ends are.

In any case, when we go to higher dimensions and longer intervals, we run quickly into infinitely many uncountable cardinals, ZFC, and large cardinals. So looking for ways to metamathematically analyze "transfinite recursion on ω_1" seem less compelling.