UNIFORM DESCENT AND TANGIBLE INCOMPLETENESS announcement

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ABSTRACT. We present deterministic processes for constructing sequences of finite sets of nonnegative integers whose maximum elements strictly decrease. Thus these processes terminate, and we establish very strong uniform bounds on the lengths of these sequences. The results can be proved only in impredicative systems. Thus we have independence results of a strong thematic character, with much left to be investigated.

Let $F_1,\ldots,F_n:N^k \to N$, $n \geq 1$. We start a deterministic process with a nonempty finite $A_0 \subseteq N$. We define $A_{i+1} = \cup_j F_j[A_i^k] \cap [0,\max(A_i))$.

It is clear that the max(Ai) strictly descend, and so we eventually reach $A_i = \emptyset$. We call this i the length of $F_1,\ldots,F_n$ on A_0. What can we say about this length?

We shall see that for a particular important class of multivariate functions, these lengths are bounded as a function of only the cardinality of the starting set A_0. This demonstrates an important uniformity.

Actually, the uniformity is far stronger. For a particular important class of multivariate functions, length depends only on the cardinality of the starting set A_0 and the number and arity of the functions.

The relevant functions $F:N^k \to N$ are called the monotonically dominating functions. This means that for all $x,y \in N^k$, if $x \leq y$ coordinatewise, then $\max(x) \leq F(x) \leq F(y)$.

UNIFORM DESCENT. Let $t \gg k,n,r$. For all $A \subseteq N$ with at most r elements, and monotonically dominating $F_1,\ldots,F_n:N^k \to N$, the length of $F_1,\ldots,F_n$ on A is at most t.

THEOREM 1. Uniform Descent is provably equivalent to $Con(\Pi^1_2\text{-}TI_0)$ over RCA_0, the same as the strength of Kruskal's Theorem with no labels.

Monotonic dominance makes perfectly good sense for $F:\{0,r]^k \to N$.

FINITE UNIFORM DESCENT. Let $t \gg k,n,r$, and $s \in N$. For all $A \subseteq N$ with at most r elements, and monotonically dominating $F_1,\ldots,F_n:[0,s]^k \to N$, the length of $F_1,\ldots,F_n$ on A is at most t.

This is explicitly Π^0_3. We can be careful to make this explicitly Π^0_2 as follows.

FINITE UNIFORM DESCENT. For all k,n,r there exists t so large that the following holds. For all $A \subseteq [t!]$ with at most r elements, and monotonically dominating $F_1,\ldots,F_n:[0,t!]^k \to N$, the length of $F_1,\ldots,F_n$ on A is at most t.

Note that monotonically dominating functions are closed under substitution. There are a great many natural monotonically dominating functions in mathematics. We can apply Uniform Descent to these functions. What logical strengths and growth rates do we get? It does appear that if we use addition, multiplication, and exponentiation, then we obtain at least 1-Con(PA).