HIGHER TANGIBLE INCOMPLETENESS
SIDES, N TAILS, FALSIFIABILITY
TANGIBLE INCOMPLETENESS SERIES
GENT LECTURE NOTES NUMBER 6

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So in Lecture 5 we have proved the following.

STABLE MAXIMAL EMULATION/Q[-1,1]. SME/Q[-1,1]. Every subset of $Q[-1,1]^k$ has a ush stable maximal r-emulator.

Let's quickly over all of the relevant definitions in case you have forgotten.

1. Q[-1,1] = Q ∩ [-1,1].
2. S is an r-emulator of $E \subseteq Q[-1,1]^k$ if and only if the concatenation of any r elements of S is order equivalent to the concatenation of some r elements of E.
3. S is a maximal r-emulator of $E \subseteq Q[-1,1]^k$ if and only if S is an r-emulator of $E \subseteq Q[-1,1]^k$ that is not a proper subset of any r-emulator of $E \subseteq Q[-1,1^k$.
4. The upper shift ush of $S \subseteq Q[-1,1]^k$ is obtained by adding 1 to all nonnegative coordinates of the elements of S.
5. We can also define ush in stages: ush:Q → Q, ush:$Q^k \to Q^k$, ush:$\wp(Q^k) \to \wp(Q^k)$.
6. $S \subseteq Q[-1,1]^k$ is ush stable if and only if (∀x)(x,ush(x) $\in Q[-1,1]^k$ → (x ∈ S ↔ ush(x) ∈ S)).

We proved BSME/Q[-1,1] last time using transfinite recursion on $\omega_1 \times \omega$. With k fixed, we used $\omega_1 \times k$. The proof is the same for r = 2 general r.

Gifted High School realm is k = r = 2, and where $E \subseteq Q[-1,1]^2$ has cardinality at most 3. Proofs effective there, getting a low level computable $S \subseteq Q[-1,1]^2$. Proof for even $|E| \leq 4$ is not effective.

This lecture and the next concerns proofs of forms of Stable Maximality. There is something stronger than Emulators that is particularly. These are what we call Sides.

DEFINITION 1. S is a side of $E \subseteq X^{2k}$ if and only if $S \subseteq X^k$ and $S^2 \subseteq E$.

Think of $E \subseteq X^{2k}$ as a "square" so clearly $S \subseteq X^k$ is like a "side" as in geometry.

DEFINITION 2. S is an r-side of $E \subseteq X^{kr}$ if and only if $S \subseteq X^{kr}$ and $S^r \subseteq E$.

These r-sides are more general than r-emulators in a careful sense.

Emulators of $E \subseteq Q[-1,1]^k$ are like Sides of order invariant $E' \subseteq Q[-1,1]^{2k}$.
r-emulators of $E \subseteq Q[-1,1]^k$ are like r-sides of order invariant $E' \subseteq Q[-1,1]^{kr}$.

So far we have only been using the crucial order equivalence relation in the definition of emulators and r-emulators. We have not been using order invariant sets.

DEFINITION 3. $E \subseteq Q[-1,1]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[-1,1]^k$, $x \in E \rightarrow y \in E$.

Because we are dealing with equivalence relations, invariance and stability are the same. I.e., we can replace $\rightarrow$ above with $\leftrightarrow$.

Recall ot(k) is the number of cosets of Q^k under order equivalence. In an earlier Lecture I told you about its webpage in the Encyclopedia of integer sequences under Fubini numbers and other names.

THEOREM 1. The number of order invariant subsets of Q^k is $2^{ot(k)}$. The number of order invariant subsets of Q^4 is 2^{75}.

THEOREM 2. The emulators of any $E \subseteq Q[-1,1]^k$ are the same as the sides of some order invariant $E' \subseteq Q[-1,1]^{2k}$.

THEOREM 3. The (maximal) r-emulators of any $E \subseteq Q[-1,1]^k$ are the same as the (maximal) r-sides of some order invariant $E' \subseteq Q[-1,1]^{kr}$.

Take $E' = \{u \in Q[-1,1)^{kr}$: u is order equivalent to some element of $E^r\}$.

Is this an exact correspondence? No. But it actually turns out that it is fairly and usefully close to an exact correspondence. We take this topic up when we get into the REVERSALS starting June 9.

We now have the sharper Stable Maximality Theorem:

STABLE MAXIMAL EMULATION/$Q[-1,1]^k$. SME/$Q[-1,1]^k$. Every subset of $Q[-1,1]^k$ has a ush stable maximal r-emulator.

STABLE MAXIMAL SIDES/$Q[-1,1]^{kr}$. SMS/$Q[-1,1]^{kr}$. Every order invariant subset of $Q[-1,1]^{kr}$ has a ush stable maximal r-side.

Same proof using $\omega_1 \times \omega$ recursion. At the core of the proof was the GREEDY maximal r-emulator of a transfinite k dimensional E. Here we instead use the corresponding GREEDY maximal r-emulator of an order invariant transfinite kr dimensional E.

Of course while we are concerned with proofs of Stable Maximality, we are better off using SIDES than EMULATORS. For reversals, obviously Emulators are better than Sides. For Gifted High School, Emulators are better than Sides because it avoids the extra abstraction: order invariant sets.

Note that SMS involves the $2^{ot(kr)}$ order invariant subsets of $Q[-1,1]^{kr}$. This is already 2^{75} order invariant subsets of $Q[-1,1]^4$ in the case k = r = 2. Of course, there are tremendous yet to be discovered opportunities for consolidating cases and reducing the number 2^{75} considerably. But still it probably remains rather high.

CONJECTURE. SMS/Q[-1,1]^4 is effectively true, and provable in RCA_0.

WEAKNESS OF THE UPPER SHIFT ON Q]-1,1]^2
WHAT HAPPENS ON Q[-2,2]^2?

Note how WEAK the ush is on Q[-1,1]^k. Note that from the definition of stable, we are only using x,ush(x) $\in$ Q[-1,1]^k. Then either max(x) < 0 in which case ush(x) = x and stability says nothing, or max(x) = 0 and ush(x) merely changes the 0's to 1's. We can't use any x with max(x) > 0 because then ush(x) $\notin$ Q[-1,1]^k.

So when we go to

STABLE MAXIMAL SIDES/Q[-2,2]^4? SMS/Q[-2,2]^4?. Every order invariant subset of Q[-2,2]^4 has a ush stable maximal side.

we have a statement of a rather different character. Here the ush relevantly moves Q[0,1] onto Q[1,2].

I don't believe that we can use the upper shift on even Q[-2,2]^2 but I haven't finished proving this yet. I certainly don't expect to make much progress here with ush.

So where do we go from here?

Well, in looking at the upper shift on Q[-1,1]^2, we can interpret it instead as shifting the nonnegative INTEGER coordinates. This would of course be the same as the upper shift, when insisting as we always do, that we remain in the space we are working in. In this case Q[-1,1]^2.

So when we go to Q[-2,2]^k we can try to simply shift the nonnegative integer coordinates. However, this is very bad, because look at

(0,1/2) goes to (1,1/2)

There is a general theorem that no stability can be imposed here between even any two specific tuples of different order types. Not even invariance.

DEFINITION 4. $R \subseteq Q[-n,n]^{2k}$ is order preserving if and only if $x R y \rightarrow x,y$ are order equivalent.

THEOREM 4. Let $R \subseteq Q[-n,n]^{2k}$. Suppose every order invariant $E \subseteq Q[-n,n]^{2k}$ has an R invariant maximal side. Then R is order preserving.

Proof: Let R be as given and let u R v. Define $x E y \leftrightarrow x,y$ are order equivalent to u. Obviously E is order invariant, and there is exactly one maximal side, namely S = {w: w is order equivalent to x}. Then S is R invariant and $u \in S$. Hence $v \in S$. Therefore u,v are order equivalent. QED

So what kind of shifting of nonnegative integers do we want to do?

DEFINITION 5. Let $x \in Q^k$. The N Tail of x are the coordinates x_n such that every $x_m \geq x_n$ lies in N.

With the N Tail, we keep these specified coordinates in position in the tuple x. There will in general be repetitions. Sometimes the N tail is empty.

(-3/2,5,4,3,13/2). N tail is empty.
(-3/2,5,4,3,7/2). N tail is marked here: (-3/2,5•,4•,3,7/2).

DEFINITION 6. Let $x \in Q^k$. The N Tail Shift of x results from adding 1 to all coordinates in the N Tail of x. We write Ntsh(x).

Ntsh(-3/2,5,4,3,13/2) = (-3/2,5,4,3,13/2)
(-3/2,5,4,3,7/2). N tail is marked here: (-3/2,5•,4•,3,7/2).

Here is what we are going to prove from large cardinals next Wednesday.

STABLE MAXIMAL SIDES/Ntsh. SMS/Ntsh. Every order invariant subset of $Q[-n,n]^{kr}$ has an Ntsh stable maximal r-side.

Note that for k = r = 2 we have already proved this using transfinite recursion on $\omega_1 \times \omega$.

In my posting on FOM, #883, I wrote, by mistake, "We also give a proof of this using transfinite recursion on omega_1 x omega." We will actually use large cardinals for this next week.

There is also a little bit stronger formulation which is what we actually will prove.

DEFINITION 7. Let $x,y \in Q^k$. x,y are N Tail Related if and only if x,y are order equivalent, and their N Tails occupy the same positions. We write x Ntr y.

Obviously x Ntr Ntsh(x).

STABLE MAXIMAL SIDES/Ntr. SMS/Ntr. Every order invariant subset of $Q[-n,n]^{kr}$ has an Ntr invariant maximal r-side.

Note that the front end of SMS/Ntr is very concrete as there are only finitely many order invariant subsets of $Q[-n,n]^{kr}$. But the back end is not: generally speaking the side is infinite. So SMS/Ntr is explicitly Σ^1_1. HOWEVER, this hides the rather Tangible nature of SMS/Ntr in the following sense.

DEFINITION 8. Here the language of second order arithmetic L(Z2) is as usual with exponentiation (for ease of formulation of EFA). T is adequate if and only if T is in $L(Z_2)$, is presented with finitely many axioms and axiom schemes, and proves EFA. Let $\varphi \in L(Z_2)$ and T be adequate. φ is implicitly Π^0_1 over T if and only if there is a Π^0_1 sentence ψ such that T proves $\varphi \leftrightarrow \psi$.

Of course, being actual Π^0_1, or what we call explicitly Π^0_1, is "better". But implicitly Π^0_1 sentences φ over T have a very important property.

DEFINITION 9. Let $\varphi \in L(Z_2)$ and T be adequate. φ is intrinsically falsifiable over T if and only if
i. T is given by finitely many axioms and axiom schemes.
ii. T + EFA proves "if φ is false then φ is provably false in T".

Intrinsic falsifiability is something that is very highly valued in scientific circles. The idea is that theories are to have consequences that can be tested by experimentation. The experiments don't prove that the theory is true, but an experiment can refute the theory. In fact, it is generally viewed in science that a meaningful theory has to be intrinsically falsifiable in this experimental sense.

Up to now, there has not really been mathematically interesting examples of intrinsically falsifiable statements independent of any of our usual f.o.m. formal systems. All of the statements we have been discussing in Tangible Incompleteness here are intrinsically falsifiable over WKL_0. This is because they are all implicitly Π^0_1 over WKL_0.

LEMMA 5. Let $\varphi \in L(Z_2)$ and T be adequate. If φ is implicitly Π^0_1 over T then φ is intrinsically falsifiable over T.

Proof: Let T prove $\varphi \leftrightarrow (\forall n)(H(n) = 0)$, H a primitive recursive function symbol. We argue in T. Suppose $(\exists n)(H(n) \neq 0)$, and fix n. Then verify $H(n) \neq 0$ by computation, showing that T (and much less) proves $\neg(\forall n)(H(n) = 0)$. Hence T proves $\neg\varphi$, plugging in the proof in T of $\varphi \leftrightarrow (\forall n)(H(n) = 0)$. QED

DEFINITION 10. IND is full induction in $L(Z_2)$. Let $\varphi \in L(Z2)$. IND/φ is induction for all subformulas of φ.

=LEMMA 6. Let $\varphi \in L(Z_2)$ and T be adequate. If φ is intrinsically falsifiable over T then φ is implicitly Π^0_1 over T + IND. In fact, over T + IND/φ.

Proof: Let φ,T be as given. Let φ be intrinsically falsifiable over T. We use Con(T + φ). We have to show that T + IND proves $\varphi \leftrightarrow$ Con(T + φ). We have T proves Con(T + φ) $\rightarrow \varphi$. So it remains to prove $\varphi \rightarrow$ Con(T + φ) in T + IND. This is well known. And by using Cut Elimination in predicate calculus, T + IND/φ suffices. QED

THEOREM 7. Let $\varphi \in L(Z_2)$ and T be adequate. Each implies the next over EFA.
1. φ is implicitly $\Pi^0 1$ over T.
ii. φ is intrinsically falsifiable over T.
iii. φ is implicitly Π^0_1 over T + IND/φ.
iv. φ is implicitly Π^0_1 over T + IND.

Proof: By Lemmas 5,6. QED

We apply this general theory to Stable Maximal Sides/Ntr and WKL_0/WKL. WKL is by definition WKL_0 + IND.

THEOREM 5. Let φ be a sentence in second order arithmetic. Each of the following implies the nest, over EFA.
1. φ is implicitly Π^0_1 over WKL_0.
ii. φ is falsifiable over WKL_0.
iii. φ is implicitly Π^0_1 over WKL.
iv. φ is falsifiable over WKL.

We now examine this:

STABLE MAXIMAL SIDES/Ntr. SMS/Ntr. Every order invariant subset of $Q[-n,n]^{kr}$ has an Ntr invariant maximal r-side.

We can see by a nice application of Gödel's Completeness Theorem that SMS/Ntr is implicitly Π^0_1 over WKL_0. In fact, this is true for any given choice of k,r,n and order invariant $E \subseteq Q[-n,n]^{kr}$. Therefore this holds of single sentence SMS/Ntr or we can fix some parameters and universally quantify over others.

So fix k,r,n and order invariant $E \subseteq Q[-n,n]^{kr}$. Let $u_1,...,u_s \in \{0,...,kr\}^{kr}$ be such that the order types of the x's are exactly the order types of the elements of E.

We now form the following theory T in predicate calculus with symbols =,<,-n,0,1,...,n, and the k-ary relation symbol S. The axioms of T are as follows.

1. < is a dense linear ordering with endpoints -n,n, and elements -n < 0 < 1 ... < n.
2. For all k-tuples $x_1,...,x_r$ with S, $(x_1,...,x_r)$ is order equivalent to some u_i.
3. Suppose 2 holds if we adjoin k-tuple x into S. Then S(x).
4. If k-tuples x,y are N Tail Related, with the N Tails defined in terms of 0,...,n, then $S(x) \rightarrow S(y)$.

It is clear that T has a countable model if SMS/Ntr is true for the given parameters. For the other direction, let M be a countable model of T. Make an isomorphism h onto $Q[-n,n]^k$ sending the constants -n,0,...,n to actual -n,0,...,n. h[S] is the desired r-side. There are some issues about the base theory of this argument that can be handled in several ways. I will address them in the proof below.

THEOREM 6. SMS/Ntr, with none or some or all of the various parameters fixed, is implicitly Π^0_1 over WKL_0. Hence it is also intrinsically falsifiable over WKL_0.

Proof: The T constructed above is a single sentence. SMS/Ntr ↔ Con(T) is provable, but we need it to be provable in WKL_0. We look at the two directions separately.

Con(T) → SMS/Ntr. From Con(T) we obtained a model M of T with domain ω using WKL_0. Extracting the maximal r-emulator S from M poses no difficulties in RCA_0.

SMS/Ntr → Con(T). Here there is a problem. The obvious proof of this implication uses induction for all formulas of $L(Z_2)$ as they may appear in a proof in T of 1 = 0. Cut elimination would cut these down, and Σ01-incudtion is available in WKL0, rather than show that Σ01-induction is all that is needed, we proceed a bit differently. T is in ∀...∀∃...∃ form, and we can introduce function symbols Skolem functions for this in the obvious way, replacing T by a single ∀...∀ sentence T*. Obviously Con(T*) → Con(T). By cut elimination, available in RCA_0, from ¬Con(T) we have a cut free proof of ¬T* in predicate calculus using only existentially quantified formulas. Using the maximal r-emulator S we can give a Σ^0_1 truth definition for these existentially quantified formulas, and then use induction for them in order to get a contradiction from ¬Con(T). This only uses Σ^0_1 induction and thus can be formalized in RCA_0. QED

It should be noted that SMS/Nts is equivalent to a Π^0_1 sentence with the ← direction proved in WKL_0 and the → direction proved in RCA_0. This is true of all of our implicitly Π01 statements in Tangible Incompleteness.