THE EMERGENCE OF (STRICT) REVERSE MATHEMATICS

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ABSTRACT. We discuss the emergence of Reverse Mathematics (RM) and Strict Reverse Mathematics (SRM). These grew out of our attempts to impress mathematicians at the afternoon teas in the Stanford University Mathematics Department with the robustness and fundamental mathematical relevance of various formal systems arising in mathematical logic - episodically through my presence there in 1967-69, 1971. We gradually saw the less demanding RM vision as more appropriate for initiating a long term foundational investigation into the logical structure of mathematical practice. My ICM paper [Fr75] clearly presented this RM vision, using RCA, WKL, ACA, ATR, Π^1_1-CA, and some other subsystems of Z_2, with RCA as the base theory, all based on two sorts: numbers and sets of numbers. Soon thereafter, in order to realize at least some of the more demanding SRM vision, we first experimented with replacing the induction scheme in these theories with the strictly mathematical set induction. For ACA and up, this results in the systems ACA_0, ATR_0, Π^1_1-CA_0, and others, in numbers and sets, that we use today as benchmarks for RM. We realized that for RCA (and WKL) the resulting system was inappropriately weak and we needed to transfer to numbers and functions (arity ≤ 3) in order to add some additional basic strictly mathematical axioms. This resulted in the systems RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0, and others, with numbers and functions, of [Fr76]. As RM took hold and SRM remained dormant, these RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0, and others, from [Fr76], were transferred from numbers and functions back to numbers and sets, resulting in the familiar systems we know today with the same names (naught notation), in numbers and sets. This transfer back to L[set] lost some of the strict mathematical character, but adds convenience for the practical development of RM. All of these transfers were done at an intuitive level, with the idea that the transfers were evidently content preserving. Here we treat the transfers formally in terms of arithmetic preserving synonymy in the sense of de Bouvere (having an arithmetic preserving common definitional extension), justifying the intuitively grasped content preservation in various ways. Thus the systems RCA$_0$, WKL$_0$, ACA$_0$, ATR$_0$, Π^1_1-CA$_0$, and others from [Fr76], are respectively arithmetic preserving synonymous with the systems RCA$_0$, WKL$_0$, ACA$_0$, ATR$_0$, Π^1_1-CA$_0$, and others from current Reverse Mathematics - and these synonymies are very natural and uniform. Furthermore, there is a nonobvious claim in [Fr76] that the ETF and RCA$_0$ of [Fr76] are logically equivalent. We left this result undocumented because of the dominance of RM over SRM. We fill this gap here and exploit it, concluding also that the strictly mathematical ETF in numbers and functions has an arithmetic preserving synonymy with the current RCA$_0$ in numbers and sets. This shows how [Fr76] initiates SRM (or more specifically CSRM = countable SRM), and that current RM can be construed as the special case of SRM (or CSRM) with base theory ETF. From this standpoint, the much later [Fr09] initiates FSRM = finite strict reverse mathematics. We conclude with an overview of developments and projects ins Strict Reverse Mathematics.

1. INTRODUCTION

RM and SRM (Reverse Mathematics, Strict Reverse Mathematics) were born out of my attempts to impress Stanford mathematicians at afternoon teas, of the robustness and fundamental mathematical relevance of various formal systems arising in mathematical logic. This occurred during my presence there in

1967-69, 1971, where I felt compelled to give the more ambitious SRM idea the most emphasis in those early days.

We discuss the emergence of Reverse Mathematics (RM) and Strict Reverse Mathematics (SRM), whose origins are concurrent and intertwined. We include a detailed analysis of the founding papers [Fr75], [Fr76]. We also discuss a new consolidation of numbers, sets, and functions, and also an overview of some projects in Strict Reverse Mathematics.

This history involved first the presentation in [Fr75] of a number of two sorted systems in numbers and sets. Then the transfer of these systems from two sorted numbers and sets, to four sorted numbers and functions (arities 1,2,3) in [Fr76]. Then, finally, the transfer of these systems from [Fr76] back to numbers and sets. Our transfer from numbers and sets to numbers and functions was motivated by the SRM vision, which we partially realized in [Fr76]. The transfer from numbers and functions back to numbers and sets went against the SRM vision in favor of convenience for the practical development of RM.

These transfers were totally natural and made at the intuitive level essentially without comment. In fact, Simpson in [Si85], p. 149, writes:

"(These versions of RCA_0 and WKL_0 [current RM versions in numbers and sets] are superficially different from, but essentially equivalent to, Friedman's formulation in [Fr76]. The superficial discrepancy arises because [Fr76] uses a language with functions variables rather than set variables. Thus Friedman is able to formulate RCA_0 and WKL_0 in terms of a quantifier free induction axiom. Then, with the help of some other axioms for the existence of functions (including primitive recursion and the μ-operator), he is able, within his version of RCA_0 to prove Σ^0_1 induction.)"

Actually, as will see in section 5, it is the Rudimentary Induction Axiom and not quantifier free induction, and it is the Permutation Axiom and not the μ-operator, both notable improvements over what Simpson wrote, from the Strict Reverse Mathematical point of view.

In section 7 we verify that the systems in L[set] arising from these transfers are in fact logically equivalent to the standard systems in numbers and sets currently in use for RM.

We also analyze these transfers formally. All of the systems are based on the well known version of first order predicate calculus with established syntax and semantics and a complete system of axioms and rules of inference. More specifically, we use many sorted logic with function, relation, and set variables under the Henkin semantics, going back to [He50]. For this purpose, we use the de Bouvere synonymy, and the fundamental equivalents of model synonymy and Visser synonymy, adapted to the many sorted logic with point, function, relation, and set variables. In section 8 we provide a brief background concerning synonymy in our context. In section 9 we give the natural common synonymies for each of the systems in [Fr76] and each of the usual systems for RM. I.e., common synonymies for each of RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and others/f of [Fr76], and the corresponding RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and others/s of current RM.

NOTE: In order to avoid confusion for readers, we use /f to indicate that the systems are in the language of numbers and functions as in [Fr76], and /s to indicate that the systems are in the language of numbers and sets as in [Fr75] and in current RM.

In [Fr76] there is a nonobvious claim that the strictly mathematical system ETF in numbers and functions already logically implies, and therefore is logically equivalent to RCA_0/f. Precise information about the power of the strictly mathematical theory ETF in [Fr76] did not ultimately play any role in this history as the transfer back into numbers and sets resulting in the present RM setup rendered the status of ETF itself, moot historically. Thus this nonobvious claim was not documented because RM took hold while SRM remained dormant. We document this claim and more in section 10.

In addition to [Fr75],[Fr76] being the decisive initiation of RM, we now regard [Fr76] as the initiation of SRM (strict reverse mathematics) - at least modulo the nonobvious claim there we just alluded to.

The much later [Fr09] was presented as the initiation of SRM. We have now rethought this matter, and we reconcile these two views as follows. [Fr76] initiates CSRM (countable strict reverse mathematics, with the help of this section 10), whereas [Fr09] initiates FSRM (finite strict reverse mathematics).

We now give some more details about this paper's content.

There are two formal languages relevant to this history. One is the two sorted language of Z_2 with numbers and sets (for the emergence of RM), and the other is the four sorted language of ETF (elementary theory of functions) with numbers and functions of arities 1,2,3 (crucial for the emergence of SRM).

DEFINITION 1.1. L[set] is the relevant language with sets. It has two sorts $\omega,\wp(\omega)$, with constant 0 of sort ω, 1-ary function symbol S on sort ω, 2-ary function symbols $+,\bullet$ on sort ω, 2-ary relation symbols $<,=$ on sort ω, and 2-ary relation symbol $\in$ between sort ω and sort SET[ω]. Variables $n_1,n_2,\ldots$ of sort ω and $A_1,A_2,\ldots$ of sort SET[ω]. Terms of sort ω are defined inductively. The atomic formulas are $s < t$, $s = t$, $t \in A_i$, where s,t are terms of sort ω and $i \geq 1$.

DEFINITION 1.2. L[fcn] is the relevant language with functions. It has four sorts ω,FCN[1],FCN[2],FCN[3], with constant 0 of sort ω, 1-ary function symbol S on sort ω, and = on sort ω. Variables $n_1,n_2,\ldots$ of sort ω, $F^1{}_n$, $n \geq 1$, of sort FCN[1], $F^2{}_n$, $n \geq 1$, of sort FCN[2], $F^3{}_n$, $n \geq 1$, of sort FCN[3]. Terms of sort ω are defined inductively. The atomic formulas are $s = t$, where s,t are terms of sort ω.

These languages come with well known complete Hilbert style axiomatizations for the logic under the Henkin semantics going back to [He50].

In the development of RM, there has really been no practical need to get into any details concerning the underlying logic. A main reason is that RM is developed in the usual semiformal mathematical style that is the standard throughout contemporary pure mathematics, where (legitimate) logical manipulations are taken for granted without discussion. Another reason is the full realization that such languages can be flattened out to ordinary one sorted predicate calculus treating sorts as unary predicates. This is particularly evident with L[set] with its particularly transparent $n \in A$. This is less clear with L[fcn] with its nested $f(n,g(n,k,k)) = h(h(r))$. Flattening out requires the introduction of function symbols for "application".

Our history is based on these two languages only. However, there are a few minor variants of L[set], the language of Z_2, in the literature. Some use constant symbol 1 of sort ω instead of S, some do not use <, and some use = also on sort SET[ω].

The L[fcn] of [Fr76] is a minor variant of L[fcn] here. S is used here as a 1-ary function symbol on sort ω. In [Fr76], N is used instead of S, and N is used as a constant of sort FCN[1].

For my 1974 ICM address, appearing as [Fr75], we focused entirely on the RM perspective, realizing its special suitability for a workable long term foundational investigation into the logical structure of mathematical practice. In my ICM paper [Fr75], we used various subsystems of Z_2, most of these going back to Feferman and Kreisel, all in L[set], with full induction.

After [Fr75], we refocused efforts on the SRM vision, beginning with a strictly mathematical replacement for the full induction present in all of the systems in [Fr75]. We realized that simply replacing full induction with set induction (if a set contains 0 and is closed under successor then it contains ω) would go a long way. We realized that this was a satisfactory improvement from the SRM standpoint, for the systems in [Fr75] from ACA and up. However, we felt that for the base theory RCA of [Fr75] (and WKL), this was likely to be too weak to be satisfactory. Evidently one cannot even develop exponentiation. So we sought to replace full induction with something stronger than set induction, at least for RCA and WKL.

We realized that there would be greater flexibility in finding strictly mathematical axioms if we moved from numbers and sets to numbers and multivariate functions. We realized that with 1,2,3-ary functions, the axiom of primitive recursion (for constructing 1,2-ary functions by primitive recursion from arbitrary 2,3-ary functions) was strictly mathematical and powerful, along with rudimentary induction (induction with induction hypothesis f(n) = g(n)). Thus the system ETF (elementary theory of functions) was introduced in [Fr76] in the language L[fcn], using some additional strictly mathematical axioms.

RCA in L[set] was the base theory for RM used in [Fr75], and in [Fr76] we proposed the new base theory RCA_0/f for RM in L[fcn]. This RCA_0 was nearly strictly mathematical as it was defined to be the strictly mathematical ETF plus: Δ^0_1-CA/f transferred from

L[set] to L[fcn]. Thus [Fr76] introduces RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and others/f, in L[fcn], RCA_0/f being regarded as the base theory in [Fr76], incorporating the strictly mathematical ETF.

To avoid confusion, in this paper we continue to write RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and others/f, for the systems in L[fcn] in [Fr76], "f" indicating functions. Also we write RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and others/s, for the corresponding systems in L[set] that are the benchmarks for the current RM. "s" indicates sets. These "f" and "s" do not appear in the literature, and are used here to avoid confusion in this historical account.

This move from [Fr75] to [Fr76] is discussed in section 6. The transfers used there from L[set] to L[fcn] are the intuitively obvious ones, which we document formally here.

At the time, RM started to take hold, with SRM falling by the wayside. So the advantages in terms of SRM of having moved in [Fr76] to L[fcn] to allow for the formulation of the strictly mathematical ETF, faded. Hence researchers moved from the L[fcn] of [Fr76] back to the L[set] in [Fr75]. This resulted in the systems RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and others/s, in L[set], that we know and use today for RM, "s" indicating sets.

There is a claim in [Fr76] that ETF is in fact equivalent to RCA_0/f. This is not obvious and remained undocumented as RM took hold, and SRM remained dormant. In the development of RM, with the systems transferred back to L[set], ETF became irrelevant to RM. However ETF is highly relevant to SRM and we fill the gap here in section 10. It appears that some earlier work in [Go54] in a somewhat different context may be historically and technically relevant. For a detailed examination of the relationship between ETF, RCA_0/f, RCA_0/s, see sections 10,11.

In light of section 11, we can view the current RM as a branch of SRM with the strictly mathematical base theory ETF. Just remark that all of the RM work to date is done with the strictly mathematical ETF as the base theory, using standard RM coding. Much of the coding is no longer needed if we use the strictly mathematical ETF[FSRA] as the base theory, presented in section 11. ETF[FSRA] is a common definitional extension of ETF $\equiv$ RCA_0/f, and RCA0/s. For more about the SRM vision and projects in SRM, see sections 12, 13.

2. TWO PRECURSORS

There are two well known examples of investigations that have the same
general structure as RM and greatly precede RM.

A. There is the logical analysis of statements in classical geometry in the sense of Euclid. A second order axiomatization of Euclidean Geometry was given by David Hilbert and later a first order axiomatization of Euclidean Geometry was given by Alfred Tarski. Hilbert's is complete in the second order sense, while Tarski's iscomplete in the first order sense. The notorious parallel axiom became very controversial and there came to be a clear division between Euclidean Geometry and a fragment of Euclidean Geometry called Absolute or Neutral Geometry. Second and first order axiomatizations of the Absolute Geometry evolved, and served as a base theory for what we now view as Reverse Geometry studies. Over Absolute Geometry, which statements prove which statements, with special attention to which statements prove the parallel axiom. I was looking for a robust delineation of Absolute Geometry (second order and first order) so that the particular axioms for it being used would not look to have any ad hoc or accidental historical features. I didn't readily find that, and I would appreciate it if some readers can inform me. Hilbert's axiomatization is 1899. Non Euclidean Geometries go back to around 1830. Tarski's first order axiomatization is 1959. So there really are two forms of Reverse Geometry here. One is second order and one is first order. The first of these uses second order Absolute Geometry axioms, whereas the second of these uses first order Absolute Geometry axioms. But note that this whole precursor is only aimed at rather special but quite interesting mathematical statements.

B. There is the logical analysis of statements in set theory without choice. The base theory is ZF and the main focus is whether one is reversing to the AxC or what natural fragments of AxC. The classic is Rubin and Rubin, 1963. Here the scope is of course quite limited for a somewhat different reason. The amount of mathematics that can be proved in ZF, therefore under the radar screen, is enormous. Also there is an interesting analog to the question I raised in A above of characterizing exactly what Absolute Geometry should mean 2nd order and 1st order. Namely, suppose we are sitting in ZFC. Then what is ZF? Is there a theorem to the effect that a sentence φ is provable in ZF if and only if it is provable in ZFC and has certain properties? Or a sentence φ is provable in ZF if and only if some associated sentence φ* is provable in ZFC? Of course, there are uninteresting answers to these questions, but we want informative answers. Or sharpen these questions. Same for Absolute Geometry and Euclidean Geometry as in A, both second order and first order. Also notice that in A, I was careful to mention BOTH 2nd order and 1st order formulations. What I am talking about here in B is normally discussed only for 1st order. But what happens if we discuss 2nd order ZF and 2nd order ZFC? Is this even interesting? If so, what would a discussion of this look like? If not, why not? Of course one big difference is that Euclidean Geometry is 2nd order categorical and ZF is not 2nd order categorical.

So the foundational interest of a brand of Reverse Mathematics depends crucially on the base theory being used. That largely determines the target and scope of the mathematics being reversed.

Note that in the case of Absolute Geometry and ZF in A,B above, the mathematics being reversed is extremely limited. For Absolute Geometry, it is statements in axiomatic geometry, which are not provable in Absolute Geometry and do not assume the parallel postulate, and specifically excludes all of mathematics which presupposes the discrete ordered ring of integers. For ZF, reversals are limited to mathematics not provable in ZF, that require some use of the Axiom of Choice. This is a very considerable limitation.

3. ORIGINAL INTENT

We now present two somewhat different perspectives on RM = Reverse Mathematics from two well known researchers who have done extensive work in the area. One is S.G. Simpson, who was my closest associate in logic during the early 70's to late 80's. I talked freely and openly with Simpson during those years, without hesitation.

Much later Simpson wrote [Si99,09], the first book and principal text on RM. The other perspective on RM is a much more recent one from R.A. Shore, who has been very active in the field in recent years, and authored [Sh10]. Also much more recent are two other books on RM, [St18] and [Hi19].

[Fr75],[Fr76] are frequently cited as the founding papers for what I later called Reverse Mathematics. According to Simpson, [Si99,09], p. 34,

"The slogan "Reverse Mathematics" was coined by Friedman during a special session of the American Mathematical Society organized by Simpson"

The name was immediately adopted. With a little bit of digging, it appears that [FSS83] is likely the first appearance of the phrase "Reverse Mathematics" in print. We have been able to trace back my using the phrase during a meeting of the three authors of [FSS83] in 1981 in Columbus, Ohio.

Simpson puts my role this way on the first page of [Si85]:

"Subsystems of second order arithmetic can be examined from two related but disparate points of view. On the one hand, such systems have many interesting metamathematical properties... On the other hand, subsystems of second order arithmetic are a natural vehicle for the formal axiomatic study of ordinary mathematics. ...

Harvey Friedman has made important contributions to both of [these]. But it is in the second area, that of the relationship between subsystem of second order arithmetic and ordinary mathematical practice, that Friedman's insights have been particularly influential and indeed decisive for later developments."

According to [Si85], p. 144,

"But apparently Friedman [Fr75] was the first to initiate a systematic study of precisely which theorems of ordinary mathematics are provable in precisely which subsystems of Z_2."

And [Si85], page 145, underline from original:

"The main theme of Friedman [Fr75] is that, when a theorem of ordinary mathematics is provable in one of the above systems, then surprisingly often, the theorem is in fact provably equivalent to the principal axiom of the system needed to prove it. This then is nowadays known as Reverse Mathematics [list of references]"

Also see [Si85a], p. 467, capitalization from the original:

“The above theorem says, in particular, that the Bolzano-Weierstrass theorem is equivalent to ACA_0 (provably in the weak system RCA_0). …

The above theorem exemplifies a phenomenon which pervades our subject. Very often, if a theorem of ordinary mathematics is proved from the weakest possible set existence axioms, it will be possible to “reverse” the theorem by proving that it is equivalent to those axioms over a weak base theory. This phenomenon is known as REVERSE MATHEMATICS.

The pervasiveness of the reverse mathematics phenomenon was first emphasized by Harvey Friedman [Fr75]. The equivalence of (1), (3), and (6) above, as well as several other equivalences of a similar nature, are due to Friedman [references]. Other examples of reverse mathematics are due to Simpson, Smith and Steel [references]."

Simpson identifies the origin of Reverse Mathematics this way in [Si99,09], p.34:

"Historically, Reverse Mathematics may be viewed as a spin-off of Friedman's work [[Fr71, Fr[71a], [Fr81], [Fr83], [Fr98] attempting to demonstrate the necessary use of higher set theory in mathematical practice."

Shore [Sh10] traces the inception of RM back to [Fr67] and [Fr71]. He writes, p. 378:

"The general enterprise of calibrating the strength of classical mathematical theorems in terms of the axioms (typically of set existence) needed to prove them was begun by Harvey Friedman in [1971] (see also [1967]). His goals were both philosophical and foundational. What existence assumptions are really needed to develop classical mathematics and what other axioms and methods suffice to carry out standard constructions and proofs? In the [1971] paper, Friedman worked primarily in the set theoretic settings of subsystems (and extensions) of ZFC. As almost all of classical mathematics can be formalized in the language of second order arithmetic and its theorems proved there, he moved [1975] to the setting of second order arithmetic and subsystems of its full theory Z_2 (i.e., arithmetic with the full comprehension axiom as described below)."

Now let me talk about my own perspective which is again somewhat different, though complementary. During almost my entire career from 1967 (Ph.D.) I have been engaged in so called reversals, where I show that in some relatively weak formal systems, various mathematical statements imply or are provable equivalent to various metamathematical statements such as key components of subsystems of second order arithmetic, or the consistency of formal systems. See some documentation of this below.

A lot of motivation for these early somewhat scattered reversals came from tea in the Stanford Mathematics Department. My original Stanford appointment was in Philosophy but my office was in the Math Department, 1967-69 and 1971. I was surrounded mainly by mainstream mathematicians - particularly from analysis - who had no interest in and were rather skeptical of mathematical logic and the foundations of mathematics. I recall their general feeling that formal systems were some sort of uninteresting technical construct having nothing to do with real mathematics.

I remember wanting to impress them that these formal systems arising in mathematical logic were somehow robustly canonical with deep connections with actual mathematics. This caused me to start developing a story for these stubborn mathematicians. This desire to impress the mathematicians made me formulate with increasing care the basic ideas of RM.

However, there were always some problems with my story. I always had to say "over some weak basic system" and necessarily hide the details of that weak basic system we now refer to as a base theory. So the ideal of having the base theory itself be strictly mathematical was firmly planted in my head during those teas. By the way I don't think I made much of an impression on these analysts at tea but I felt I had learned a lot by trying.

This mindset of trying to impress mathematicians with metamathematical phenomena, and thereby rethinking and fine tuning the story to make it more mathematical, is something that I have consistently practiced not only in my development of RM (and the more ambitious SRM as discussed below), but also with regard to Tangible Incompleteness, or what I formerly called Concrete Mathematical Incompleteness. Also in many other contexts ranging from my 1976 equivalence of relative consistency with interpretability (see [Sm85], p. 216-229, [Fr76b], [Fr80]) to my 2021 No Interpretation form of Gödel's Second Incompleteness Theorem (see [Fr21], p. 5).

In this way, my original intent, with regard to what I later called Reverse Mathematics, or RM, was more ambitious than the RM we know today. Unfortunately, I don't have any relevant manuscripts before [Fr75]. But I have returned some to my original intent, encapsulated by my phrase SRM = strict reverse mathematics. It requires that all base theories be strictly mathematical, that all target theories be strictly mathematical, and that all reversed statements be strictly mathematical and unaltered by coding. It is incomparably less developed than the now rather robust and vibrantly active RM. Various developments in and prospects for SRM are discussed in section 12. In particular there are suggestions for SRM research.

Although I have lost manuscripts and correspondence during 1969 - 1974, I do have [Fr75a] scanned on my website, which has the following recollection right up front in I of [Fr75a]:

"1. In 1969, I discovered that a certain subsystem of second order arithmetic based on a mathematical statement (that every perfect tree which does not have at most countably many paths, has a perfect subtree), was provably equivalent to a logical principle (the weak Π^1_1 axiom of choice) modulo a weak base theory (comprehension for arithmetic formulae)."

I don't have documents for this but the above is what I wrote in [Fr75a] about the very early years (my Ph.D. was in 1967). The underlines are from the original [Fr75a], the first in the four manuscript series. So these ideas were gestating many years before my [Fr75], [Fr76]. There is also my vivid recollection of my struggles at the Stanford Mathematics teas discussed above.

In contrast, Simpson wrote this in [Si99,09] p, 34:

"The Theme of Reverse Mathematics in the context of subsystems of Z_2 first appeared in Steel's thesis [Ste76] (an outcome of Steel's reading of Friedman's thesis [Fr67] under Simpson's supervision [Si73]) and in Friedman [Fr75], [Fr76]; see also [Si85]."

Simpson also wrote this in [Si85]:

"RCA proves the equivalence of [various] to the determinacy of [various]...

Thus Steel's results foreshadowed the theme of Reverse Mathematics as inaugurated in Friedman [Fr75]. (Steel's results were not widely circulated until the appearance of his 1976 Ph.D. thesis [Ste76]]"

Here, particularly in the first Simpson quote, Simpson is not taking into account my efforts since at least 1969 as referred to in the quote above from [Fr75a]. During this period I was quite open talking freely with many prominent logicians at many meetings, including Feferman, Kreisel, Simpson, and others.
We now discuss this "inauguration" paper [Fr75] in some detail.

4. ICM PAPER

[Fr75] (along with [Fr76]) launches what later became known as RM in rather explicit terms, couched in terms of the base theory RCA, which has full induction. It is the ICM paper appearing in the Proceedings of the 1974 International Congress of Mathematicians. The lecture was given in August 1974.

At that time I was fully conscious of the fact that large amounts of mathematics can be proved in RCA, and therefore not subject to reversals over RCA. But I also remember being fearful of going too weak, endangering the ideal of a limited number of equivalence classes under provable equivalence. So I felt that RCA was a good compromise for an initial format.

[Fr75] very explicitly states the main theme of RM in this way on page 235:

"I. When the theorem is proved from the right axioms, the axioms can be proved from the theorem."

Whereas one can question what "right" axioms exactly means here, the "reversal" idea is crystal clear.

Already in this paper there are quite a number of reversals. Many of these reversals were accumulated over many years going back to 1969 as indicated by the quote from [Fr75a] in section 3 above.

Here is the overall structure of the 1974 ICM paper. All of the formal systems presented in [Fr75] are in the two sorted language L[set] from Definition 1.1. So in this section we will not bother to write /s after the names of the formal systems.

0. Introduction with two principal themes stated explicitly, the first being the RM theme, we discussed earlier.

"When the theorem is proved from the right axioms, the axioms can be proved from the theorem."
1. Axioms for arithmetic sets. Here RCA, WKL, ACA are discussed, the first and third being I think due to Kreisel and Feferman. RCA is stated with Δ^0_1 comprehension and full induction. ACA is stated with arithmetic comprehension and full induction. WKL is stated with "every infinite 0,1 finite sequence tree has an infinite path" and full induction. KL is also considered where 0,1 is dropped. The following reversals are given:

KL
SHB (sequential Heine Borel).
SLUB (sequential least upper bound)
MLUB (monotone least upper bound).
SBW (sequential Bolzano Weierstrass)

Reversal of all five above over ACA is claimed, except for SHB. The reversal of SHB to WKL over RCA is claimed. Also reversals of several logical statements about propositional and predicate calculus, to WKL over RCA, are claimed.

Conservation of WKL over RCA is stated for all sentences starting with universal set quantifiers followed by anything arithmetical. I definitely had in mind the following argument using Scott systems. Let M be a countable model of RCA. We define within M a complete diagram W for a countable nonstandard model of EFA in a Δ^0_2 fashion with the help of induction in M. Then we can define the Scott system of M and get a model of WKL with the same arithmetic part. This
will show conservation for arithmetic sentences. To handle the universal set quantifier, we first assume in M that we have a set X without the arithmetic property in question; We repeat the argument but making sure that the nonstandard complete diagram W incorporates X (many convenient ways to do that).

So the ICM paper starts off with the first three of the "big five" but with full induction.

2. Axioms for hyperarithmetic sets. My idea was to move to systems reflecting relative arithmeticity to systems reflecting

relative hyperarithmeticity. I introduce HCA, HAC, HDC, which are the Feferman Kriesel systems Δ^1_1-CA, Σ^1_1-AC, Σ^1_1-DC with full induction. Then I give two mathematical statements ABW (arithmetic Bolzano Weierstrass) and SL (sequential limit system). I claim the second reverses to HAC over RCA, and claim the first is proved in HAC.

For a long time, this level of RM, which is off the levels of the "big five", was virtually inactive. Indeed, Simpson wrote in [Si85], p. 148,

"It is noteworthy that the systems Δ^1_1-CA, Σ^1_1-AC, Σ^1_1-DC, and BI [I write TI] have played a relatively minor or nonexistent role in in Reverse Mathematics. While these systems have many very interesting metamathematical properties, they appear to be of little or no interest from the viewpoint of the study of ordinary mathematics."

However, in more recent years, starting around 2006 with A. Montalban (see link below), this level of hyperarithmetic analysis is becoming significant for Reverse Mathematics, and I am glad I included this level in my ICM paper. See, for example,

https://math.berkeley.edu/~antonio/slides/kyotoh.pdf
https://people.math.wisc.edu/~jgoh/files/goh_irt_slides.pdf
https://pi.math.cornell.edu/~shore/papers/pdf/ATHA32.pdf
https://www.worldscientific.com/doi/10.1142/S0219061306000517

and [Co12], [CGT13].

3. Axioms for arithmetic recursion. I present ATR, weak Π^1_1-AC, and then three reversals. CWO for comparability of well orderings, PST for perfect subtree theorem, and CS for countability of discrete sets. I claimed that all of these are provably equivalent over RCA. I claimed that ATR proves HAC but not HDC, and this was proved in my PhD Thesis.
ATR is the fourth of the big five systems with full induction.

4. Axioms for transfinite induction. TI is RCA together with transfinite induction on all well orderings. It is more commonly referred to as BI and is probably due to G. Kreisel with his analysis of
C. Spector's famous paper on bar recursive functionals. TI does not fit so nicely in the big five picture. However, there is an important fragment of TI which is TI for Π^1_2 formulas, which according to Rathjen and Weiermann, is provably equivalent to KT with well quasi ordered labels. See [RW93].

It would seem likely that TI restricted to Π^1_n formulas, for other n, should be targets of interesting reversals.

I also introduce RFN, a reflection principle, that says that any second order property that holds of certain sets holds in some countable ω structure of those given sets. I claim that RFN and TI are equivalent over RCA.

5. Axioms for the hyperjump. I present Π^1_1-CA which is the Π^1_1 comprehension axiom scheme on top of RCA. This system is due to S. Feferman and G. Kreisel. I claim two equivalents over RCA, one for the perfect kernel theorem for trees of finite sequences of natural numbers, and the arithmetic least upper bound principle. This is the fifth of the big five with full induction.

I finish the paper by referring to work on restricting induction pointing to the two JSL abstracts which can be conveniently downloaded from item 1 in my Downloadable Manuscripts https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/

In summary, the ICM paper presents and discusses some major reversals over RCA to mostly the these five systems with full induction: RCA, WKL, ACA, ATR, Π^1_1-CA. It also discusses some key systems with limited reversals strictly between ACA and ATR: HCA, HAC, HDC. This hyperarithmetic level has proved to be increasingly active in recent years for RM. Also fragments of TI has proved important for the RM of Kruskal's theorem and variants.

There is a small technical error in [Fr75] in the definition of Δ_0 formulas. We need to say that the variable n does not appear in the term t.

5. JSL ABSTRACTS

The two JSL Abstracts, commonly mentioned along with [Fr85] as the founding papers for RM, are in [Fr76] and can be downloaded from the link listed at [Fr76].

In [Si85], Simpson wrote on p. 149:

"In [Fr76] Friedman does not explicitly discuss his reasons for concentrating on systems with restricted induction. However, with hindsight, one can say that the decision to do so was a sound one."

I remember thinking that using restricted induction was obviously the right way for me to set up RM but that it was somewhat difficult to articulate good clear reasons, and that given those difficulties, it wasn't really worth the serious effort to articulate good clear reasons. Stepping back I can say what my reasons where much more clearly than I could back then.

To begin with, the full system Z_2/s and its use for formalizing mathematics seems to go back to Hilbert and Bernays. And relying on my recollections of interactions with S. Feferman and G. Kreisel while I was at Stanford mainly 1967-69, the systems RCA/s, ACA/s, Π^1_1-CA/s, as well as many other systems with full induction like Δ^1_1-CA/s, Σ^1_1-AC/s, Σ^1_1-DC/s, are due to them. This does not include WKL/s and ATR/s, although Feferman's IR has an ATR *rule* as a principal component.

According to [Si99,09], p. 132,

"The formal system WKL_0 was first defined in [Fr76]"

and I surmise that this attribution for WKL_0/f applies to WKL/s first defined in print in [Fr75] and also WKL_0/s.

The systems ATR/s and ATR_0/f were first presented by us in [Fr75], [Fr76] respectively. According to [Si82], p. 239,

“The specific system ATR_0 was introduced by [Fr76] and was studied in some detail by
[FMS82]. (A stronger system ATR, consisting of ATR_0 plus full induction on the natural numbers, had been introduced earlier by [Fr75] and had been studied by [Fr67] and [Ste76].“

Again I surmise that this attribution for ATR_0/f and ATR/s applies to ATR_0/s. According to [Si82a], p. 255,

“By ATR_0 we mean the formal system of arithmetical transfinite recursion with quantifier free induction on the natural numbers. … It was first isolated by H. Friedman [Fr75], [Fr76].”

Feferman and Kreisel always used full induction in their systems (as far as I know). This was because they were principally concerned with truth and self evidence. Full induction was considered unproblematic and implicit in what is being axiomatized by the various subsystems of Z_2. So Feferman and Kreisel apparently and quite naturally saw no reason to restrict induction.

Whereas I had no quarrel with that reason for including full induction, I was focused on another concern and that was the axiomatic structure of actual mathematics. Furthermore, we were driven by the idea of treating not only strictly mathematical theorems to be reversed, but having a strictly mathematical base theory. It was clear that RCA was not really satisfactory from this standpoint. An obvious improvement would be to use only set induction

$$0 \in A \wedge (\forall n)(n \in A \rightarrow S(n) \in A) \rightarrow n \in A$$

We saw that this replacement of full induction with set induction preserved the essential mathematical power of ACA/s, ATR/s, Π^1_1-CA/s, and others/s. However, this was apparently not the case with RCA/s, WKL/s. There the resulting system appeared insufficient to develop even exponentiation. We wished to avoid a foundation for RM at the levels of around ACA and higher, separate from that at the level of RCA (and WKL), separate in the sense of having different languages. We aimed for uniformity of language.

So we were inexorably drawn to the advantages of moving to functions. We fixed on the four sorted language L[fcn] in Definition 1.2. Although it has more sorts than L[set], it is simpler than L[set] in that it does not have the primitives $+,\bullet,<$.

The obvious version of set induction for unary functions would be the strictly mathematical Rudimentary Induction axiom

$$f(0) = g(0) \wedge (\forall n)(f(n) = g(n) \rightarrow f(S(n)) = g(S(n)) \rightarrow f(n) = g(n)$$

and the strictly mathematical Primitive Recursion

$$(\exists f)(f(0) = m \wedge (\forall n)(f(S(n)) = g(n,f(n)))).$$
$$(\exists f)(f(n,0) = g(n) \wedge (\forall m)(f(n,S(m)) = h(n,m,f(n,m))))$$

allowing exponentiation and much more to be developed. This is the axiom of Primitive Recursion rather than an introduction rule as in Kleene's well known method of obtaining the recursive functions in recursion theory (with his μ-operator). (In [Fr76], this Rudimentary Induction is called Atomic Induction).

However, there was Δ^0_1-CA/f, which remains far from being strictly mathematical, even in its language L[fcn]. Nevertheless, at least some progress was being made toward the SRM vision by retaining only a strictly mathematical form of induction.

At this point we fixed on the strictly mathematical base theory ETF = elementary theory of functions.

There are four sorts, ω, 1-ary, 2-ary, 3-ary functions. We have variables over each sort, constant 0 of sort ω, 1-ary function symbol S on ω, and = between terms (of sort ω), with the usual connectives and quantifiers over all four sorts. Here terms are all of sort ω. ([Fr76] uses a 1-ary function constant N instead of S). Axioms for ETF there were as follows:

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axioms.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

For full details see section 6.

At that point, we decided that the proper replacement for the base theory RCA used in [Fr75] would be ETF + Δ^0_1-CA/f, and therefore with all of the induction coming from ETF (Rudimentary Induction). We therefore gave it the name RCA_0 (writing this here as RCA_0/s). Of course the name RCA_0/s now refers to the well known base theory of current RM in L[set]. In fact, in [Fr76], we used the following replacements for the main systems from [Fr75]:

RCA/s replaced by ETF + Δ^0_1-CA/f, written RCA_0/f
WKL/s replaced by ETF + Δ^0_1-CA/f + weak Konig's Lemma transferred to L[fcn], written WKL_0/f
ACA/s replaced by ETF + arithmetic comprehension axioms transferred to L[fcn], written ACA_0/f
ATR/s replaced by ETF + arithmetic comprehension axioms + arithmetic transfinite recursion transferred to L[fcn], written ATR_0/f
Π^1_1-CA/s replaced by ETF + Π^1_1 comprehension axioms transferred to L[fcn], written Π^1_1-CA_0/f
and analogously from other systems in [Fr75] to corresponding systems in [Fr76]

Thus the notation RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and others, was introduced in [Fr76] for new systems in L[fcn] with base theory ETF. (Again, the /f is used here to avoid confusion).

Later, these names began to be used in their present sense for the main systems of current RM in L[set]. How did that come about and why?

Since that time, RM took hold as a ready made long term foundational investigation into the logic of mathematical practice, and the vision of SRM fell by the wayside. Of course, the targets for reversals in RM were generally strictly mathematical, but there was no focus on having a strictly mathematical base theory. Consequently, the language L[set] was regarded as simpler and formal systems in L[set] were simpler, and coding issues were regarded correctly as largely unproblematic for the main targets of RM. So it was obvious at the time that according to the prevailing RM perspective, the systems RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and others/f, ought to be transferred from L[fcn] to L[set], and then streamlined as much as possible (reaxiomatization) for practical use in the development of RM, yielding the usual RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and others/s.

Simpson takes note in [Si85], p. 149, of the difference between the RCA_0/s and WKL_0/s for current RM in L[set] and the RCA_0/f and WKL0/f in [Fr76] as follows:

"(These versions of RCA_0 and WKL_0 [in L[set])] are superficially different from, but essentially equivalent to, Friedman's formulation in [Fr76] [in L[fcn]]. The superficial discrepancy arises because [Fr76] uses a language with functions variables rather than set variables. Thus Friedman is able to formulate RCA_0 and WKL_0 in terms of a quantifier free induction axiom. Then, with the help of some other axioms for the existence of functions (including primitive recursion and the μ-operator), he is able, within his version of RCA_0 to prove Σ^0_1 induction.)"

Note that Simpson does not specifically mention the SRM vision, which was partially realized by [Fr76], although he obviously recognized some significance to the radially more basic form of induction present in [Fr76] than in current RCA_0/s WKL_0/s. Actually, the Rudimentary Induction in ETF is strictly mathematical in a way that even quantifier free induction, let alone Σ^0_1 induction, is not. Also [Fr76] does not use the μ-operator, but something more strictly mathematical, namely the inversion of permutations of ω (Permutation Axiom).

The transfer of the RCA_0/f of [Fr76] to present RCA_0/s is facilitated by two related easy observations discussed in sections 6,7.

1. ETF + Δ^0_1-CA/f can first be transferred to L[set], where, e.g., the primitive recursion in ETF is formulated in terms of set coding. Then this is easily seen to derive Σ^0_1-IND/s. See Theorem 7.1.
2. ETF + Δ^0_1-CA/f easily proves Σ^0_1-IND/fcn. This is explicitly stated (in very strong form) in Theorem 2 of [Fr76] - see note at end of this section. Then the transfer of the RCA_0/f of [Fr76] to L[set] yields the transfer of ETF + Δ^0_1-CA/f + Σ^0_1-IND/f to

L[set], which is immediately seen to be equivalent to the present RCA_0/s in L[set]. See Theorem 7.2.

Our bringing in Σ^0_1-IND/f in [Fr76] is not surprising in light of the then well known relationship between primitive recursion and Σ^0_1-IND in the setting of arithmetic (see [Pa70], [Fe05]). But there primitive recursion is schematic and the relationship is one of equal logical strength and conservation, not outright equivalence.

There is an important claim in [Fr76], in Theorem 6, concerning the strength of ATR_0. We state that ATR_0 and Feferman's IR prove the same Π^1_1 sentences. This obviously implies that ATR_0 has proof theoretic ordinal Γ_0. In [Si85], p. 143,

"In [FMS82], Friedman computed the proof-theoretic ordinal of a subsystem of Z_2 known as ATR_0. This is a system with restricted induction whose principal axiom is the Π^1_2 sentence ... "

Accounts of our proofs appeared in [FMS82] and [Si82]. The account in [FMS82] conforms to our original proof. The account in [Si82] builds on two systems of Jaeger, ATR_0^J and ATR^J, which are closer to Feferman's IR than our ATR_0/s and ATR/s. Jaeger had computed the proof theoretic ordinal of his ATR_0^J and ATR^J, respectively, as Γ_0 and Γ_{ϵ_0}, in [Ja80]. In [Si82] we find our proof that ATR_0 and ATR prove, respectively, the same Π^1_1 sentences as ATR_0^J and ATR^J, so the two computations are valid for ATR_0 and ATR as well. Our original proof, presented in [FMS82], going back to at least 1976, really amounts to incorporating ideas seen in the much later [Ja80] into the proof. However, the computation $|ATR| = \Gamma_{\epsilon\_0}$ does rely on [Ja80].

There is another important claim in [Fr76], in Theorem 1, that has far reaching implications for the SRM vision. Namely, that the RCA_0/f of [Fr76] is in fact logically equivalent to ETF. This claim was never documented because as RM quickly took hold, the SRM vision fell by the wayside. Here we will fill this gap in section 10. Specifically, ETF proves $\Delta^0{}_1$-CA/f + $\Sigma^0{}_1$-IND/f. Here the much earlier work of [Go54] in a different context may be relevant.

NOTE: Theorem 2 of [Fr76] actually states that ETF proves $\Pi^0{}_1$-IND/f, which is an obvious equivalent to $\Sigma^0{}_1$-IND/f.

6. FROM [Fr75] to [Fr76] HISTORICALLY

We now provide more details concerning the move from [Fr75] to [Fr76]. The system ETF (elementary theory of functions) in L[fcn] was introduced in [Fr76]. We now present ETF in full detail, with some minor simplifications.

DEFINITION 6.1. L[fcn] has four sorts, ω and FCN[1], FCN[2], FCN[3] for 1,2,3-ary functions from ω to ω. Variables over sorts ω, FCN1, FCN2, FCN3. Constant symbol, 0, of type ω, 1-ary function symbol S from sort ω into sort ω, and 2-ary relation symbol = on sort ω. Terms of sort ω are defined as usual (built up from variables over ω, function variables, and 0,S). Terms of other sorts have no significance, and so we refer to terms of sort ω as simply terms. Atomic formulas are of the form s = t where s,t are terms. We use connectives ¬,∧,∨,→,↔ and quantifiers ∀,∃. Formulas are built up in the usual way.

In [Fr76] we used the constant symbol N of sort FCN1 for "next function" instead of our S.

We use the usual standard complete Hilbert style logical axioms and rules for many sorted systems with function variables and equality. The nonlogical axioms are as follows.

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axioms.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

For specificity below, n,m,r are the first three variables over sort ω.

1. Successor Axioms.

i. $S(n) \neq 0$
ii. $S(n) = S(m) \rightarrow n = m$
iii. $n \neq 0 \rightarrow (\exists m)(S(m) = n)$

2. Initial Function Axioms.

i. There exists 1-ary, 2-ary, 3-ary functions that are constantly any n. Here n is a variable of sort ω.
ii. The three 3-ary projection functions exist. The two 2-ary projection functions exist. The 1-ary identity function exists.
iii. S(n) defines a 1-ary function.

Since equality between function sorts is not allowed, i,ii,iii are existence statements. Of course, extensional uniqueness is immediate.

3. Composition Axioms.

i. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n,m))$
ii. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n))$
iii. $(\exists f)(\forall n,m)(f(n,m) = g(n,m,m))$
iv. $(\exists f)(\forall n)(f(n) = g(n,n,n))$
v. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(h1(n,m,r),h2(n,m,r),h3(n,m,r)))$

The above is a simplification of Composition in [Fr76]. We see that 2,3 are appropriately sufficient in Lemma 10.1.

4. Primitive Recursion Axiom.

$(\exists f)(f(n,0) = g(n) \wedge (\forall m)(f(n,S(m)) = h(n,m,f(n,m))))$.

In [Fr76] we also use primitive recursion for constructing 1-ary functions f. That is easily derivable. See Lemma 10.2.

5. Permutation Axiom.

Every 1-ary function that maps ω one-one onto ω has an inverse.

6. Rudimentary Induction Axiom.

$f(0) = g(0) \wedge (\forall n)(f(n) = g(n) \rightarrow f(S(n)) = g(S(n))) \rightarrow f(n) = g(n)$.

In [Fr76] we also used a corresponding induction statement for arities 2,3. We also called it Atomic Induction. It is derivable. See Lemma 10.2.

In [Fr75], we used the systems

1) RCA/s, WKL/s, ACA/s, ATR/s, Π^1_1-CA/s, and others/s

In preparation for [Fr76], we first replaced the induction axioms - which are particularly at odds with the SRM vision - with set induction

$$0 \in A \wedge (\forall n)(n \in A \rightarrow S(n) \in A) \rightarrow n \in A$$

resulting in what we will refer to here as

2) RCA-/s, WKL-/s, ACA-/s, ATR-/s, Π^1_1-CA-/s, and others-/s

We accepted all but the first two of these systems as very good refinements of the corresponding systems in 1) from [Fr75], in the sense of preserving the essence of their mathematical power, while at the same time incorporating more of the SRM vision. The systems in 2) starting with ACA- are exactly the familiar systems

3) ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and $others_0$/s, in L[set]

that we use today for RM.

However, we viewed the first two, RCA-/s, WKL-/s, as too weak for the envisioned RM, in that, for example, exponentiation cannot be developed there. At that time, we were already worried about the effect of having too many incomparable RM levels, resulting in something like what is now referred to as the RM zoo [As17]. Whereas mature RM is being enlightened by this zoo, we thought that, quite rightly, RM would not survive an early zoo.

DIGRESSION: The RM zoo does not appear to contain any incomparable logical strengths, or incomparabilities under interpretability. This confirms the observed comparability of logical strengths and interpretation power in natural contexts (as long as EFA is interpreted), something we have long pointed to as deeply fundamental foundationally.

At this point we looked for axioms to strengthen RCA-/s and WKL-/s, preferably strictly mathematical. We did not see attractive possibilities in L[set], and came up with the idea of using numbers and functions instead of numbers and sets. It seemed that the mathematical richness of functions over sets would prove to be decisive.

Early in this process we sensed the importance of primitive recursion, and fixed on the idea of having only functions of arities 1,2,3, as 1,2-ary functions defined by primitive recursion seem so mathematically fundamental. So we fixed on the language L[fcn] in Definition 6.1, and saw advantages in simplicity to not having +,•,< as primitives (but necessarily retaining 0,N,=, with the 1-ary function constant N. N is replaced here by the 1-ary function symbol S).

We transferred the systems in 2) from L[set] to L[fcn]. Since RCA-/s, WKL-/s were too weak, along with any excepted transfers, we formulated and used ETF as a base in L[fcn], and in the interests of uniformity, we used ETF as a base for the transfers of ACA-/s, ATR-/s, Π^1_1-CA-/s as well, even though we recognized them as fine as they were sitting in L[set]. We didn't want some of the systems to be in L[fcn] whereas others in L[set]. (The idea of using the combined sorts fleshed out in section 11 had not occurred to us at the time). This results in the following final systems in L[fcn] from [Fr76]:

4) EFA + RCA-trans/f, EFA + WKL-trans/f, EFA + ACA-trans/f, EFA + ATR-trans/f, EFA + Π^1_1-CA-trans/f, and EFA + others-trans/f

where here we use "trans" to indicate the straightforward transfer from L[set] to L[fcn].

In [Fr76] we named these systems in 4), as

5) RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0, and $others_0$

with the same names we use now for the systems of the current RM. The idea was that these were a reworking of the systems RCA, WKL, ACA, ATR, Π^1_1-CA in L[set] in L[set] from [Fr75] that we use today for RM, over to L[fcn]. This was considerable but incomplete move away from [Fr75] towards the SRM vision.

Note that in this paper, in order to avoid confusion, we continue to write /s or /f to indicate whether the systems are in L[set] or in L[fcn]. Thus we refer to the systems 5) here as

6) RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and $others_0$,/f

We present some details of this natural transfer process from L[set] to L[fcn] in order to carefully examine the first five systems in 6) from [Fr76]. These details are used in sections 7 and 10. We need to start with 1) here from [Fr75]. This depends on the four formula classes Δ_0/s, Σ^0_1/s, Π^0_1/s, Π^0_∞/s, Π^1_1/s.

Δ_0/s contains the atomic formulas (in L[set]), and generated by the usual connectives and bounded quantifiers. Bounded quantifiers take the form $(\exists n < t)(\varphi)$, $(\forall n < t)(\varphi)$, where the variable n of sort ω does not appear in the term t of sort ω (in L[set]). Officially, these need to be expanded out to $(\exists n)(n < t \wedge \varphi)$, $(\forall n)(n < t \rightarrow \varphi)$. < (as well as $0,S,+,\bullet$) is a primitive symbol in the systems of [Fr75] but not in the systems of [Fr76].

Σ^0_1/s formulas are of the form $(\exists n)(\varphi)$ where φ is Δ_0/s.

Π^0_1/s formulas (in L[set]) are of the form $(\forall n)(\varphi)$ where φ is Δ_0/s.

Π^0_∞/s formulas are the formulas in L[set] with no set quantifiers.

Π^1_1/s formulas are the formulas $(\forall A)(\varphi)$, where A is a set variable and φ is Π^0_∞/s.

RCA/s in [Fr75]

SUCCESSOR AXIOMS. $S(n) \neq 0$, $S(n) = S(m) \rightarrow n = m$.
RECURSION AXIOMS. $n+0 = n$, $n+S(m) = S(n+m)$, $n\bullet 0 = 0$, $n\bullet S(m) = n\bullet m + n$, $n < m \leftrightarrow (\exists r)(r \neq 0 \wedge n+r = m)$.
Δ^0_1-CA/s. $(\forall n)(\varphi \leftrightarrow \psi) \rightarrow (\exists A)(\forall n)(n \in A \leftrightarrow \varphi)$, where φ is Σ^0_1/s, ψ is Π^0_1/s, and A is not in φ,ψ.
INDUCTION. $\varphi[n/0] \wedge (\forall n)(\varphi \rightarrow \varphi[n/S(n)]) \rightarrow \varphi$, where φ is a formula of L[set].

WKL/s in [Fr75]

RCA/s.
WEAK KONIG'S LEMMA. In L[set]. Every infinite finite sequence tree of 0's and 1's has an infinite path. Stated with finite sequence coding in L[set].

ACA/s in [Fr75]

RCA/s.
Π^0_∞-CA/s. $(\exists A)(\forall n)(n \in A \leftrightarrow \varphi)$, where φ is Π^0_∞/s.

ATR/s in [Fr75]

ACA/s.
ARITHMETIC TRANSFINITE RECURSION. In L[set]. The Kleene H-set on every well ordering of a subset of ω exists. Stated with ordered pairing coding in L[set].

Π^1_1-CA/s in [Fr75]

ACA/s.
$(\exists A)(\forall n)(n \in A \leftrightarrow \varphi)$, where φ is Π^1_1/s and A is not in φ.

We could have had greater uniformity of presentation by using RCA/s instead of ACA/s for these last two systems. I remember being concerned with nuisance details about the coding of H-sets (starting at finite levels) just having RCA/s and not ACA/s. But it doesn't appear that any worries along these lines are warranted with RCA/s.

RCA-/s in preparation for [Fr76]
rejected for weakness

RCA/s with Induction replaced by
SET INDUCTION. $0 \in A \wedge (\forall n)(n \in A \rightarrow S(n) \in A) \rightarrow n \in A$.

WKL-/s in preparation for [Fr76]
rejected for weakness

WKL/s with Induction replaced by set induction.

ACA-/s in preparation for [Fr75]
accepted but later
transferred for uniformity of language in [Fr76]

ACA/s with Induction replaced by set induction.

ATR-/s in preparation for [Fr75]
accepted but later
transferred for uniformity of language in [Fr76]

ATR/s with Induction replaced by set induction.

Π^1_1-CA-/s in preparation for [Fr75]
accepted but later
transferred for uniformity of language in [Fr76]

Π^1_1-CA/s with Induction replaced by set induction.

Again note that ACA-/s, ATR-/s, and Π^1_1-CA-/s are the same as what we now use and call ACA_0, ATR_0, and Π^1_1-CA_0 in RM (in the language L[set]), and written here as ACA_0/s, ATR_0/s, and Π^1_1-CA_0/s to avoid confusion.

We now made the transfers of the above five systems to L[fcn]. Here is where we constructed ETF (elementary theory of functions) in [Fr76] in L[fcn] in order to remedy the weakness of RCA-/s and WKL-/s.

For this we need to transfer the formula classes we defined in L[set] to L[fcn]. We make the most obvious transfer by essentially copying from the formula class definitions given above for l[set] appearing just before the presentation of RCA in [Fr75]:

Δ_0/f contains $t_1 = t_2$, $t_1 < t_2$, t_1,t_2 terms, and generated by the usual connectives and bounded quantifiers. Bounded quantifiers take the form $(\exists n < t)(\varphi)$, $(\forall n < t)(\varphi)$, where the variable n of sort ω does not appear in the term t of sort ω (in L[fcn]). Here these need to be expanded out to $(\exists n)(n < t \wedge \varphi)$, $(\forall n)(n < t \rightarrow \varphi)$. Recall that L[fcn] does not use $+,\bullet$. It also does not use <, and so it is important to see the remark below about the use of < here.

Σ^0_1/f formulas are of the form $(\exists n)(\varphi)$ where φ is Δ_0/f.

Π^0_1/f formulas are of the form $(\forall n)(\varphi)$ where φ is Δ_0/f.

Π^0_∞/f formulas are the formulas in L[fcn] with no function quantifiers.

Π^1_1/f formulas are the formulas $(\forall f)(\varphi)$, where f is a function variable (arities 1,2,3) and φ is Π^0_∞/f.

REMARK. But there is something important missing here. < is in L[set] but not in L[fcn]. So we need to give a natural treatment in L[fcn] of < that naturally preserves the essential meaning of < from L[set]. That's the idea of "transfer".

We take s < t to abbreviate "there exists 2-ary f such that $(\forall n)(f(n,0) = 0) \wedge (\forall n,m)(f(n,S(m)) = S(0) \leftrightarrow f(n,m) = S(0) \vee n = m) \wedge f(s,t) = S(0)$".

The idea here is that f is intended to be the characteristic function of the relation <. Such an extensionally unique f is easily proved to exist in ETF.

Note that we are not allowing the use of +,• in the above formula classes, as they are not in L[fcn]. However, +,• are properly incorporated through the function parameters. But < is needed to identity the formulas to be given special treatment as bounded formulas. In fact, for Π^0_∞/f and Π^1_1/f we didn't need to use < because of function parameters and because bounded quantifiers have no role in the definition of those more encompassing formula classes. But Σ^0_1, Π^0_1 are more delicate.

We are now ready to present the transfers from L[set] to L[fcn] made in [Fr76].

RCA_0/f

ETF.

Δ^0_1-CA/f. Let φ be Σ^0_1/f, ψ be Π^0_1/f, variable f not in φ,ψ.
$(\forall n,m)(\varphi \leftrightarrow \psi) \wedge (\forall n)(\exists! m)(\varphi) \rightarrow (\exists f)(\forall n)(\varphi[m/f(n)])$.

REMARK 1. The Δ^0_1-CA/s of [Fr75] asserts that a virtual set that is Δ^0_1 exists as a set. [Fr76] correspondingly takes Δ^0_1-CA_0/f to assert that a virtual function that is Δ^0_1 exists as a function, referring explicity to [Fr75]. Actually this formulation is rather robust. If we use the ostensibly weaker "a virtual function that is Δ_0 (bounded) exists as a function", or the ostensibly =stronger "a virtual function that is Σ^0_1 exists as a function", then we get logically equivalent forms of RCA0/f.

REMARK 2. There is, however, an interesting weaker form of Δ^0_1-CA/f that is also natural but not used in [Fr76]. The weak form asserts that "every virtual relation that is Δ^0_1 is the zero set =of a function". This is like Δ^0_1-CA/s from [Fr75] transferred to L[f] without respecting the difference between sets and functions. More formally, weak Δ^0_1-CA is as follows:

WΔ^0_1-CA/f. Let φ be Σ^0_1/f, ψ be Π^0_1/f, variable f not in φ,ψ.
$(\forall n)(\varphi \leftrightarrow \psi) \rightarrow (\exists f)(\forall n)(\varphi \leftrightarrow f(n) = 0)$.

REMARK 3. All of these distinctions, and indeed the whole present of Δ^0_1-CA/f gets washed away by the power of ETF, which was claimed in [Fr76] but not documented until this section 10. This is because of the powerful interactive presence of Primitive Recursion and Permutation in ETF. If re remove Primitive Recursion in ETF, then we still get the same RCA_0/f provided we use formulations in Remark 1. However, with the formulation in Remark 2, and without Permutation, then we get a system weaker than RCA0/f. These matters and their connection with SRM are discussed in section 13.

WKL_0/f

RCA_0/f.
WEAK KONIG'S LEMMA. In L[fcn]. Every infinite finite sequence tree of 0's and 1's has an infinite path. Stated with finite sequence coding in L[fcn].

ACA_0/f

RCA_0/f.
$(\forall n)(\exists!m)(\varphi) \rightarrow (\exists f)(\forall n)(\varphi[m/f(n)])$, where φ is Π^0_∞/f.

ATR_0/f

RCA_0 in L[fcn} above.
ARITHMETIC TRANSFINITE RECURSION. In L[fcn]. The Kleene H-set on every well ordering of a subset of ω exists. Stated with ordered pairing coding in L[fcn].
We were needlessly frightened that we needed ACA_0/f to state the existence of Kleene H-sets in L[fcn]. RCA_0/f clearly suffices.

Π^1_1-CA_0/f

RCA_0 in L[fcn} above.
$(\forall n)(\exists!m)(\varphi) \rightarrow (\exists f)(\forall n)(\varphi[m/f(n)])$, where φ is Π^1_1 in L[fcn].

Note that in this transfer to L[fcn], we ignored the ∈,+,• of L[set], although we did transfer the < of L[set] by treating < as an abbreviation (not a primitive) in L[fcn] with

s < t abbreviating "there exists 2-ary f such that $(\forall n,m)(f(n,m) = 0 \vee f(n,m) = S(0)) \wedge (\forall n)(f(n,0) = 0) \wedge (\forall n,m)(f(n,S(m)) = S(0)$ if $f(n,m) = S(0) \vee n = m$; 0 otherwise) $\wedge f(s,t) = S(0)$".

We had to make this actual abbreviation because of the need to transfer the formula classes to L[fcn], two of which critically involve < from L[set], which needed to be reflected in the transfer. But were the RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and $others_0$/f, in L[fcn] of [Fr76] really strong enough transfers? I.e., is RCA_0/f really a strong enough base theory for RM?

I remember at the time I made some quick checks that gave me confidence. First of all, the abbreviation of < would only have force and be appropriate if in fact RCA_0/f proves an extensionally unique f such that $(\forall n)(f(n,0) = 0) \wedge (\forall n,m)(f(n,S(m)) = S(0)$ if $f(n,m) = S(0) \vee n = m$; 0 otherwise). It is clear that this roughly comes from the Primitive Recursion in ETF assuming we have a simple but a bit messy 3-ary function defined with equations and propositional combinations. But we have Δ^0_1-CA/f as a sledgehammer here.

Also was ∈,+,• of L[set], in addition to the <, properly transferred? We can transfer $t \in A$ to $A(t) = 0$ transferring sets in L[set] to 1-ary functions in L[fcn]. Then also set induction transfers to a special case of the Rudimentary Induction in ETF. In addition, +,• transfers through the extensional uniqueness of 2-ary f satisfying the recursion equations for +,•. We did not have to incorporate +,• as abbreviations as we needed to for <, because we can replace +,• with two additional function parameters.

Another check that things are fitting together properly is how the direct transfer of the ACA-/s, ATR-/s, Π^1_1-CA-/s, and others-/s, as subsystems of the ACA/s, ATR/s, Π^1_1-CA/s, and others/s, in [Fr75], to L[fcn], compares to the ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and $others_0$/f, in [Fr76]. I.e., direct transfer in the sense of not adding ETF to buttress the strength. In the former case the transfer to L[fcn] is utterly slavish, with the only induction involved being set induction going to Rudimentary Induction. In the latter case we use RCA_0/f = ETF + Δ^0_1-CA/f, as the base, for a uniform treatment. But in the presence of Π^0_∞-CA/f, RCA_0/f is obviously equivalent to Rudimentary Induction. So these match exactly at the level of arithmetic comprehension and higher.

Of course, at the time, all of these transfers and considerations were thought through only at an intuitive level. Here we analyze these moves more formally through the natural common definitional extensions given in section 9, and the development in section 10. This natural common definitional extension is simplified in section 11 in a particularly satisfactory way, consolidating numbers, sets, and functions.

In fact, the story gets retold in a considerably simpler and more powerful way in light of the claim in [Fr76] that ETF and RCA_0/f are logically equivalent! As RM took hold while SRM remained dormant, we did not document this result as it was playing no role in the emergence of RM. For RM, the transfer was made from the RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and $others_0$/f, of [Fr76] to the current RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and $others_0$/s, of today, with ETF and L[fcn] disappearing into obscurity.

7. FROM [Fr76] TO CURRENT RM HISTORICALLY

The transfer from the RCA_0/f, WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and $others_0$/f, from [Fr76], to the present RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s, and $others_0$/s of today is particularly simple and smooth.

This transfer is back to the same language L[set] which we used in [Fr75]. In L[set], the 1-ary, 2-ry, and 3-ary functions are treated as certain subsets of ω.

The 1-ary functions are treated as the sets $A \subseteq \omega$ such that $(\forall n)(\exists! m)(2^n 3^m \in A)$.

The 2-ary functions are treated as the sets $A \subseteq \omega$ such that $(\forall n,m)(\exists! r)(2^n 3^m 5^r \in A)$.

The 3-ary functions are treated as the sets $A \subseteq \omega$ such that $(\forall n,m,r)(\exists! s)(2^n 3^m 5^r 7^s \in A)$.

Application is defined by
i. A(n) = the unique m such that $2^n 3^m \in A$.
ii. A(n,m) = the unique r such that $2^n 3^m 5^r \in A$.
iii. A(n,m,r) = the unique s such that $2^n 3^m 5^r 7^s \in A$.

Now we already have as a start, RCA-/s. We know that RCA-/s is too weak since it can't even develop exponentiation. We can expect the Δ^0_1-CA/s to be sufficient to transfer to Δ^0_1-CA/f, but the real issue is the transfer of ETF to L[set].

Δ^0_1-CA/s together with set induction is enough to easily transfer all of ETF except Primitive Recursion and Permutation. A closer look at Permutation reveals that there is no difficulty with this either. So there remains Primitive Recursion.

So we can complete the transfer by using RCA-/s + Primitive Recursion in L[set]. This has the following four components.

CRUDE RCA_0/s

Successor Axioms.
Recursion Equations. For $+,\bullet,<$.
Set Induction.

Δ^0_1-CA/s.
Primitive Recursion in L[set].

But it is fairly clear that we can simply replace "primitive recursion in L[set]" here with Σ^0_1-IND/s and obtain logical equivalence. See Theorem 7.1 below.

The relevance of Σ^0_1-IND/f was readily recognized in [Fr76] on the L[fcn] side. In Theorem 2 there, we state that ETF proves Π^0_1-IND/f, which is trivially equivalent to Σ^0_1-IND/f. This relied on the nonobvious claim that ETF proves Δ^0_1-CA/f (see section 10 here), but is otherwise very easy. I.e., ETF + Δ^0_1-CA_0/f readily implies Σ^0_1-IND/f. See Theorem 7.2 below.

Analysis of the proof of Primitive Recursion in L[set] from RCA reveals that Σ^0_1-IND suffices. So this suggests that we should replace Primitive Recursion in L[set] by Σ^0_1=IND. This is all the more compelling because we shall show below that Primitive Recursion in L[set] and Σ^0_1-IND in L[set] are logically equivalent over the other three axioms of Crude RCA_0 in L[set]. See Theorem 7.1 below.

RCA_0/s

Successor Axioms.
Recursion Equations. For +,•,<.
Δ^0_1-CA/s.
Σ^0_1-IND/s.

This formulation of RCA_0/s is now the standard base theory RCA_0 for current RM.

It is natural to follow the same setup in [Fr76], that is to use RCA_0/s as the base theory for the remaining systems.

WKL_0/s

RCA_0/s
WEAK KONIG'S LEMMA. In L[set]. Every infinite finite sequence tree of 0's and 1's has an infinite path. Stated with finite sequence coding in L[set].

ACA_0/s

RCA_0/s
Π^0_∞-CA/s.

ATR_0/s

RCA_0/s
ARITHMETIC TRANSFINITE RECURSION. In L[set]. The Kleene H-set on every well ordering of a subset of ω exists. Stated with ordered pairing coding in L[set].

Π^1_1-CA_0/s

RCA_0/s
Π^1_1-CA in L[set] (just the comprehension)

The above five systems RCA_0/s, WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s are the basic five systems used in RM today.

It was evident at the time that these ACA_0/s, ATR_0/s, Π^1_1-CA/s are logically equivalent to the ACA-/s, ATR-/s, Π^1_1-CA-/s that came so easily after [Fr75] in preparation for [Fr76]. In retrospect, only RCA-/s, WKL-/s needed to have been revamped.

THEOREM 7.1. Crude RCA_0/s is logically equivalent to standard RCA_0/s.

Proof: Assume crude RCA_0/s. For Σ^0_1-IND/s, assume $(\exists m)(\varphi)[n/0]$ and $(\forall n)((\exists m)(\varphi) \rightarrow (\exists m)(\varphi)[n/S(n)])$, where φ is Δ_0/s. The idea is to first fix all the parameters of either sort, and let A = $\{<n,m>: \varphi\}$ by Δ^0_1-CA/s. Then fix m such that $<0,m> \in A$. Given

<n,r> ∈ A we can find <S(m),r'> ∈ A, and using Δ^0_1-CA/s and set induction, we can go after the <S(m),r'> ∈ A with r' minimized, and realize this procedure by a set coded function. This sets up a Primitive Recursion resulting in a set coded function h where h(0) = <0,m>, and whenever h(n) is some <n,r> ∈ A, h(n+1) is <n+1,r'> ∈ A with r' minimized. By more Δ^0_1-CA/s and set induction, we see that each h(n) is some <n,r> ∈ A. So (∀n)(∃m)(φ).

Now assume the usual base theory RCA_0/s for current RM. It is very well known how to obtain Primitive Recursion in RCA_0/s. QED

THEOREM 7.2. RCA_0/f proves Σ^0_1-IND/f.

Proof: This is proved using the same line of reasoning as in the first paragraph of the proof of Theorem 7.1. Also see Lemma 10.28. QED

As discussed at the end of section 5. Theorem 7.2 was stated in Theorem 2 of [Fr76] in the stronger form using ETF instead of RCA_0/f, obviously relying on that nonobvious claim in Theorem 1 that ETF ≡ RCA_0/f. (Π^0_1-IND was written but that is trivially equivalent to Σ^0_1-IND both in L[fcn] and in L[set]). This nonobvious claim is documented in section 10 here.

8. SYNONYMY

We will give a treatment of synonymy with some perhaps novel points, for the form of many sorted logic used in RM and its emergence. But first we want to give this thorough treatment for the uncluttered ordinary first order predicate calculus with equality.

A language L consists of any set of constant, relation, and function symbols, including the special binary relation symbol =. Every theory S comes with its language L(S) where all axioms of S are in L(S). S,T are logically equivalent if and only if they have the same language and each is provable from the other (in terms of their universal closures).

DEFINITION 8.1. K is a literal definitional extension of S if and only if
i. L(K) is L(S) extended by zero or more constant, relation, and function symbols.
ii. The axioms of K are the axioms of S together with statements of definitions of L(T)\L(S) in L(S) in standard form (called defining axioms).
iii. There is exactly one of these defining axioms in T for every constant, relation, and function symbol in L(T)\L(S).
iv. In the case of the new constant and function symbols, it is required that S prove the relevant existence and uniqueness in S.
K is a definitional extension of S if and only if K is logically equivalent to a literal definitional extension of S.

No one definition of synonymy (synonymous theories) has become entirely standard. The closest that we have for a standard is due to de Bouvere. See [Bo65a], [Bo65b]. However, more recently, there has been interest in a less restrictive notion of synonymy called Morita equivalence in the philosophy of science community. See [BH16].

DEFINITION 8.2. S,T are Bouvere synonymous if and only if they have a common definitional extension. I.e., there is a theory K such that K is a definitional extension of S and K is a definitional extension of T.

Note that Bouvere synonymy appears to be the strongest reasonable notion of synonymy. For it is obvious that Bouvere synonymy is at least as strong as any equivalence relation on theories which takes any theory to be synonymous with its definitional extensions. That Bouvere synonymy is an equivalence relation drops out from the development below, but also can be proved from scratch based on the idea that definitions from definitions are definitions.

For our purposes, we want to set up slightly different terminology because we want to consider another more model theoretic notion of synonymy, and also apply these in section 9 where there are strong uniformities.

DEFINITION 8.3. A literal definitional system (lds) is a quadruple (L_1,L_2,K_0,K_2), where
i. L_1,L_2 are languages in our logic.
ii. $L(K_1) = L(K_2) = L_1 \cup L_2$.
iii. K_1 consists of definitions of nonlogical symbols in L2\L1 by formulas in L1. These take the form $R(x_1,\ldots,x_n) \leftrightarrow \varphi(x_1,\ldots,x_n)$, $x = c \leftrightarrow \varphi(x)$, $F(x_1,\ldots,x_n) = y \leftrightarrow \varphi(x_1,\ldots,x_n,y)$, where R,c,F are from $L_2\backslash L_1$, and φ is a formula of L_1 whose free variables are among the variables shown on the left side of the $\leftrightarrow$.
iv. K_2 consists of definitions of nonlogical symbols in $L_1\backslash L_2$ by formulas in L_2. These take the form $R(x_1,\ldots,x_n) \leftrightarrow \varphi(x_1,\ldots,x_n)$, $x = c \leftrightarrow \varphi(x)$, $F(x_1,\ldots,x_n) = y \leftrightarrow \varphi(x_1,\ldots,x_n,y)$, where R,c,F are from $L_1\backslash L_2$, and φ is a formula of L_2 whose free variables are among the variables shown on the left side of the $\leftrightarrow$.

DEFINITION 8.4. An lds for S,T is an lds $(L(S),L(T),K_1,K_2)$, where S proves existence and uniqueness for all constant and function symbols defined by elements of K_1, and T proves existence and uniqueness for all constant and function symbols defined by elements of K_2.

As you can see, no parameters or multidimensionality is allowed, and equality is preserved (R is not allowed to be the equality symbol).

Having an ids for S,T is weak in that there is no interaction between S,T involved. There is just the legitimacy of the definitions of the constants and functions.

LEMMA 8.1. S,T are Bouvere synonymous if and only if there is an lds for S,T where $S \cup K_1$ and $T \cup K_2$ are logically equivalent.

Proof: Here $S \cup K_1$ and $T \cup K_2$ are the desired common definitional extensions of S,T, respectively. We can obviously eliminate any nonlogical symbols outside $L(S) \cup L(T)$ entirely from the picture. QED

DEFINITION 8.5. A Bouvere synonymy of S,T is an lds $(L(S),L(T),K_1,K_2)$ for S,T where $S \cup K_1$ and $T \cup K_2$ are logically equivalent.

We discuss another notion of synonymy with a model theoretic flavor. Although this notion is obviously extremely attractive and particularly robust, we still think it is important to have a syntactic based notion such as Bouvere synonymy because of the motivating proof theoretic conception behind synonymy.

DEFINITION 8.6. Let $(L(S),L(T),K_1,K_2)$ be a literal definitional system for S,T. $mmap_1$ is the function (a proper class) that maps (all and only) models of S (each are a set) into models of $S \cup K_1$ by expansion. $mmap_2$ is the function that maps (all and only) models of T into models of $K_2 \cup T$ by expansion. Let $mmap_1\#$ be the function that maps models of S into models of L(T) by reduction after $mmap_1$. Let $mmap_2\#$ be the function that maps models of T into models of L(S) by reduction after $mmap_2$. mmap is read "model map".

DEFINITION 8.7. S,T are model synonymous if and only if there is a literal definitional system $(L(S),L(T),K_1,K_2)$ for S,T such that
i. For all models M of T, $mmap_1\#(map_2\#(M)) = M$.
ii. For all models M of S, $mmap_2\#(map_1\#(M)) = M$.
A model synonymy of S,T is an lds for S,T with i,ii.

LEMMA 8.2. Let $(L(S),L(T),K_1,K_2)$ be a literal definitional system for S,T. Suppose $S \cup K_1$ and $T \cup K_2$ are logically equivalent. Then $(L(S),L(T),K_1,K_2)$ is a model synonymy of S,T.

Proof: Let S,T,K_1,K_2 be as given. By Definition 8.5, $mmap_1\#$ maps models of S into models of T and $mmmap_2\#$ maps models of T into models of S.

Let M be a model of S. Then $mmap_1(M)$ is a model of K_1, and its L(T) part is $mmap_1\#(M)$. Since $mmap_2(mmap_1\#(M)))$ is a model of K_2, its L(S) part is $mmap_2\#(mmap_1\#(M))$.

Now $mmap_2(mmap_1\#(M)))$ is a model of K_1. So its L(T) part is $mmap_1\#(M)$. So $mmap_2(mmap_1\#(M)))$ is a model of K_1 with the same L(T) part as $mmap_1(M)$, which is also a model of K_1. Hence $mmap_2(mmap_1\#(M)) = mmap1(M)$. So both of these have the same L(S) parts. Hence $mmap_2\#(mmap_1\#(M))$ is the L(S) part of $mmap_1(M)$ which is M.

So we have shown that for every model M of S, $mmap_2\#(map_1\#(M)) = M$, establishing i, By symmetry we also have established ii. QED

LEMMA 8.3. Let $(L(S),L(T),K_1,K_2)$ be a model synonymy of S,T. Then $S \cup K_1$ and $T \cup K_2$ are logically equivalent.

Proof: Let M be a model of $S \cup K_1$.

CLAIM 1. The L(S) part of the expansion of $M|L(T)$ using K_2 is $M|L(S)$.

Let M' be $M|L(S)$. Then M' is a model of S. Therefore $mod_2\#(mod_1\#(M')) = M'$.

1. $mmap_1\#(M')$ is the L(T) part of the expansion of M' by K_1. By definition.
2. $mmap_1\#(M')$ is the L(T) part of M. Because the expansion of M' by K_1 is M.
3. The L(S) part of the expansion of $mod_1\#(M')$ using K_2 is $M' = M|L(S)$. By the above equation.

4. The L(S) part of the expansion of M|L(T) using K_2 is M|L(S). By 2,3.

CLAIM 2. The L(T) part of the expansion of M|L(T) using K_2 is M|L(T). By definition.

CLAIM 3. The expansion of M|L(T) using K_2 is M. In particular, M satisfies K_2. Immediate from Claims 1,2.

So we have shown that every model of $S \cup K_1$ is a model of $T \cup K_2$. By symmetry every model of $T \cup K_2$ is a model of $S \cup K_1$. Hence $S \cup K_1$ and $T \cup K_2$ are logically equivalent. QED

THEOREM 8.4. S,T are Bouvere synonymous if and only if they are model synonymous. The Bouvere synonymies are the same as the model synonymies.

Proof: By Lemmas 8.2, 8.3. QED

Albert Visser in [Vi06] presents another form of synonymy which we call Visser synonymy, which can be viewed as a syntactic form of the model synonymy we are using. So the equivalence of Visser synonymy with model synonymy and therefore Bouvere synonymy can be seen through the completeness theorem for predicate logic. According to Visser, the equivalence of Bouvere synonymy and Visser synonymy has been around in the folklore.

We now extend this treatment to the many sorted logic that we use here in our account of the emergence of RM and SRM.

L[set] and L[fcn] are two languages in many sorted logic with set and function variables respectively. We use a fairly general version of languages in many sorted logic. A theory is a set of sentences equipped with a language. Two theories are logically equivalent if and only if they have the same models, or equivalently, because of completeness theorems, they have the same theorems. If two theories are logically equivalent then they must have the same language. We now give the details concerning the version of many sorted logic we are using.

A language L will have one or more labeled sorts. The sorts are partitioned into the point sorts, the non point sorts. The non point sorts are further partitioned into the relation sorts, the function sorts, and the set sorts. The intuitive idea is that the point sorts hold points, the relation sorts hold arity ≥ 1 relations on points, the function sorts hold arity ≥ 1 functions from points to points, and the set sorts hold sets of points. The nonlogical symbols are the constant, relation, and function symbols.

The point sorts of L are labeled by a nonempty ASCI word not using the comma or brackets. The function sorts of L are labeled FCN[x_1,...,x_{n+1}], the relation sorts of L are labeled REL[x_1,...,x_n], and the set sorts of L are labeled SET[x], where x_1,...,x_{n+1},x are point sort labels of L and $n \geq 1$. There can be repetitions among the x_i.

The nonlogical symbols of L are constant, relation, and function symbols of L. The constant symbols of L are labeled con[x,α], the relation symbols of L are labeled rel[x_1,...,x_n,α], $n \geq 1$, the function symbols of L are labeled fcn[x_1,...,x_{n+1},α], $n \geq 1$, where x_1,...,x_{n+1} are point sort labels of L, and α is a nonempty ASCI word not using the comma or brackets.

The sorts and the nonlogical symbols of L, collectively, have distinct labels. The labels of the non point sorts of L start with capital letters and use brackets, whereas the labels of the nonlogical symbols of L start with lower case letters and use brackets. The labels of the point sorts of L do not use brackets.

To simplify matters, we assume that for each point sort x of L, there is a relation symbol rel[x,x,=], the so called equality relation on sort x. As usual in logic, it will be given special treatment syntactically and semantically. However, there is no equality relation on non pointed sorts, and indeed no constant, relation, or function symbols involving the non pointed sorts. Note that the equality symbols of L all use different point sorts so there is no equality facility across sorts.

Note that the non pointed sorts have their own built in defined extensional equality.

We have variables v_n^x of L ranging over the sort in L labeled x, where $n \geq 1$. Here x may be a point sort or a non point sort.

The terms of L are inductively defined as follows.

1. All variables v_n^x in L where x is a point sort of L is a term of L of sort x.
2. All constant symbols labeled con[x,α] of L are terms of L of sort x.
3. If f is a function symbol of L labeled fcn[$x_1,\ldots,x_{n+1},\alpha$], or v_n^x is a variable where x is fcn[$x_1,\ldots,x_{n+1}$] of L, and $t_1,\ldots,t_n$ are terms of L of sorts $x_1,\ldots,x_n$, then $f(t_1,\ldots,t_n)$ and $x(t_1,\ldots,t_n)$ are terms of L of sort x_{n+1}.

The atomic formulas are as follows. Let $t_1,\ldots,t_n,t$ be terms of L of sorts $x_1,\ldots,x_n,x$.
i. $r(t_1,\ldots,t_n)$, where r is a relation symbol of L, rel[$x_1,\ldots,x_n,\alpha$].
ii. $v_n^x(t_1,\ldots,t_n)$, where x is REL[$x_1,\ldots,x_n$], a relation sort of L.
iii. $t \in$ SET[x], where SET[x] is a set sort of L.

The formulas are built up inductively in the usual way using the connectives $\neg,\wedge,\vee,\rightarrow,\leftrightarrow$ and the quantifiers $\forall,\exists$.

The semantics is defined using models and assignments. Models assign a nonempty set to each point sort. No requirements whatsoever on these such as disjointness. SET[x] is assigned any nonempty set of subsets of the set assigned to the sort x of L. REL[x_1,...,x_n] is assigned any nonempty set of elements of the Cartesian product of what is assigned to the sorts x_1,...x_n of L. FCN[x_1,...,x_{n+1}] is assigned any nonempty set of functions from the sets assigned to sorts x_1,...,x_n in L into the set assigned to sort x_{n+1} in L. The equality relation on each point sort, rel[x,x,=) must be interpreted as actual equality on sort x. Details concerning ordered tuples are not important; nor is overlaps a problem between elements of the sets assigned to the various sorts. We don't even have to worry about functions in the set assigned to a function sort in L being identical to an element of the set assigned to a point sort in L.

Also constant, relation, and function symbols are assigned elements, relations, and functions of the appropriate kind in the usual way. Collisions with these and the entities in the previous paragraph again are of no consequence. It is required that the relation symbol assigned to rel[x,x,=] in L be the equality relation on the set assigned to sort x. This completes the definition of model for language L.

The satisfaction relation depends on an assignment. The assignment maps the ω many variables over each sort of L to an element of the set assigned to that sort. Thus point variables are assigned points, relation variables are assigned relations, function variables are assigned functions, and set variables are assigned sets. But these points, relations, functions, sets must be elements of the point sorts, relation sorts, function sorts, and set sorts of L used by the variable of L.

There is a standard Hilbert style system which is complete for this semantics. It has the usual finite number of propositional schemes that take care of all five connectives. There are the axiom schemes of universal instantiation and existential generalization, where the quantifiers range over point, relation, function, and set sorts. There are also the rules of conditional universal generalization and conditional existential instantiation, where the quantifiers range over point, relation, function, and set sorts. Then there are a few quantifier free equality axioms for each equality relation on each point sort. Finally there is modus ponens.

The Completeness Theorem can be proved from scratch using say Henkin's method (Henkin constants), or it can be made to follow from the usual Completeness Theorem for the usual one sorted logic (no set, relation, function variables or sorts).

We now come to our treatment of synonymy in the context of our many sorted logic setup above. The core concept that drives everything are our literal definitional extensions. For have been using them for ordinary predicate calculus - see Definition 8.1. We now indicate how this needs to be extended to our version of many sorted logic.

We have to expand on the forms we allowed in iii,iv in Definition 8.3. First of all, the R,c,F, being nonlogical symbols in L_1 or L_2, need to be appropriately sorted with point sorts, which are required to be the same for L_1 and L_2. However, the non pointed sorts of L_1,L_2 can overlap in any way.

Now we come to the case where we are adding a new relation, function, or set sort. In the case of a new relation sort $REL[x_1,...,x_n]$, the elements are going to be certain relations on the Cartesian product of point sorts $x_1,...,x_n$. But which relations? If we just say "all that obey a definition in L" then we are getting perhaps all such relations, and we are really doing second order logic, which is totally inappropriate here. What we need is a formula $\varphi(z_1,...,z_r,w_1,...,w_n)$, where $w_1,...,w_n$ are variables of sorts $x_1,...,x_n$, and $z_1,...,z_r$ are variables of any sorts in L, with the idea that $REL[x_1,...,x_n]$ is to consist of the relations obtained from $\varphi(z_1,...,z_r,w_1,...,w_n)$ by fixing any choice of $z_1,...,z_r$. This way, the relations in the new $REL[x_1,...,x_n]$ are tied down to objects under the purview of S. For $FCN[x_1,...,x_{n+1}]$ we proceed in the same way, taking the functions obtained from $\psi(z_1,...,z_r,w_1,...,w_{n+1})$ by fixing any choice of $z_1,...,z_r$. Here we state the uniqueness using the equality relations on the relevant point sorts. For SET[x], we use $\rho(z_1,...,z_r,w)$, and use the sets obtained by fixing any choice of $z_1,...,z_r$. It is important to allow the z's to come from both point sorts and from relation, function, and set sorts, in L. Application of course works as required.

Now the previous paragraph really is more directly an explanation of how we go from a model of S to an expansion of the model of S accommodating either a new symbol or a new sort. In the case of adding a new symbol, it is obvious what we are doing syntactically to form S' - we are adding the obvious explicit definition. For adding a relation sort, syntactically, we are saying that every object of the new relation sort acts exactly like the formula says it should for some fixing of the z's, and for every fixing of the z's there exists an object of the new relation sort that acts exactly ss prescribed by the formula. Analogously for a new set sort. For a new function sort, we use only fixings of the z's resulting in the graph of a function (existence and uniqueness condition), using the required equality relations on the relevant sorts.

DEFINITION 8.8. For our version of many sorted logic. A literal definitional system (lds) is a quadruple (L_1,L_2,K_1,K_2), where
i. L_1,L_2 are languages in our many sorted logic with the same point sorts.
ii. $L(K_1) = L(K_2) = L_1 \cup L_2$.
iii. K_1 consists of definitions of nonlogical symbols and non point sorts in $L_2\backslash L_1$ by formulas in L_1 as discussed above.
iv. K_2 consists of definitions of nonlogical symbols and non point sorts in $L_1\backslash L_2$ by formulas in L_2 as discussed above.

DEFINITION 8.9. An lds for S,T is an lds $(L(S),L(T),K_1,K_2)$, where S proves existence and uniqueness for all constant and function symbols defined by elements of K_1, and T proves existence and uniqueness for all constant and function symbols defined by elements of K_2.

Note that in a lds $(L(S),L(T),K_1,K_2)$ for S,T, we have that S,T have the same point sorts, different non point sorts and different nonlogical symbols.

We extend Definition 8.5, 8.7 to our version of many sorted logic. We extend Bouvere, model, synonymy and synonymies to our version of many sorted logic.

THEOREM 8.5. The following holds in our many sorted logic. S,T are Bouvere synonymous if and only if they are model synonymous. The Bouvere synonymies are the same as the model synonymies.

Visser synonymy [Vi06] also extends naturally to our many sorted logic, and can be viewed as a syntactic form of the model synonymy we are using in many sorted logic. So the equivalence of Visser synonymy with model synonymy and therefore Bouvere synonymy can again be seen through the completeness theorem for our many sorted logic.

9. THE NATURAL UNIFORM SYNONYMIES

With this background information behind us in section 8, we now take up the two natural literal definitional systems $(L[fcn],L[set],K_1,K_2)$. Recall

L[fcn]. Pointed sort ω. Equality on ω. Nonlogical symbols 0,S. Non pointed sorts FCN[1], FCN[2], FCN[3]. Standard application facility for using function aorts.

L[set]. Pointed sort ω. Equality on ω. Nonlogical symbols 0,S,+,•,< on sort ω. Non pointed sort SET[ω]. Standard relation ∈ between sort ω and sort SET[ω].

This all fits smoothly into our many sorted logic with trivial changes of notation. First of all the names of the nonlogical symbols and sorts are changed to conform to the naming conventions in many sorted logic, which of course nobody pays any attention to. The binary relation ∈ in L[set] is not a relation symbol in many sorted logic but rather a special symbol used in connection with all sorts SET[x], x a point sort.

Note that L[fcn],L[set] meet the requirements of Definition 8.8 involving the two languages L_1,L_2 there, having the same point sorts.

We now define K_1,K_2 so that we have a literal definitional system (L[fcn],L[set],K_1,K_2).

DEFINITION 9.1. We define K_1 as follows. L(K_1) = L[fcn] ∪ L[set] defines the nonlogical symbols and non point sorts in L[set]\L[fcn] by formulas in L[fcn]. These are +,•,<,and SET[ω]. The SET[ω] sort in L[set] properly formally specified as the formal extensions of the formula f(n) = 0 where n ranges over sort ω and f ranges over FCN[1]. The 0,S of L[set] are the same as the 0,S of L[fcn]. n+m = r in L[set] is defined by there exists 2-ary f satisfying the recursion equations for + such that f(n,m) = r, in L[fcn]. n•m = r in L[set] if and only if there exists 2-ary f satisfying the recursion equations for + such that f(n,m) = r, in L[fcn]. n < m in L[set] if and only if there exists 2-ary f satisfying the recursion equations for the characteristic function of < such that f(n,m) = S(0) in L[fcn].

The way we set things up it is not a problem that it takes some proper axioms to prove that these definitions of +,• are legitimate (existence and uniqueness) or that this definition of < has anything like the intended meaning. That comes in only when we speak of (L[fcn],L[set],K_1,K_2) being a literal definitional system **for** theories S,T.

DEFINITION 9.2. We define K_2 as follows. $L(K_2)$ = L[fcn] ∪ L[set] defines the nonlogical symbols and non point sorts in L[fcn]\L[set] by formulas in L[set]. These are the sorts FCN[1],FCN[2],FCN[3] only. The FCN[1] sort in L[fcn] consists of the functions formally specified by the formula $2^n3^m \in A$ for A ranges over sort SET[ω] in L[set]. Recall the way this works is that formally we look for all sets A such that $2^n3^m \in A$ defines the graph of a 1-ary function and use that graph as a 1-ary function. The FCN[2] sort in L[fcn] uses the formula $2^n3^m5^r \in A$ in the same way. Also the FCN[3] sort in L[fcn] uses the formula $2^n3^m5^r7^s \in A$ in the same way. Now of course, this arithmetic is only part of the vocabulary of L[set]. So these formulas are conjuncted with a formula calling for the binary exp function satisfying the usual recursion equations which are then used to eliminate the 2^n3^m, $2^n3^m5^r$, $2^n3^m5^r7^s$, and stay within L[set].

The way we set things up it is not a problem that it takes some proper axioms to prove that these definitions of the function sorts have anything like their intended meaning - an apparent problem in light of the need for proper axioms to see that the treatment of +,•,exp is at all correct. However, this is not an issue for (L[fcn],L[set],K_1,K_2) being a literal definitional system, but only when we speak of (L[fcn],L[set],K_1,K_2) being a literal definitional system **for** theories S,T.

LEMMA 9.1. (L[fcn],L[set],K_1,K_2) is a literal definitional extension for RCA_0/f and RCA_0/s.

Proof: This means that the existence and uniqueness of new constant and function symbols is proved in the required theory. In this case, this only refers to RCA_0/f proving existence and uniqueness in connection with the defined +,•. This follows from RCA_0/f proving the existence and uniqueness of 2-ary functions satisfying the recursion conditions for +,• and the characteristic function of <. QED

LEMMA 9.2. (L[fcn],L[set],K_1,K_2) is a model synonymy of RCA_0/f and RCA_0/s. Hence it is also a Bouvere synonymy and a Visser synonymy of RCA_0/f and RCA_0/s.

Proof: Below when we write "comes from Primitive Recursion etc." we mean "comes from Primitive Recursion Axiom, Initial Functions Axiom, Composition Axiom".

Let M = (ω,FCN[1],FCN[2],FCN[3],0,S) be a model of RCA_0/f. Write $mmap_1$#(M) = (ω,SET[ω],0,S,+,•,<) according to K_1. The successor axioms in $mmap_1$#(M)) comes from the successor axioms in M. The recursion equations for +,•,< have extensionally unique solutions in $mmap_1$#(M) come from Primitive Recursion and Rudimentary Induction in M. Set Induction in $mmap_1$#(M) comes from Rudimentary Induction in M. Δ^0_1-CA/s in $mmap_1$#(M) comes from Δ^0_1-CA/f in M. Σ^0_1-IND/s in map_1#(M) comes from Σ^0_1-IND/f in M. So $mmap_1$#(M) satisfies RCA_0/s.

Let M = (ω,SET(ω),0,S,+,•,<) be a model of RCA_0/s. Write $mmap_2$#(M) = (ω,FCN[1],FCN[2],FCN[3],0,S) according to K_2. The successor axioms in $mmap_2$#(M) comes from the successor axioms in M. The Initial Functions and Composition and Permutation Axioms in $mmap_2$#(M) come from Δ^0_1-CA/s in M. The Primitive Recursion Axioms in $mmap_2$#(M) come from the whole of RCA_0/s in M via standard coding arguments in RM. The Rudimentary Induction axiom in $mmap_2$#(M) comes from Set Induction (with very modest Δ^0_1-CA/s) in M. So $mmap_2$#(M) satisfies RCA_0/f.

Let M = (ω,FCN[1],FCN[2],FCN[3],0,S) be a model of RCA_0/f. Go to $mmap_1$#(M) = (ω,SET[ω],0,S,+,•,<) according to K_1, which satisfies RCA_0/s. The sets we get in $mmap_1$#(M) are of the form {n: f(n) = 0} for 1-ary f in M. Now we want to apply $mmap_2$# and show we get back M. The functions that we get back are those whose graphs are $\{2^n3^m: 2^n3^m \in A\}$, $\{2^n3^m5^r: 2^n3^m5^r \in A\}$, $\{2^n3^m5^r7^s: 2^n3^m5^r7^s \in A\}$ for sets A in $mmap_1$#(M). The sets A came from M in an explicit way, and so we can use Δ^0_1-CA/f in M to see that these functions whose graphs are of those forms must exist in M. So every function we get back belongs to M. Now let f be a function in M, 1,2,3-ary. The sets in $mmap_1$#(M) are of the form {n: f(n) = 0}, f 1-ary in M. Now f can be put in the form $\{2^n3^m5^r7^s: g(2^n3^m5^r7^s) = 0\}$ using Primitive Recursion etc.

Let M = (ω,SET[ω],0,S,+,•,<) be a model of RCA_0/s. Go to $mmap_2$#(M) = (ω,FCN[1],FCN[2],FCN[3],0,S) according to K_2. Now we want to apply $mmap_1$# and show we get back M. The functions that we get in $mmap_2$#(M) are exactly the functions that are coded in M in the usual sense of RM. Using standard RCA_0/s reasoning from RM, the sets {n: f(n) = 0} obtained from 1-ary such functions are exactly the sets in M. Also there are unique functions satisfying the recursion equations for +,•,< among these functions coded in M, and this must be the actual +,•,< in RM. So with this reasoning, we get back exactly M. QED

THEOREM 9.3. Our literal definitional extension (L[fcn],L[set],K_1,K_2) is a model synonymy for the following pairs of theories:
RCA_0/f and RCA_0/s
WKL_0/f and WKL_0/s
ACA_0/f and ACA_0/s
ATR_0/f and ATR_0/s
Π^1_1-CA_0/f and Π^1_1-CA_0/s
others/f and others/s
Furthermore (L[fcn],L[set],K_1,K_2) is a Bouvere synonymy and a Visser synonymy.

Proof: The first was done in Theorem 9.2. The second is almost identical with the coded weak Konig's Lemma going back and forth between L[fcn] and L[set]. The others are easier since on the function side, Initial Functions, Composition, Primitive Recursion, and Permutation Axioms are redundant, and on the set side, Δ_{01}-CA/s and Σ^0_1-IND/s are reduntant. QED

Notice the very heavy uniformity of the synonymies in Theorem 9.3.

However, there is another aspect to this. From our theory in section 8, we get common definitional extensions. With the help of section 10, there is a common definitional extension of RCA0/f and RCA0/s that is strictly mathematical (RCA_0/f and RCA_0/s are decidedly not). This is the ETF[FSRA] discussed in section 11.

10. ETF = RCA_0 IN L[fcn] - newly documented

In this section we prove the undocumented claim in [Fr76] that ETF proves Δ^0_1-CA/f, and hance ETF and RCA_0/f are logically equivalent (see Theorem 1 of [Fr76]). Furthermore, [Fr76] claims that ETF proves both Δ^0_1-CA/f and Σ^0_1-IND/f (see Theorem 2 of [Fr76]), but "RCA_0/f proves Σ^0_1-IND/f" is relatively straightforward - see Theorem 7.2. ([Fr76] uses Π^0_1-IND/f instead of Σ^0_1-IND/f which is trivially equivalent).

We repeat the presentation of the system ETF from [Fr76], this time in greater detail and with some explanatory notes.

The language of ETF is L[fcn]. See Definition 1.2. There are four sorts, ω, FCN[1], FCN[2], FCN[3]. The intended interpretation is for FCN[i] to be the i-ary functions from ω into ω.

We have variables over each sort, constant 0 of sort ω, 1-ary function symbol S on sort ω, and equality = on sort ω. Terms are all of sort ω, and defined inductively using the function variables, 0,S, and function application. The atomic formulas are of the form s = t where s,t are terms. We use the usual logical connectives $\neg,\wedge,\vee,\rightarrow,\leftrightarrow$ and quantifiers $\forall,\exists$ over each sort. Formulas are built inductively in the obvious way starting with atomic formulas. We have the usual Hilbert style axioms and rules of logic. In [Fr76] we used a constant symbol N of sort FCN[1] which we eliminate in favor of the 1-ary function symbol S on sort ω.

The proper axioms for ETF are as follows:

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axiom.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

Here are some explanatory notes. There are some very minor changes from [Fr76] as indicated.

For specificity below, n,m,r are the first three variables over ω.

Successor Axioms.
i. $S(n) \neq 0$
ii. $S(n) = S(m) \rightarrow n = m$
iii. $n \neq 0 \rightarrow (\exists m)(S(m) = n)$

In PA, one normally only sees i,ii as iii is derivable. But the induction hypothesis is quantified, which does not fit into the SRM point of view. However iii itself is strictly mathematical. And i-iii are also used in the basic system of arithmetic Q (see [HP93], p. 28).

Initial Functions Axioms.
i. There exists 1-ary functions that are constantly any n. Here n is a variable of sort ω.
ii. The three 3-ary projection functions exist.
iii. S(n) defines a 1-ary function.

The above is a simplification of Composition in [Fr76].

Since equality between function sorts is not allowed, i,ii,iii are existence statements. Of course, extensional uniqueness is immediate.

Composition Axioms.

i. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n,m))$

ii. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n))$

iii. $(\exists f)(\forall n,m)(f(n,m) = g(n,m,r))$

iv. $(\exists f)(\forall n)(f(n) = g(n,m,r))$

v. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(h_1(n,m,r),h_2(n,m,r),h_3(n,m,r)))$

The above is a simplification of Composition in [Fr76].

Primitive Recursion Axiom. Consist of two assertions, for arity 2.

$(\exists f)(\forall n)(f(n,0) = g(n) \wedge (\forall m)(f(n,S(m)) = h(n,m,f(n,m))))$.

In [Fr76], Primitive Recursion for arity 1 is also included. This is derivable.

Permutation Axiom. Asserts that every 1-ary function that maps ω one-one onto ω has an inverse. I.e., let f be a permutation of ω. There exists g such that $(\forall n)(f(g(n)) = n)$.

Rudimentary Induction.

$f(0) = g(0) \wedge (\forall n)(f(n) = g(n) \rightarrow f(S(n)) = g(S(n))) \rightarrow f(n) = g(n)$

In [Fr76] we also used a corresponding statement for arities 2,3. Here we derive those.

ETF uses only Rudimentary Induction. This suggests a possible connection with work done in a different, purely first order context; namely see [Go45], [Go54], [Cu41]. We have not yet explored this.

We show that ETF derives Δ^0_1-CA/f and Σ^0_1-IND/f. We first work in ETF\PERM, as Permutation will play no role until we get to the advanced stages of the development.

LEMMA 10.1. Let n,m,r be distinct variables and t be a term. ETF\PERM proves

i. $(\exists f)(\forall n,m,r)(f(n,m,r) = t)$.

ii. $(\exists f)(\forall n,m)(f(n,m) = t)$.

iii. $(\exists f)(\forall n)(f(n) = t)$.

Proof: Fix distinct variables n,m,r. We first show that for all t, ETF\PERM proves i $\rightarrow$ ii and i $\rightarrow$ iii. Let t be given and assume i. Let f be given by i. Use Composition Axioms iii,iv to derive ii,iii here.

We now prove i above by external induction on t. It is a provability statement. If t is 0 or a variable other than n,m,r then we use 3-ary constant functions from Initial Functions. These 3-ary constant functions are obtained from the 1-ary constant functions and Composition Axiom ii. It t is among n,m,r, use 3-ary projection functions from Initial Functions. Let t be S(s1), $f_1(s_1)$, $f_2(s_1,s_2)$, $f_3(s_1,s_2,s_3)$. By induction hypothesis, let

1. $g_1(n,m,r) = s_1$
2. $g_2(n,m,r) = s_2$
3. $g_3(n,m,r) = s_3$

hold for all n,m,r. We claim that there exists h_0,h_1,h_2,h_3 such that

a. $h_0(n,m,r) = S(g_1(n,m,r))$
b. $h_1(n,m,r) = f_1(g_1(n,m,r))$
c. $h_2(n,m,r) = f_2(g_1(n,m,r),g_2(n,m,r))$
d. $h_3(n,m,r) = f_3(g_1(n,m,r),g_2(n,m,r),g_3(n,m,r))$

hold for all n,m,r. Here the h's depend internally on the g's, variables of sort ω in t other than n,m,r, and functions in t. Here d is from Composition Axioms v. For c we convert f2 to a 3-ary function f_2' by Composition Axioms i, (dummy argument), and use f_2' for the f_3 in d. For b, we convert f_1 to a 3-ary function f_1' by Composition Axiom$_s$ ii, (two dummy arguments), and use f_1' for the f_3 in d. For a, convert function symbol S to a 1-ary function f_1 by Initial Functions Axioms iii and apply b. QED

DEFINITION 10.1. For each n, n* is S...S0, where there are n S's. 0* is 0.

This n* notation is useful and clarifying when n is an arbitrary external element of ω. However, when n is a specific element of ω, say 1,2, we will simply use 1,2 instead of 1*,2*.

LEMMA 10.2. ETF\PERM proves induction for any equation s = t on any variable n of sort ω. ETF/PERM proves (∃f)(f(0) = r ∧ (∀m)(f(S(m)) = h(m,f(m)))).

Proof: Let s = t be an equation and n be a variable of sort ω. By Lemma 10.1, let f(n) = s and g(n) = t, where f,g depend internally on the functions in s,t and the variables over ω in s,t other than n. Now apply Rudimentary Induction to f(n) = g(n). For the second claim, let f'(n,0) = r ∧ (∀m)(f'(n,S(m)) = g(m,f'(n,m)))), which we can obtain by Primitive Recursion using the 1-ary constant function r, and g'(s,m,t) = g(m,t) by Lemma 10.1. Set f(m) = f'(n,m) by Lemma 10.1 (dummy variable n). QED

We call the induction in Lemma 10.2, equational induction. The second claim (1-ary primitive recursion) along with the Primitive Recursion Axiom (2-ary) will be referred to as Primitive Recursion.

LEMMA 10.3. (ETF\PERM) There exist functions +,•,P,-,F,G such that the following holds with the variables of sort ω universally quantified.
i. n+0 = n, n+S(m) = S(n+m).
ii. n•0 = 0, n•(S(m)) = n•m + n.
iii. P(0) = 0, P(S(n)) = n.
iv. n-0 = n, n-S(m) = P(n-m).
v. F(0) = 1, F(S(n)) = 0.
vi. G(0) = 0, G(S(n)) = 1.
In each of i-vi, the function introduced is extensionally unique in the sense that any two such functions agree everywhere.

Proof: Use Primitive Recursion and Lemma 10.1 for existence, and Equational Induction for extensional uniqueness. QED

We will not expand the language of ETF to incorporate $+,\bullet,P,-,F,G$. Instead, use of these symbols in a formula φ is taken as a form of abbreviation, abbreviating "φ holds of some functions $+,\bullet,P,-,F,G$ obeying i-vi". Thus because of the function variables in ETF, we don't need to be expanding the language in order to really use $+,\bullet,P,-,F,G$.

LEMMA 10.4. (ETF\PERM)
i. F is 1 at 0 and 0 elsewhere.
ii. G is 0 at 0 and 1 elsewhere.
iii. $1-S(n) = 0$.
iv. $2-SS(n) = 0$.
v. $n = 0 \vee n = 1 \vee 2-n = 0$.
vi. If $m \neq 0$ then $P(n) = m \leftrightarrow n = S(m)$.
vii. $0-n = 0$.
viii. $S(n) = n+1$.
ix. $0+n = n$.
x. $S(n)+m = n+S(m)$.
xi. $n+m = m+n$.
xii. $(n+m)+r = n+(m+r)$.
xiii. $n+m = 0 \leftrightarrow n = 0 \wedge m = 0$.
xiv. $n-1 = 0 \leftrightarrow n = 0 \vee n = 1$.
xv. $1-n = 0 \leftrightarrow n \neq 0$.
xvi. $n = 1 \leftrightarrow (n-1)+(1-n) = 0$.
xvii. $0\bullet n = 0$.
xviii. $(2-n)\bullet n$ is 1 at 1 and 0 elsewhere.

Proof: For i,ii use Successor Axioms.

For iii, we use Equational Induction on n. $1-S(0) = P(1-0) = P(1) = 0$. Suppose $1-S(n) = 0$. Then $1-SS(n) = P(1-S(n)) = P(0) = 0$.

For iv, we use Equational Induction on n. $2-SS(0) = P(2-S(0)) = PP(2-0) = PP(2) = PPSS(0) = PS(0) = 0$. Suppose $2-SS(n) = 0$. Then $2-SSS(n) = P(2-SS(n)) = P(0) = 0$.

For v, let n not be 0,1. We use Successor Axioms. Write $n = S(m)$. Then $m \neq 0$. Write $m = S(r)$. Then $n = SS(r)$. Hence $2-n = 2-SS(r) = 0$ by iv.

For vi, let $m \neq 0$ and $P(n) = m$. Then $n \neq 0$ and write $n = S(r)$. Then $PS(r) = m = r$. So $n = S(m)$. Conversely, $n = S(m) \rightarrow P(n) = m$ holds unconditionally.

For vii, use Equational Induction on n. 0-0 = 0 is trivial. Suppose 0-n = 0. Then 0-S(n) = P(0-n) = P(0) = 0.

For viii, n+1 = n+S(0) = S(n+0) = S(n).

For ix, use Equational Induction on n. 0+0 = 0. Suppose 0+n = n. Then 0+Sn = S(0+n) = S(n).

For x, use equational induction on m. S(n)+0 = S(n) = n+S(0). Suppose S(n)+m = n+S(m). Then S(n)+S(m) = S(S(n)+m) = S(n+S(m)) = n+SS(m).

For xi, use equational induction on m. n+0 = 0+n from ix. Assume n+m = m+n. Then n+S(m) = S(n+m) = S(m+n) = m+S(n) = S(m)+n by x.

For xii, use equational induction on r. (n+m)+0 = n+(m+0) is immediate. Suppose (n+m)+r = n+(m+r). Then (n+m)+S(r) = S((n+m)+r) = S(n+(m+r)) = n+S(m+r) = n+(m+S(r)).

For xiii, suppose m ≠ 0. Write m = S(r). n+m = n+S(r) = S(n+r) ≠ 0. Suppose n ≠ 0. Write n = S(u). n+m = m+n = S(m+u) ≠ 0.

For xiv, let n ≠ 0,1. Write n = S(m), m ≠ 0. Write m = S(r). Then n = SS(r), n-1 = PSS(r) = S(r) ≠ 0.

For xv, if n = 0 then obvious. Suppose n ≠ 0 and write n = S(m). We want 1-S(m) = 0 ↔ S(m) ≠ 0. Both sides are true by iii.

For xvi, suppose (n-1)+(1-n) = 0. By xiii, n-1 = 1-n = 0. By xiv, n = 0 ∨ n = 1. By 1-n = 0, we have n ≠ 0. Hence n = 1. The forward direction is trivial.

For xvii, we use Equational Induction on n. 0•0 = 0. Suppose 0•n = 0. Then 0•S(n) = 0•n + 0 = 0+0 = 0.

For xviii, suppose n ≠ 1. By v, n = 0 ∨ 2-n = 0. Hence (2-n)•n = 0. Also (2-1)•1 = 1•1 = 1•0 + 1 = 0 + 1 = 1. QED

DEFINITION 10.2. An open formula is a propositional combination of equations. A singular open formula is a propositional combination of equations of the form t = n*.

We need to obtain induction for all open formulas. We first obtain induction for the singular open formulas.

LEMMA 10.5. $n = 0 \leftrightarrow F(n) = 1$ is provable in ETF\PERM. For $m \geq 2$, $n = m^* \leftrightarrow P...P(n) = 1$ is provable in ETF\PERM, where there are m-1 P's. Every singular open formula is provably equivalent to a propositional combination of equations of the form $t = 1$, over ETF\PERM.

Proof: The first claim is by Lemma 10.4i.

For the second claim, n is a variable of sort ω and m is an external nonnegative integer. Note that the forward direction is obviously provable in ETF\PERM. We establish the provability of the converse by external induction on $m \geq 2$. The basis case is $P(n) = 1 \rightarrow n = 2$, with $m = 2$. Suppose $P(n) = 1$. Then $n \neq 0,1$. As in the proof of Lemma 10.4v, write $n = SS(r)$. Then $P(n) = S(r) = 1$. Hence $r = 0$, and so $n = SS(0) = 2$. Now suppose $P...P(n) = 1 \rightarrow n = m^*$, $m \geq 2$, is provable in ETF\PERM, where there are m-1 P's. Then $P...P(P(n)) = 1 \rightarrow P(n) = m^*$ is provable in ETF\PREM by substituting P(n) for n. By Lemma 10.4vi, $P...P(n) = 1 \rightarrow n = (m+1)^*$ is provable in ETF\PERM, where there are m P's.

The last claim is now clear by replacing the atomic formulas $t = n^*$, $n \geq 2$, in singular open formulas, by formulas $P...P(n) = 1$, and atomic formulas $t = 0$ by $F(t) = 1$. QED

LEMMA 10.6. Every singular open formula φ is provably equivalent over ETF\PERM to an equation $t = 0$. ETF\PERM proves induction for φ on n.

Proof: By external induction on the singular open formula. We assume the singular open formulas are propositional combinations of atomic formulas $t = 1$, by Lemma 10.5.

The basis case is handled by $t = 1 \leftrightarrow (t-1)+(1-t) = 0$ from Lemma 10.4xvi. Suppose $\varphi \leftrightarrow t = 0$ is provable. Then $\neg\varphi \leftrightarrow t \neq 0 \leftrightarrow G(t) = 1$ by Lemma 10.4ii. Suppose $\varphi \leftrightarrow s = 0$ and $\psi \leftrightarrow t = 0$ are provable. Then $\varphi \wedge \psi \leftrightarrow s+t = 0$ is provable by Lemma 10.4xiii. The second claim immediately follows by Equational Induction. QED

LEMMA 10.7. ETF\PERM proves the following.

i. $n \neq 0 \rightarrow SP(n) = n$.

ii. $P(n) = 0 \rightarrow n = 0 \vee n = 1$.

iii. $S(n)-S(m) = n-m$.

iv. $(n+m)-m = n$.

v. $n+r = m+r \rightarrow n = m$.

Proof: For i, let $n \neq 0$ and write $n = S(m)$. Then $SP(n) = SP(S(m)) = S(m) = n$.

For ii, let $\neg(n = 0 \vee n = 1)$, and write $n = SS(m)$. Then $P(n) = S(m) \neq 0$.

For iii, use Equational Induction on m. $S(n)-S(0) = P(S(n)-0) = P(S(n) = n = n-0$. Suppose $S(n)-S(m) = n-m$. Then $S(n)-SS(m) = P(S(n)-S(m)) = P(n-m) = n-S(m)$.

For iv, use Equational Induction on m. Basis case is immediate. Suppose $(n+m)-m = n$. We want $(n+S(m))-S(m) = n$. If $m = 0$ then this is $(n+S(0))-S(0) = P(n+S(0)) = PS(n) = n$. So let $m = S(m')$. We have $(n+S(m'))-S(m') = n$ and we want $(n+SS(m'))-SS(m') = n$. We have $S(n+m')-S(m') = n$, and so by iii, $SS(n+m')-SS(m') = n$. Hence $(n+SS(m'))-SS(m') = n$.

For v, $n+r = m+r \rightarrow (n+r)-r = (m+r)-r \rightarrow n = m$ by iv. QED

LEMMA 10.8. (ETF\PERM) $n = m \vee r = 0 \leftrightarrow n \bullet G(r) = m \bullet G(r)$.

Proof: The forward direction is trivial. Now assume $n \bullet G(r) = m \bullet G(r)$. If $r = 0$ we are done. Otherwise $G(r) = 1$, and so $n = m$. This uses Lemma 10.4ii. QED

LEMMA 10.9. (ETF\PERM)
i. $n-n = 0$.
ii. $S(n)-n = 1$.
iii. If $n \neq 0$ then $P(n)+S(m) = n+m$.
iv. $n-m = 0 \vee (n-m)+m = n$.

Proof: For i, use Equational Induction on n. Suppose $n-n = 0$. Then $Sn-Sn = 0$ by Lemma 10.7iii.

For ii, we use equational induction on n. $S(0)-0 = 1$. Suppose $S(n)-n = 1$. Then $SS(n)-S(n) = S(n)-n = 1$ by Lemma 10.7iii.

For iii, let $n \neq 0$ and write $n = S(r)$. Then $P(n)+S(m) = PS(r)+S(m) = r+S(m) = S(r+m) = r+m+1 = r+1+m = n+m$ using Lemma 10.4xi,xii.

For iv, we use induction on the statement with respect to the variable m. This is allowed because by Lemma 10.8, the statement is equivalent to an equation in n,m, and so we are just using equational induction. $n-0 = 0 \vee (n-0)+0 = n$ is trivial. Suppose $n-m = 0 \vee (n-m)+m = n$. We want $n-S(m) = 0 \vee (n-S(m))+S(m) = n$. We want $P(n-m) = 0 \vee P(n-m)+S(m) = n$. If $n-m = 0$ then we are done. So assume $n-m \neq 0$ and $(n-m)+m = n$. By iii, $P(n-m)+S(m) = (n-m)+m = n$. QED

LEMMA 10.10. $n-m = m-n = 0 \rightarrow n = m$.

Proof: By induction on m with induction hypothesis $n-m = m-n = 0 \rightarrow n = m$. This is not a singular open formula, so this induction needs to be justified. Note that the formula can be put into the form $n-m \neq 0 \vee m-n \neq 0 \vee n = m$, which is of the form $\varphi \vee n = m$ where φ is a singular open formula. Hence by Lemma 10.6, we can put this in the form $t = 0 \vee n = m$, and then into a single equation by Lemma 10.8.

The basis case is trivial. Suppose $n-m = m-n = 0 \rightarrow n = m$. Assume $n-S(m) = S(m)-n = 0$. Clearly $n \neq 0$. By Lemma 10.7iii, $S(m)-n = m-P(n) = 0$, and $m-n = P(m-P(n)) = 0$. If $n-m = 0$ then we are done. Hence we can assume $n-m \neq 0$. Now $n-S(m) = P(n-m) = 0$, and so $n-m = 0 \vee n-m = 1$ using Lemma 10.7ii. By Lemma 10.9iv, $(n-m)+m = n$. Hence $m = n \vee 1+m = n$. $m = n$ contradicts $S(m)-n = 0$, and so $n = S(m)$. QED

LEMMA 10.11. Every open formula is provably equivalent to an equation $t = 0$ over ETF\PERM. ETF\PERM proves induction for open formulas. In ETF\PERM, we can define functions $f(n,m,r) = s$ if φ; t otherwise, with extensional uniqueness, where s,t are terms and φ is an open formula.

Proof: By Lemma 10.10, $n = m \leftrightarrow n-m = m-n = 0$ is provable in ETF\PERM. Therefore every open formula is provably equivalent to a propositional combination of equations $t = 0$. Now apply Lemma 10.6, first claim. This establishes the first and second claims here.

For $f(n,m,r) = s$ if φ; t otherwise, rewrite as $f(n,m,r) = g(n,m,r)$ if $J(n,m,r) = 0$; $h(n,m,r)$ otherwise. Define the auxiliary function $H(n,m,r) = n$ if $m = 0$; r otherwise. Note that $H(n,m,r) = n \bullet F(m) + r \bullet G(m)$ by Lemma 10.4i,ii, is thus given by a term. Then $f(n,m,r) = H(g(n,m,r),J(n,m,r),h(n,m,r))$ from Composition Axioms. Extensional uniqueness follows from the second claim. QED

We refer to induction for open formulas as open induction. We refer to this way of defining functions (1,2,3-ary) as by conditional terms.

LEMMA 10.12. (ETF\PERM) n-m = 0 ∨ m-n = 0. n-m = 0 ↔ (∃r)(n+r = m).

Proof: For the first claim, suppose n-m,m-n are nonzero. By Lemma 10.9iv, (n-m)+m = n and (m-n)+n = m. Hence (n-m)+(m-n)+m+n = m+n, and so (n-m)+(m-n) = 0, by Lemmas 10.4xi,xii and Lemma 10.7v. By 10.4xiii, n-m,m-n = 0. This is a contradiction.

For the second claim, suppose n-m = 0. If m-n = 0 then by Lemma 10.10, n = m and we are done. Assume m-n ≠ 0. By Lemma 10.9iv, (m-n)+n = m, and we can use r = m-n.

Finally suppose n+r = m. We assume n-m ≠ 0 and derive a contradiction. By Lemma 10.9iv, (n-m)+m = n. Hence (n-m)+n+r = n, and so by Lemma 10.7v, (n-m)+r = 0. By Lemma 10,4xiii, n-m = r = 0 which is a contradiction. QED

We are now prepared to define < in ETF\PERM. We want to be compatible with how we used < as an abbreviation in section 6 to make those transfers there from L[set] to L[fcn].

DEFINITION 10.3. n < m if and only if (∃f)((∀n)(f(n,0) = 0) ∧ (∀n,m)(f(n,S(m)) = S(0) ↔ f(n,m) = S(0) ∨ n = m) ∧ f(n,m) = S(0)). n ≤ m if and only if n < m ∨ n = m.

LEMMA 10.13. (ETF\PERM) The f in Definition 10.3 exists and is extensionally unique.

Proof: The f is defined by a primitive recursion based on the 3-ary function J(n,m,r) = S(0) if r = S(0) ∨ n = m; 0 otherwise. By Lemma 10.11, this J exists. The extensional uniqueness is by open induction. QED

LEMMA 10.14. (ETF\PERM)
i. ¬n < 0.
ii. n < S(m) ↔ n ≤ m.
iii. Every propositional combination of inequalities (≤,<,=) is equivalent to an equation t = 0.
iv. Induction holds for propositional combinations of inequalities (≤,<,=).
v. 0 ≤ n.

vi. $n \leq m \leftrightarrow n-m = 0 \leftrightarrow (m-n)+n = m \leftrightarrow (\exists r)(n+r = m)$.
vii. $n < m \leftrightarrow S(n) < S(m) \leftrightarrow S(n) \leq m$.
viii. $\leq$ is reflexive, connected, transitive, anti symmetric, with least element 0. mix. $\neg n < n$. $<$ is a (irreflexive) linear ordering with least element 0, where each $S(n)$ is the immediate successor of n, each $n \neq 0$ is the immediate successor of $P(n)$.
x. $n+m \leq n+r \leftrightarrow m \leq r$.
xi. $n+m < n+r \leftrightarrow m < r$.
xii. $m \leq n \rightarrow (n-m)+m = n$.

Proof: By Lemma 10.13, let f be as provided by Definition 10.3. Then i,ii are immediate.

For iii, every propositional combination of inequalities ($\leq$,$<$,$=$) is equivalent to a propositional combination of equations $t = 0$ because $n < m \leftrightarrow f(n,m) = 0$, where f is as provided by Definition 10.3.

For iv, this is immediate from iii and open induction.

For v, $0 \leq n$ is proved by induction on n. $0 \leq 0$ is immediate. Suppose $0 \leq n$. We want $0 \leq S(n)$. We want $0 < S(n) \vee 0 = S(n)$. We want $0 < S(n)$. This is equivalent to $0 \leq n$, and so $0 < S(n)$.

For vi, we show $n \leq m \rightarrow n-m = 0 \rightarrow (m-n)+n = m \rightarrow (\exists r)(n+r = m) \rightarrow n \leq m$. We prove $n \leq m \rightarrow n-m = 0$ by induction on m. Suppose $n \leq m \rightarrow n-m = 0$. Assume $n \leq S(m)$. Then $n \leq m \vee n = m$. In either case $n-m = 0$.

Suppose $n-m = 0$. By Lemma 10.10iv, $m-n = 0 \vee (m-n)+n = m$. If $m-n = 0$ then $n = m$ by Lemma 10.10, and hence $(m-n)+n = m$. If $(m-n)+n = m$ then we are done.

Suppose $(m-n)+n = m$. Set $r = m-n$. Then $n+r = m$.

We need to show $n-m = 0 \rightarrow n \leq m$. This is proved by induction on m. The basis case is from v. Now suppose $n-m = 0 \rightarrow n \leq m$. Assume $n-S(m) = 0$. Then $P(n-m) = 0$, and so $n-m = 0$ or 1.

case 1. $n-m = 0$. Then $n \leq m$. Hence $n < m \vee n = m$. Therefore $n < S(m)$. Hence $n \leq S(m)$.

case 2. $n-m = 1$. By Lemma 10.9, $(n-m)+m = n$. Hence $n = S(m)$. Therefore $n \leq S(m)$.

For vii, we show $n < m \rightarrow S(n) < S(m) \rightarrow S(n) \leq m \rightarrow n < m$. Suppose $n < m$. Then $n \leq m$, and so let $n+r = m$ by vi. Then $r \neq 0$, and so $S(n)+P(r) = m$ and so $S(n) \leq m$ by vi. Hence $S(n) < S(m)$.

Now suppose $S(n) < S(m)$. Then obviously $S(n) \leq m$.

Now suppose $S(n) \leq m$. Let $S(n)+r = m$ by vi. Hence $n+S(r) = m$, and so $n \leq m$ by vi. But $n = m$ is impossible, since it implies $S(n)+r = n$, $1+r = 0$. So $n < m$.

For viii, $n \leq n$ by vi and $n-n = 0$ (Lemma 10.9i). By Lemma 10.12, $n-m = 0 \vee m-n = 0$, and hence by vi, $n \leq m \vee m \leq n$. This establishes connectiveness. For transitivity, let $n \leq m \wedge m \leq r$. By vi, write $n+s = m \wedge m+t = r$, obtaining $n+s+t = r$, in which case $n \leq r$ by vi. For anti symmetric, let $n \leq m \wedge m \leq n$. Write $n+r = m$ and $m+s = n$ by vi. Then $n+r+s = n$, and hence $r+s = 0$, and so $r = s = 0$ and $n = m$. For least element 0, by v, $0 \leq n$.

For ix, suppose $n < n$. By vii, $S(n) \leq n$, and so by vi, let $S(n)+r = n$. Then $n+1+r = n$, $1+r = 0$, which is a contradiction. So $<$ is irreflexive, and Evidently, from vi, $<$ is transitive and has trichotomy, with least element 0. We have $n < S(n)$ by viii. Suppose $n < m < S(n)$. Then $n < m \wedge (m = n \vee m < n)$ which contradicts the linearity of $<$. Hence each $S(n)$ is the immediate successor of n in $<$. Now let $n \neq 0$. Then $n = S(Pn))$ is the immediate successor of $P(n)$ in $<$.

for x, suppose $n+m \leq n+r$ and write $n+m+t = n+r$. Then $m+t = r$, and so $m \leq r$. Suppose $m \leq r$, and write $m+t = r$. Then $n+m+t = n+r$, and so $n+m \leq n+r$.

For xi, let $n+m < n+r$. Then $n+m \leq n+r$, and so $m \leq r$. Now $m = r$ is impossible by irreflexivity of $<$. Now suppose $m < r$. Then $m \leq r$, and so $n+m \leq n+r$. If $n+m = n+r$ then $m = r$, violating irreflexity.

For xii, assume $m \leq n$. Then $n-m = 0$ by Lemma 10.14vi. Also by Lemma 10.9iv, $(n-m)+m = n$. QED

DEFINITION 10.4. A pairing system consists of functions $< >, P_1, P_2$, where $< >$ is 2-ary and P_1, P_2 are 1-ary, such that $P_1(<n,m>) = n \wedge P_2(<n,m>) = m \wedge <P_1(n),P_2(n)> = n$. 1-ary f is strictly increasing if and only if $n < m \rightarrow f(n) < f(m)$.

LEMMA 10.15. (ETF\PERM) If $< >,P_1,P_2$ is a pairing system then $< >$ is a bijection from ω^2 onto ω.

Proof: Let $< >,P_1,P_2$ be as given. Suppose $P(n,m) = P(n',m')$. Then $P_1(P(n,m)) = n = P_1(<n',m'>) = n'$ and $P_2(<n,m>) = m = P_2(<n',m'>) = m'$. Also obviously $< >$ is onto. QED

LEMMA 10.16. (ETF\PERM) Let $< >,P_1,P_2$ be such that $P_1(<n,m>) = n \wedge P_2(<n,m>) = m \wedge < >$ is onto ω. Then $< >,P_1,P_2$ is a pairing system.

Proof: Let $< >,P_1,P_2$ be as given. We have only to show that $n = <P_1(n),P_2(n)>$. Let n be given. Then $n = <m,r>$. Hence $P_1(n) = m$, $P_2(n) = r$. So $n = <P_1(n),P_2(n)>$. QED

LEMMA 10.17. (ETF\PERM) Let f be 1-ary and g be 3-ary. Assume $(\forall n,m)(g(n,m,0) = 0)$. The following functions exist.
i. the function $h(n,m)$ = max (min) of f on $\{r \leq m: g(n,m,r) = 0\}$.
ii. The function $h(n,m)$ is the greatest (least) argument where f is maximized (minimized) on $\{r \leq m: g(n,m,r) = 0\}$.
iii. The function $h(n,m) = 1$ if f attains value 1 on $[r \leq m: g(n,m,r) = 1\}$; 0 otherwise.
In ii, there are four statements.

Proof: We are not assuming that these various maxes and mins exist. That is implicit in the claims to be proven.

Let f,g be as given. For i, define $h(n,0) = f(0)$, $h(n,m+1) = \max(h(n,m),f(m+1))$ if $g(n,m+1,m+1) = 0$; $h(n,m)$ otherwise. This is a primitive recursion using a conditional term (Lemma 10.11). From the body of the primitive recursion, we see that $h(n,m) \leq h(n,m+1)$. Then prove $m \leq m' \rightarrow h(n,m) \leq h(n,m')$ by open induction on m'. We now claim that $h(n,m)$ is the max of f on $\{r \leq m: g(n,m,r) = 0\}$. Let $g(n,m,r) = 0 \wedge r \leq m$. Then $f(r) \leq h(n,r) \leq h(n,m)$.

For i, we can use the same argument with min instead of max.

For ii, first define $h'(n,m)$ to be the maximum value of f on $\{r \leq m: g(n,m,r) = 0\}$. We are looking for the $r \leq m$ such that $g(n,m,r) = 0$ and $f(r) = h'(n,m)$. But this r is the least value of $f'(r) = m\bullet((f(r)-h'(n,m))+(h'(n,m)-f(r))) + r$. So we apply i to an adjusted g. Similar arguments work for the other three variants.

For iii, define $h'(n,m)$ = minimum value of $(1-f(r))+(f(r)-1)$ for $r \leq m$ with $g(n,m,r) = 1$, by i. Define $h(n,m) = 1$ if $h'(n,m) = 0$; otherwise. QED

LEMMA 10.18. (ETF\PERM)
i. There exists extensionally unique 1-ary T such that $T(0) = 0$ and $T(n+1) = T(n)+n+1$.
ii. T is strictly increasing. $T(0) = 0$, $T(1) = 1$ and $n \geq 2 \rightarrow n < T(n)$.
iii. There exists T' such that $T'(n)$ = is the greatest m such that $T(m) \leq n$.
iv. Every n is uniquely of the form $T(m)+r$, $r \leq n$. 0 is uniquely of the form $T(m)+r$.
v. Define $<n,m> = T(n+m)+m$. $< >$ is a surjective function. There exists P_1,P_2 such that $P_1(<n,m>) = n$ and $P_2(<n,m> = m$. $< >,P_1,P_2$ is a pairing system.

Proof: For i, use Primitive Recursion and Equational Induction.

For ii, we prove $n < m \rightarrow T(n) < T(m)$ by open induction on m. Suppose $n < m \rightarrow T(n) < T(m)$. Assume $n < m+1$. If $n < m$ then $T(n) < T(m) < T(m+1)$ by i. If $n = m$ then $T(n) = T(m) < T(m+1)$ by i. For the second claim, let $n = m+2$. Then $T(m+2) = T(m+1)+m+2 > m+1$ since $T(m+1) > 0$.

For iii, by Lemma 10.17ii, the function $T^*(n)$ = the greatest $m \leq n$ such that $T(m) \leq n$ exists. Clearly T^* is the desired T' by ii.

For iv, let n be given. by iii, let m be greatest such that $T(m) \leq n$. Then $T(m+1) \leq n$ is false, and so $n < T(m+1) = T(m)+m+1$ (Lemma 10.14viii). I.e., $T(m) \leq n < T(m+1)$. Set $r = n-T(m) \leq n$ (Lemma 10.14xii). For uniqueness, let $T(m)+r = T(m')+r'$, $r \leq m \wedge r' \leq m'$. Assume $m < m'$. Then $T(m+1)+r' < T(m')+r' = T(m)+r$, and so $T(m)+m+1+r' < T(m)+r$. Hence $m+1+r' < r < m$, which is impossible. By symmetry, $m' < m$ is impossible. Therefore $m = m'$ and $r = r'$.

For v, $< >$ is a function defined by a term. Let n be given. By iv, write $T(m)+r$, m,r unique, so $n = <m,r>$. So $< >$ is a surjective r-ary function.

We now want to define a function P_2 such that $P_2(<n,m>) = m$. I.e., $P_2(T(n+m)+m) = m$. n+m has the property that $T(n+m) \leq T(n+m)+m$. This is false for n+m+1. Hence $T'(T(n+m)+m) = n+m$. So to compute $P_2(T(n+m)+m)$ we take $T'(T(n+m)+m)$ getting n+m, and taking $(T(n+m)+m)-T(n+m) = m$. Evidently, this defines P_2 by a term and therefore is legitimate.

Next we define a function P_1 such that $P_1(<n,m>) = n$. We have $P_2(<n,m>) = m$. Since $n+m = T'(T(n+m)+m)$, we subtract $P_2(<n,m>)$ to get n. This also defines P_2 by a term, securing the function P_2.

$< >,P_1,P_2$ is a pairing system by Lemma 10.16. QED

DEFINITION 10.5. $< >,P_1,P_2$ is the pairing system from Lemma 10.18, where $<n,m> = T(n+m)+m$ and P_1,P_2 are as defined by terms from the proof of 10.18v.

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DEFINITION 10.6. The Δ_0/f formulas are the least class of formulas containing $t_1 = t_2$, t1 < t2, t_1,t_2 terms, and closed under the usual connectives, and bounded quantification $(\exists n < t)$, $(\forall n < t)$, where t is a term without n, and where meant as abbreviations. The $\sum^0{}_1$/f formulas are formulas of the form $(\exists n)(\varphi)$, where φ is a bounded formula. The $\prod^0{}_1$/f formulas are formulas of the form $(\forall n)(\varphi)$, where φ is Δ_0/f.

We will treat Δ_0/formulas using functions of several variables. The only functions that we have as actual objects are only of arities 1,2,3. So we will need to develop virtual functions of any arity. This is of course crucially on our pairing system $< >,P_1,P_2$, of Lemma 10.18.

DEFINITION 10.7. $<n> = n$. For external $k \geq 2$, $<n_1,...,n_k> = <<n_1,n_2>,n_3,...,n_k>$. $P[k,1](n) = P_1(P[k-1,1](n))$, $P[k,2](n) = P_2(P[k-1,1](n)$, $P[k,i](n) = P[k-1,i-1](n)$, $3 \leq i \leq k$.

LEMMA 10.19. (ETF\PERM) $<P[k,1](n),...,P[k,k](n)> = n$, $P[k,i](<n_1,...,n_k>) = n_i$. $<n_1,...,n_k>$ is a bijection from ω^k onto ω.

Proof: Left to the reader. QED

LEMMA 10.20. (ETF\PERM) Let f be 1-ary, t be a conditional term, and $n_1,...,n_k$ be distinct variables of sort ω. There exists 1-ary function f such that $(\forall n_1,...,n_k)(f(<n_1,...,n_k>) = t)$.

Proof: We can replace any conditional term by a term with the same variables, with the help of some simple function parameters. Define $f(n) = t$ with $n_1,...,n_k$ replaced by $P[k,1](n),...,P[k,k](n)$. Then $f(<n_1,...,n_k>) = t$ with $n_1,...,n_k$

replaced by P[k,1]($<n_1,\ldots,n_k>$),...,P[k,k]($<n_1,\ldots,n_k>$), which are $n_1,\ldots,n_k$. Hence f($<n_1,\ldots,n_k>$) = t. QED

LEMMA 10.21. (ETF\PERM) Let f be 3-ary. The function g(n,m) = 1 if there exists r < m such that f(n,m,r) = 1; 0 otherwise, exists.

Proof: It clearly suffices to prove the following modification. The function g(n,m) = 0 if there exists r < m such that f(n,m,r) = 0; 1 otherwise, exists.

Let f be 3-ary. Let f*(<n,m,r>) = f(n,m,r). we want to minimize f* over an interval? g(t) is the minimum of f* at the <n,m,r> such that n = P_1(t) and

LEMMA 10.22. (ETF\PERM) Let φ be Δ_0/f. Suppose all variables of sort ω in φ are among $n_1,\ldots,n_k$. Then $(\exists f)(\forall n_1,\ldots,n_k)((\varphi \rightarrow f(<n_1,\ldots,n_k>) = 1) \wedge (\neg\varphi \rightarrow f(<n_1,\ldots,n_k>) = 0))$ is provable in ETF\PERM. We can define 1-ary f and 3-ary g by f(<n1,...,nk>) = s if φ; t otherwise, g(n,m,r) = s if φ; t otherwise, where φ is Δ_0/f.

Proof: We prove the first claim by external induction on φ.

s = t. Let $n_1,\ldots,n_k$ be distinct variables of sort ω that include the variables in s = t. We want $(\exists f)(\forall n_1,\ldots,n_k)((s = t \rightarrow f(<n_1,\ldots,n_k>) = 1) \wedge (s \neq t \rightarrow f(<n_1,\ldots,n_k>) = 0))$. By, Lemma 10.21, define f($<n_1,\ldots,n_k>$) = 1 if s = t; 0 otherwise.

s < t. The same as the case s = t with = replaced by <.

Connective combinations. Compose f with functions from {0,1} and $\{0,1\}^2$ into {0,1} corresponding to the connectives.

(∃n < t)(φ), n not in t. Let the variables of sort ω, in φ, be among the distinct variables $n,n_1,\ldots,n_k$. Assume $(\forall n,n_1,\ldots,n_k)((\varphi \rightarrow f(<n,n_1,\ldots,n_k>) = 1) \wedge (\neg\varphi \rightarrow f(n,n_1,\ldots,n_k>) = 0))$. We want 1-ary g such that $g(<n_1,\ldots,n_k>) = 1$ if $(\exists n < t)(f(<n,n_1,\ldots,n_k>) = 1)$; 0 otherwise. This can be obtained straightforwardly from Lemma 10.17iii. (In general, the variable n is not first in the list $n,n_1,\ldots,n_k$. However, this doesn't affect the argument).

The second claim follows from the first claim using conditional definitions as in Lemma 10.11. QED

LEMMA 10.23. (ETF\PERM) Induction holds for all Δ_0/f formulas on all variables of sort ω.

Proof: This follows easily from Lemma 10.22. QED

We now finally about to use PERM = Permutation Axiom. We begin with an observation about PERM.

LEMMA 10.24. (ETF\PERM) Let f be a permutation of ω. $(\forall n)(f(g(n)) = n) \leftrightarrow (\forall n)(g(f(n)) = n)$. In fact, this g is extensionally unique. Furthermore, g is a permutation of ω. The following are equivalent.
i. If f is a permutation of ω then there exists g such that $(\forall n)(f(g(n)) = g(f(n)) = n)$.
ii. If f is a permutation of ω then there exists g such that $(\forall n)(g(f(n)) = n)$.
iii. PERM.
In fact, this entire Lemma can be proved in pure logic.

Proof: Let f be a permutation of ω. Suppose $(\forall n)(f(g(n)) = n)$. Then $f(g(f(n))) = f(n)$. Since f is one-one, $f(n) = g(f(n))$.

Conversely, Suppose $(\forall n)(g(f(n)) = n)$. We claim that g is one-one. Suppose $g(m) = g(r)$. Since f is onto, write $g(f(m')) = g(f(r'))$, where $f(m') = m$ and $f(r') = r$. Then $m' = r'$, and so $m = r$. This establishes that g is one-one. Now $g(f(g(n))) = g(n)$. Since g is one-one, we have $f(g(n)) = g(n)$.

We claim g is onto. This is clear from $g(f(n)) = n$. Hence g is a permutation of ω. g is extensionally unique because $g(f(n)) = n$ determines g at every argument because f is onto.

The equivalence of i,ii,iii is immediate from what he have just proved. QED

LEMMA 10.25. (ETF\PERM) Let the characteristic function of $A \subseteq 2\omega+1$ exist. Let $h:\omega \to \omega$ map A one-one onto $2\omega+1$.
i. There is a strictly increasing surjective $f:\omega \to \omega\backslash A$ such that $(\forall n)(f(n) \geq n)$.
ii. There exists f' such that for all $n \notin A$, $f(f'(n)) = n$. f' maps $\omega\backslash A$ one-one onto ω.
iii. There exists a permutation g of ω which agrees with h on A.

Proof: Define g(n) = if n odd then n+1 else if n+1 $\notin$ A then n+1 else n+2. Then g(n) is the least number outside of A and greater than n.

Now define f(0) = 0, f(n+1) = g(f(n). Clearly $(\forall n)(f(n) \notin A)$. We claim that $n < m \rightarrow f(n) < f(m)$. This is easily proved by open induction.

Since f is strictly increasing, it is easy to prove $f(n) \geq n$ by open induction.

Let $n \notin A$. By Lemma 10.17i, there is a greatest value t of f on [0,n] that is $\leq n$. Evidently, t is also the greatest value of f on ω that is $\leq n$. Suppose $t = f(s) < n$. Then $g(t) = f(s+1) \leq n$. This is a contradiction. Hence n is a value of f. We have shown that f is surjection, and hence f is a strictly increasing surjective map from ω onto $\omega\backslash A$.

By Lemma 10.17i, for all n, the maximum of f on $\{r \leq n: f(r) \leq n\}$ exists. Suppose $n \notin A$, and this maximum is realized at r. If $f(r) < n$. Then $r < n$ and $f(r+1) = g(f(r)) \leq n$. This is a contradiction. Hence for $n \notin A$, this maximum is n. Since f is strictly increasing, the place at which this maximum is realized is unique. By Lemma 10.17ii, define f'(n) to be the maximum place at which f is maximized on $\{r \leq n: f(r) \leq n\}$. Clearly $n \notin A \rightarrow f(f'(n)) = n$, and ii is established.

Finally define g(n) = if $n \notin A$ then 2f'(n) else h(n). Then g maps $\omega\backslash A$ one-one onto 2ω, and A one-one onto $2\omega+1$. Hence g is a permutation. QED

LEMMA 10.26. (ETF)
i. Suppose $(\forall n)(\exists! m)(f(n,m) = 0)$. There exists g such that $(\forall n)(f(n,g(n)) = 0)$.
ii. Suppose $(\forall n)(\exists m)(f(n,m) = 0)$. There exists g such that $(\forall n)(g(n) = (\mu m)(f(n,m) = 0)$.
iii. Suppose $(\forall n,m)(\exists r)(f(n,m,r) = 0)$. There exists g such that $(\forall n,m)(g(n,m) = (\mu r)(f(n,m,r) = 0)$.
iv. Let φ be Δ_0/f. $(\forall n,m,r)(\exists s)(\varphi) \rightarrow (\exists g)(\forall n,m,r)(g(n,m,r) = (\mu s)(\varphi))$

Proof: For i, let $(\forall n)(\exists! m)(f(n,m) = 0)$. Let $A = \{2\langle n,m\rangle+1: f(n,m) = 0\}$. Then A has a characteristic function using the pairing system < >,P_1,P_2. Define h(2<n,m>+1) = 2n+1; 0 elsewhere. Then h maps A one-one onto $2\omega+1$. By Lemma 10.25iii, let g be a permutation of ω which agrees with h on A. Applying PERM, g^{-1} exists and for all n, $g^{-1}(2n+1)$ = 2<n,m>+1 for the unique m with f(n,m) = 0. Define g'(n) = $P_2(F^{-1}(2n)-1/2)$. Then $(\forall n)(f(n,g'(n))$ = 0.

For ii, let $(\forall n)(\exists m)(f(n,m) = 0)$. Then $(\forall n)(\exists! m)(f(n,m) = 0 \wedge (\forall r < m)(f(n,m) \neq 0))$. Let h(n,m) = 0 if f(n,m) = 0 $\wedge$ $(\forall r < m)(f(n,m) \neq 0)$; 1 otherwise. Here h exists by Lemma 10.11. Also $(\forall n)(\exists! m)(h(n,m) = 0)$ by Lemma 10.17i. Now apply the first claim.

For iii, let $(\forall n,m)(\exists r)(f(n,m,r) = 0)$. Convert to ii using our pairing system < >,P_1,P_2.

For iv, pull it back to iii using our pairing system and Lemma 10.22. QED

DEFINITION 10.8. $\Delta^0{}_1$-CA/f is the following scheme. Let φ be $\Sigma^0{}_1$/f, ψ be $\Pi^0{}_1$/f, variable f not in φ,ψ. $(\forall n,m)(\varphi \leftrightarrow \psi) \wedge (\forall n)(\exists! m)(\varphi) \rightarrow (\exists f)(\forall n)(\varphi[n/f(n)])$.

$\Delta^0{}_1$-CA/f was important for section 7. There is a stronger form of it.

LEMMA 10.27. Let φ be $\Sigma^0{}_1$/f and not mention f. ETF proves $(\forall n)(\exists m)(\varphi) \rightarrow (\exists f)(\forall n)(\varphi(m/f(n)])$. ETF proves $\Delta^0{}_1$-CA/f.

Proof: Let $\varphi = (\exists r)(\psi)$, ψ Δ_0/f. Assume $(\forall n)(\exists m)(\varphi)$ = $(\forall n)(\exists m)(\exists r)(\psi)$. Then $(\forall n)(\exists u)(\psi[m/P_1(u),r/P_2(u)])$. Let g be a witness function by Lemma 10.26iv. Then f(n) = $P_1(g(n))$ is as required. QED

DEFINITION 10.9. $\Sigma^0{}_1$-IND/f is induction for $\Sigma^0{}_1$ formulas in L[fcn] on any variable of sort ω.

LEMMA 10.28. ETF proves $\Sigma^0{}_1$-IND/f.

Proof: We use Δ^0_1-CA/f from Lemma 10.27. Assume $(\exists n)(\varphi)[m/0] \wedge (\forall m)((\exists n)(\varphi) \rightarrow (\exists n)(\varphi)[m/m+1])$, where φ is Δ_0/f. Define $f(n,m) = 0 \leftrightarrow \varphi$ by Lemma 10.20. We have

1) $(\exists n)(f(n,0) = 0) \wedge (\forall n,m)(f(n,m> = 0 \rightarrow (\exists n)(f(n,m+1) = 0)$.

2) $(\exists n)(f(n,0) = 0) \wedge (\forall n,m)(\exists r)(f(n,m) = 0 \rightarrow f(r,m+1) = 0)$.

Fix c with f(c,0) = 0. By Lemma 10.26iv, let g go from <n,m> to <n+1,r> so that if f(n,m) = 0 then f(n+1,r) = 0. Then we can define 1-ary h by primitive recursion, where h(0) = <c,0>, and the induction step is by g. Then an obvious equational induction shows that each h(m) is of the form <n,m> with f(n,m) = 0. QED

REMARK. There is a much easier, comparatively straightforward, result that ETF + Δ^0_1-CA/f proves Σ^0_1-IND/f. Look at the proof of Lemma 10.28. There is a main primitive recursion, obviously in ETF, but also characteristic functions for Δ_0/f formulas and using the μ operator from Lemma 10.26iv. These are trivially obtained in ETF + Δ^0_1-CA/f. ETF + Δ^0_1-CA/f is the RCA_0/f of [Fr76].

We have been relying on a standard development of arithmetic, but we now want to robustly state the relationship between ETF and RCA_0, we need to proceed more abstractly.

DEFINITION 10.10. An arithmetic is a system (0,1,+',•',<'), where
i. 0 is the primitive of ETF.
ii. n+'1 = S(n).
iii. +',•' are 2-ary functions from ω into ω.
iv. <' is a 2-ary characteristic function of a linear ordering with least element 0.
v. It forms a linearly ordered commutative semiring with zero 0 and unit 1.

LEMMA 10.29. (ETF) There is an extensionally unique arithmetic.

Proof: From the particular development of arithmetic here, and Lemmas 10.3, 10.4, 10.7, 10.9, 10.12, 10.14, we have an arithmetic (0,1,+,•,<). Now let (0,1,+',•',<') be an arithmetic. We prove n+'m = n+m by induction on m. Assume n+'m = n+m. Then n+'(S(m)) = n+'(m+'1) = (n+'m)+'1 = (n+m)+'1 = S(n+m) = n+S(m).

We prove n•'m = n•m by induction. n•'(S(m)) = n•'(m+'1) = n•'m +' n = n•m + n = n•(S(m)).

We prove n <' m ↔ n < m by open induction on m. Assume n <' m ↔ n < m. Then n <' S(m) ↔ n <' m+'1 ↔ n <' m ∨ n = m ↔ n < m ∨ n = m ↔ n < S(m). QED

LEMMA 10.30. ETF proves
i. There is a unique arithmetic 0,1,+,•,<.
ii. Δ^0_1-CA/f.
iii. Σ^0_1-IND/f.
Here ii,iii are formulated with <.

Proof: By Lemmas 10.27, 10.28, 10.29. QED

Recall the following definition from [Fr76] which was used in sections 6,7.

DEFINITION 10.11. RCA_0 = ETF + Δ^0_1-CA/f. This is the RCA_0 of [Fr76],

THEOREM 10.31. ETF and RCA_0 of [Fr76] are logically equivalent. They logically imply Σ^0_1-IND/f.

Proof: By Lemma 10.30. QED

Theorem 10.31 is as claimed in [Fr76], but not needed for the development of RM.

Recall that the derivation of Σ^0_1-IND in L[fcn] from RCA_0 of [Fr76] is relatively simple and well under control in 1976, as discussed at the end of section 7.

11. STRICTLY MATHEMATICAL BASE THEORIES

In this section we present a way of construing present RM as a case of partly SRM = strict reverse mathematics, in the sense of working over a strictly mathematical base theory. There are other criteria that need to be met for us to say that we are doing full blown SRM, and these additional criteria are discussed in section 12.

For this discussion it is particularly useful to define an extension of the usual RCA_0/s we use today as the base theory for RM. The theory that we present is also constructed in order to naturally flow into further developments in section 12.

DEFINITION 11.1. The language L[FSRA] = functions, sets, relations, arithmetic, has the eight sorts ω, FCN[1], FCN[2], FCN[3], SET[ω], REL[1], REL[2], REL[3], constant 0 of sort ω, 1-ary function symbol S from sort ω into sort ω, 1-ary function symbols +,• from sort ω into sort ω, 2-ary relation symbol < on sort ω, and equality = on sort ω. Syntax, semantics, and a complete Hilbert style system are defined in the standard way. REL[i], $1 \leq i \leq 3$, has intended interpretation the set of all i-ary relations on ω.

In an ultimate form of what we are doing, we also use free logic (logic with undefined terms), but we will not do this here. There are other things missing here, that we might want in an ultimate form, such as special sorts and operations for the ordered ring of integers, and the ordered field of rationals. And then at a more advanced stage, quite a number of additional features. Exactly where the natural stopping places are for the various kinds of Reverse Mathematics Investigations, are yet to be investigated.

DEFINITION 11.2. RCA_0/FSRA, WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, Π^1_1-CA_0/FSRA are the obvious adaptations to L[FSRA], with Δ^0_1-CA/FSRA asserting not only the existence of sets, but the existence of functions and relations by appropriate pairs of formulas in FSRA.

Of course, despite this extended language L[FRSA], the above five systems remain quite far from being strictly mathematical. However, we will now show that they have strictly mathematical axiomatizations.

DEFINITION 11.3. ETF/FSRA has language L[FSRA] with the following nonlogical axioms.
1. ETF.
2. $(\exists f)(\forall n)(n \in A \leftrightarrow f(n) = 0)$.
3. $(\exists A)(\forall n)(n \in A \leftrightarrow f(n) = 0)$.
4. $(\exists R)(\forall n)(R(n) \leftrightarrow f(n) = 0)$.
5. $(\exists f)(\forall n)(R(n) \leftrightarrow f(n) = 0)$.
6. $(\exists R)(\forall n,m)(R(n,m) \leftrightarrow f(n,m) = 0)$.
7. $(\exists f)(\forall n,m)(R(n,m) \leftrightarrow f(n,m) = 0)$.
8. $(\exists R)(\forall n,m,r)(R(n,m,r) \leftrightarrow f(n,m,r) = 0$
9. $(\exists f)(\forall n,m,r)(R(n,m,r) \leftrightarrow f(n,m,r) = 0)$.
10. Usual quantifier free axioms for 0,S,+,•,<.
11. +,• define 2-ary functions and < defines a relation.

Obviously ETF/FSRA is a literal definitional extension of ETF.

THEOREM 11.1. ETF/FSRA and RCA_0/FSRA are logically equivalent. They are definitional extensions of ETF, RCA_0/f, RCA_0/s. Also the four theories WKL_0/f, ACA_0/f, ATR_0/f, Π^1_1-CA_0/f, and the four theories WKL_0/s, ACA0/s, ATR0/s, Π^1_1-CA_0/s, are respectively synonymous. In fact, we can use a common synonymy for all of these five pairs of systems (in the sense in section 8 of a common literal definitional system). RCA_0/FSRA, WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, Π^1_1-CA_0/FSRA are, respectively, common definitional extensions of all of these five pairs of systems.

Proof: By Theorem 10.31, ETF proves Δ^0_1-CA/f + Σ^0_1-IND/f. By 1-6 above, we easily obtain Δ^0_1-CA/FSRA + Σ^0_1-IND/FSRA, yielding RCA_0/FSRA. Conversely, 1-6 immediately follow from Δ01-CA/FSRA, according to the text of Definition 11.2. The synonymies are based on the K_1,K_2 from section 9, extended to incorporate REL. QED

We can construe current RM as being conducted with ETF/FSRA as the strictly mathematical base theory. At the beginning, Δ01-CA/FSRA and Σ^0_1-IND/FSRA are proved following section 10 here (or whatever simplifications or improved expositions arise). At this point we proceed with the whole development of RM unchanged, with the advantage that some (but by no means all) of the coding is no longer needed.

12. ADVENTURES IN STRICT REVERSE MATHEMATICS

The phrase "strictly mathematical" is not meant to indicate a strict dividing line between what is or not strictly mathematical. Here it is meant to apply predominantly to the choice of logical theories, particularly base theories. Generally speaking, the strictly mathematical ranges greatly from the obviously not, to a place where there is a big jump into the at least arguably strictly mathematical, and on to greater mathematical elegance, familiarity, importance, and usefulness. These further steps after the big jump should be recognized as significant improvements. Thus sometimes when we achieve the strictly mathematical, we can go on to the yet more strictly mathematical.

In RM as practiced, the theorems being targeted for reversal generally meet the criteria of being strictly mathematical. Thus the choice of reversal candidates A is not at issue. However, the base theories and also the targets of the reversals generally are not strictly mathematical. Also the reversal candidates A are generally modified via coding and thereby normally lose their status as strictly mathematical on the way to being reversed. In SRM, all formal systems are strictly mathematical, and the reversal candidates are not modified.

We now more carefully distinguish four types of Reverse Mathematics Investigations.

I. RMI. Reverse Mathematics Investigations. This is how current RM proceeds. We identity two formal systems S,T, where T is an extension of S in the same language. The investigation consists of looking at strictly mathematical theorems A. Typically, A is not in the language of S,T. Then we modify A, preserving the "mathematical meaning", through coding, so that A' is in the language of S,T, and seek

S + A' logically equivalent to T

In RMI, there is no requirement that S or T be strictly mathematical and there is no requirement that A' = A. Intuition and experience from Recursion Theory are used, as well as certain checks and balances, to "ensure" the "legitimacy" of the coding used to "preserve the mathematical meaning". When issues arise, one generally cautions with "A' is A as coded". In current RMI, generally, but by no means always, the S,T are drawn from RCA0/s, WKL0/s, ACA0/s, ATR0/s, ∏11-CA0/s, with RCA0/s overwhelmingly most commonly chosen for S.

II. RMI/base. Reverse Mathematics Investigations with Strictly Mathematical Base Theory. These are the same as RMI except that the base S is required to be strictly mathematical. In section

11, we have proposed ETF/FSRA as a suitable strictly mathematical base theory, where we can construe present RMI as RMI/base.

III. RMI/sys. Reverse Mathematics Investigations with Strictly Mathematical Systems. These are the same as RMI except that both theories S,T are required to be strictly mathematical. We will discuss how we can construe present RMI as RMI/sys. This amounts to finding strictly mathematical equivalents of RCA_0/FSRA, WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, $\prod^1_1$-CA_0/FSRA. We have already done so for RCA_0/FSRA with our strictly mathematical ETF/FSRA in section 11.

IV. RMI/nocode. Reverse Mathematics Investigations with No Coding of Statements. Here the requirement is that the statements A to be reversed must be strictly mathematical and reversed with no coding involved. There are two strategies involved to accomplish this. One is to cleverly reformulate or adjust A so that it is now in the language of S,T and still strictly mathematical. The other is to use systems S,T in richer languages, so that A, without adjustment, is in the richer language. Of course, both strategies can be used together.

V. SRMI. Strict Reverse Mathematics Investigations. Here we require S,T,A to all be strictly mathematical, with A not altered by any coding when we seek a proof that S + A is logically equivalent to T. Thus SRMI combines all of the restrictions present in RMI/sys and RMI/nocode.

Current RMI does not meet the criteria for RMI/base, RMI/sys, RMI/nocode, SRMI. We have already seen this and discussed, in section 11, how we can recast current RMI as RMI/base through ETF/FSRA. Recasting current RMI as RMI/sys requires that we find strictly mathematical versions of WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, $\prod^1_1$-CA_0/FSRA. One satisfactory way of doing this in discussed below. But, like all of the projects in this section, there are many ways of accomplishing this, some of which will represent advances in SRM.

Recasting current RMI as RMI/nocode is more involved and requires a different kind of effort than recasting current RMI as RMI/sys. Here we need to significantly expand the language of the base theory to accommodate a wide range of mathematical statements unaltered by coding. But we want to keep the base theory manageable logically and also proof theoretically weak like RCA_0/s.

We discuss how to recast a significant but limited portion of the current RM development as RMI/nocode, with ongoing research. This does involve recasting many reversed A as A', where A' preserves the mathematical meaning, and is coding free. There is a fundamental realm of mathematics where the RMI/nocode is particularly successful and comparatively straightforward. This is the realm of Countable Mathematics. We discuss a particularly comprehensive way of achieving RMI/nocode in Countable Mathematics, with a plan to give strictly mathematical reaxiomatizations of the five basic systems involved. This promises to achieve SRMI for Countable Mathematics.

When we step outside of Countable Mathematics, achieving RMI/nocode is particularly challenging with many unexplored avenues. Here we discuss an experience of Simpson which indicates some of the challenges that need to be met here. We then discuss some promising approaches.

Differing approaches to RMI, RMI/base, RMI/sys, RMI/nocode, SRMI should not be viewed as in conflict with one another. The basic vision of Strict Reverse Mathematics is to take the mathematics being reversed organically from the mathematics, and not shoehorned into a preordained formal frameworks. Thus in the minds of different mathematical minds, the same mathematical terrain might unravel differently into diverse basic components. One mathematician may view their real analysis as based on real numbers as Dedekind cuts with standard equality, whereas another mathematician may be thinking in terms of isomorphic real number systems, whereas another thinking of Cauchy sequences of rationals with prescribed estimates, and yet another on Cauchy sequences of rationals with no prescribed estimates. The various direct strictly mathematical formulations of fundamental real analysis will have differing reversals and logical status, including different proof theoretic strengths. At this elementary level, most of the distinct outcomes are washed away at the ACA_0 level. But it remains to explore systematically what happens here at the RCA_0 and WKL_0 levels.

We begin by discussing RMI/sys. A good uniform way of achieving RMI/sys is to use ETF/FSRA as the base theory, and add a single strictly mathematical statement P_1,P_2,P_3,P_4 to ETF/FSRA to get logical equivalence with WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, Π^1_1-CA_0/FSRA.

We define A of sort SET[ω] to be infinite if and only if $(\forall n)(\exists m)(n < m \wedge m \in A)$.

$A \subseteq \omega$ is halving closed if and only if $(\forall n)(2n \in A \vee 2n+1 \in A \rightarrow n \in A)$. I.e., the floor of half of every element of A is an element of A.

P_1. Every subset of ω has an inclusion maximum finite halving closed subset or an inclusion minimal infinite halving closed subset.

P_2. The nonnegative difference set of every subset of ω exists.

P_3. For any two well orderings of ω, one is isomorphic to an initial segment of the other.

P_4. In any linear ordering of ω there is a longest well ordered initial segment.

THEOREM 12.1. The following are logically equivalent.
WKL_0/s and RCA_0/s + P_1
ACA_0/s and RCA_0/s + P_2
ATR_0/FSRA and EFA/FSRA + P_3
Π^1_1-CA_0/FSRA and EFA/FSRA + P_4

Proof: By standard techniques from RM. QED

Note that P_1 is strictly mathematical in the language L[set]. In fact, in only $(\omega,SET[\omega],\in,<,0,S,+)$. We came up with P_1 in order to address the challenge:

FIND A STRICTLY MATHEMATICAL SENTENCE IN L[SET] WHICH REVERSES TO WKL_0/s OVER RCA_0/s.

Obviously from the well known enormous power of L[set] we know we can express just about anything, but the challenge here is to do so strictly mathematically. There can be other, yet more interesting ways of meeting this challenge. Here is a related challenge we have not explored:

FIND A STRICTLY MATHEMATICAL SENTENCE IN L[FCN] WHICH REVERSES TO WKL_0/f OVER RCA_0/f.

We can't be using Weak Konig's Lemma itself for such problems as that involves finite sequence coding. Without some special ideas, this would look hopelessly contrived in comparison to P_1.

With the more flexible challenge,

FIND A STRICTLY MATHEMATICAL SENTENCE IN L[FCN,SET] WHICH REVERSES TO WKL_0/fs OVER RCA_0/fs.

we can arguably introduce an improved level of strict mathematical naturalness. In fact we can obtain an interesting example in $(\omega,SET[\omega],FCN[1],<)$.

P_1'. Let f:ω → ω be finite to one obeying f(n) ≤ n. Every subset of ω has an inclusion maximum finite f closed subset or an inclusion minimal infinite f closed subset.

FIND A STRICTLY MATHEMATICAL SENTENCE IN (ω,FCN[1],SET[ω]) WHICH REVERSES TO WKL_0/sf OVER RCA_0/fs.

could be rather challenging.

Our example P_2 raises the question of what the status is of P_2 if we replace binary - by other numerical functions.

For P_3,P_4, note that orderings of ω are directly treated using REL[2], and isomorphisms are directly treated using REL[2], FCN[2].

QUESTION. Can we find such strictly mathematical P_3,P_4 (for theorem 12.1) that are in L[set]? In L[fcn]? Linear orders can be defined using a binary characteristic function, which is arguably flawed. Part of the challenge is to avoid this.

Having discussed RMI, RMI/base, RMI/sys, we now turn to RMI/nocode. Very little of current RMI is RMI/nocode, let alone SRMI.

In fact, this adherence to various choices of coding in current RM is really where current RM becomes what I call "arguably problematic". Calling it "problematic" is too strong because I don't believe that current RM should be overhauled or is significantly flawed. It's just that there is a lot missing that will be supplied as RMI/nocode and SRMI get developed.

The first place where current RM is arguably problematic occurs with the introduction of reals and infinite sequences of reals. In Chapter 2 of [Si99,09], Simpson defines "real numbers" as

"a sequence of rational numbers [satisfying successive term Cauchy convergence with 2^{-n}].
Two real numbers are equal if [they satisfy the associated termwise estimate]"

Right here the choice of using sequences of rationals rather than Dedekind cuts, an ad hoc but commonly used choice of convergence rates, and the decision to use an equivalence relation instead of identity, with a particular choice of estimate in the relation.

I'm not suggesting this is a wrong or bad way to do real numbers. I'm suggesting that the SRM approach can hope to do much better. To take actual fundamental mathematics that uses real numbers as the sole driver and see what we get metamathematically.

The situation gets even more critical and more revealing when Simpson later defines "sequences of real numbers" as

"a function $f:N \times N \to Q$ such that for each $n \in N$
the function $(f)_n:N \to Q$ defined by $(f)_n(k) = f((k,n))$ is a real number"

Having the real numbers interact appropriately with sequences of real numbers is what causes rejection of alternative definitions. Simpson became well aware of issues surrounding the various choices made about reals and sequences of reals. We see these quotes from Chapter 2:

"Note that we are taking equality between real numbers ... to be an equivalence relation
rather than true identity. This choice is dictated by our goal of developing mathematics
within subsystems of second order arithmetic such as RCA_0 and ACA_0.

One might consider alternative definitions under which a real number would be an equivalence class as a representative of an equivalence class. Both alternatives turn out to be inappropriate. Equivalence classes would require the language of third order arithmetic, and the use of representatives would demand a strong form of the axiom of choice which is not available even in full second order arithmetic Z_2."

(As a minor remark, the use of representatives would not demand any axiom of choice, if one is using a definite rate of convergence, as one can use the lexicographically least representative. Even some other treatments don't require use of the axiom of choice to pick out representatives. I have never seen this situation fleshed out anywhere and it would be interesting to sort it out.)

"Our treatment of the real number system in the context of RCA_0 is analogous to that of Aberth [2] in the somewhat different context of recursive analysis. ... In an earlier paper Simpson [234] developing mathematics within RCA0, we defined a real number to be the set of smaller rational numbers. This alternative definition, although in a sense equivalent to definition II.4.4 above, turns out to be inappropriate for other reasons, as explained in Brown/Simpson [26, section 3]"

In [BS86], there is a section on "the real numbers in RCA_0". They give a definition of real number that "is different from, but equivalent to, the definition used in [Si84]."

Then [BS86] comes to sequences of real numbers. They give the definition we have quoted above from [Si99,09], Chapter 2, and write

"The above definition of sequences of real numbers is *not* equivalent to the corresponding definition in [Si84]. Under the old definition, it would be consistent with RCA_0 that there exists a sequence of real numbers $(x_n: n \in N)$ such that $(x_n + \pi: n \in N)$ is not a sequence of real numbers. We thank Ian Richards for pointing out this defect of the old definition. Our new definition, given above, is adopted in order to remove this defect. All of the arguments and results of [Si84] remain correct under the new definition."

The "old definition" from [Si84] is

"Within RCA_0 we define a (code for a) sequence of real numbers to be a set $S \subseteq Q \times N$ such that, for each $n \in N$, $(X)_n = \{q:(q,n) \in X\}$ is a (code for a) real number."

So the "old definition" is based on straightforward adaptation of Dedekind cut real numbers to sequences, and is discarded because RCA_0 doesn't prove basic facts, whereas the "new definition" is based on straightforward adaption of Cauchy sequence real numbers to sequences and did not appear to have difficulties in the RCA_0 context.

Although there is nothing flawed in the way Simpson winds up on this in [Fr99,09], it is clear that this situation practically begs for the systematic principled approach of Strict Reverse Mathematics. Various fundamental statements and theorems about reals and infinite sequences of reals and associated notions involving integers and rationals are logically analyzed in their own right, taking various approaches to reals and infinite sequences of reals. One can take this approach with the integers and the rationals as well, but significant issues do not seem to arise.

The SRM approach does is consider a range of treatments of infinite sequences of reals with no preference given to any coding based treatment such as above, but particular attention paid to treatments where infinite sequences of real numbers are taken as primitive. But as emphasized earlier, this matter really becomes critical when we get to reals/sequences of reals.

In fact, we surmise that there is a very critical dividing line here for RMI/nocode. In what we call Countable Mathematics, this kind of issue - with reals/sequences of reals - does not arise. Insisting on RMI/nocode is still interesting and important, and does lead to other kinds of interesting issues. So we will discuss two satisfactory approaches to RMI/nocode within the realm of Countable Mathematics, both of which have their clear merits.

In the first approach to RMI/nocode for Countable Mathematics, we simply use ETF/FSRA, WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, Π^1_1-CA_0/FSRA from section 11. The most fundamental notion from Countable Mathematics that is arguably only awkwardly handled here is finite sequences from ω. Yes, they are naturally truncations of $f:\omega \to \omega$, so can arguably bring them under the tent that way. But one also need sets of finite sequences from ω, and that is really awkward without using 2-ary functions. This is all arguably more palatable than resorting to the arithmetic based approach in [Si99,09], Chapter 2. When a mathematician makes statements involving finite sequences from ω they are generally not thinking of elementary number theory involvement, but rather a much purer clearly visually picture of an actual finite sequence.

So under this more conceptual approach to finite sequences from ω, they would be $f:\omega \to \omega$ together with n (the length for truncation). Various devices need to be brought in for sets of such, finite sets of such, and so forth, none involving any arithmetic except $<$.

With the FSRA approach, though only partially satisfactory from the SRM point of view, we do have strictly mathematical axiomatizations, through ETF/FSRA, ETF/FSRA + P_1, ETF/FSRA + P_2, ETF/FSRA + P_3, ETF/FSRA + P_4. However, when it comes to doing ultimate SRMI for Countable mathematics, we really need a richer language than FSRA. But then of course the Strictly Mathematical Axiomatizations need to be extended.

We now discuss the more powerful second approach to RMI/nocode for Countable Mathematics.

The starting point is the realization that these issues with regard to the formalization of concepts like finite sequences from ω, and finite/finite sets of such, and more, already arise with Finite Mathematics. We view Countable Mathematics as the combination of these parts:

1. Finite mathematics, where all objects are finitary, and there are denumerably many finite objects, comprising a universe FIN of finite mathematical objects.
2. Partial functions on FIN into FIN.
3. Subsets of FIN.
4. Relations on FIN.

In consonance with mathematical practice, the functions have arities 1,2,3, and the relations have arities 1,2,3. This is admittedly a somewhat arbitrary cutoff in arity. Higher arities are handled with <...> on FIN, which is incorporated into the finite mathematics.

Thus in this conception of Countable Mathematics, our infinite objects are immediately grounded entirely in the finite. There are definitely more general conceptions of Countable Mathematics where countable objects are not so immediately grounded, and we basically have what set theorists refer to as hereditarily countable sets. In one approach to reals/infinite sequences of reals, reals are immediately grounded in the finite (Dedekind cuts), whereas infinite sequences of reals are only immediately grounded in the countable that is immediately grounded in the finite.

In the RMI/nocode adventures, the integer and rational number systems are part of finite mathematics, and we will, in this limited preliminary report, define integers and rationals as certain ordered pairs from ω. In fact, here, we will take a fairly uncritical stance to FIN relying on the substantial power we pack into FIN. This is reflected in a quite principled way of formalizing Finite Mathematics as about objects in FIN - a universe of finite mathematical objects. FIN has the following critical closure properties. If A is a set in FIN then the set of subsets of A, the set of partial functions of arities 1,2,3 from A into A, and the set of relations on A of arities 1,2,3, form sets. We use = on FIN and extensionality in FIN. We use operation symbols for the Boolean ring operations, and composition, and these power operations, and more. The key axiom will be Δ_0/FIN induction where Δ_0 allows all of the primitive operations, with $<,\in,\subseteq$ doing the bounding. This is a conservative extension of the arithmetic system EFA. This system needs to be given a strictly mathematical axiomatization. Probably the key to this is forms of finite primitive recursion, where the primitive recursion is perhaps not along just <.

NOTE: For Finite Mathematics, see [Fr09]. There I wasn't trying to be comprehensive, but rather show how one can at least achieve bounded arithmetic and EFA minimalistically. This was the inception of FSRM = finite strict reverse mathematics.

From now on to the end of this paper, we will basically take the Finite Mathematics part of Countable Mathematics as a black box, supporting any ordinary finite mathematical constructions in an arguable strictly mathematical way.

The move to partial functions on FIN from just total functions on FIN is important for RMI/nocode. For instance, a big topic is infinite sequences of rationals, and these are f:ω → Q, and certainly f is not total on FIN, FIN being incomparably richer than ω. Of course Q ⊆ FIN. But the partiality is taken to be tame, in the sense that domains of partial f:FIN → FIN are subsets of FIN.

In this region of RMI/nocode we want a base theory like RCA_0/s, only for Countable Mathematics. And we will also want to give a strictly mathematical axiomatization like we did with ETF/FSRA. We are not prepared to give any full details of this, but sketch some ideas that we expect to carry out later.

The more detailed architecture of the base theory CM = Countable Mathematics, would look like this:

1. Autonomous finite mathematics with the formal umbrella universe FIN. This is a conservative extension of EFA as indicated above with operations like the power set.
2. Sorts PF_1(FIN), PF_2(FIN), PF_3(FIN), SET(FIN), REL_1(FIN), REL_2(FIN), REL_3(FIN).
3. FIN is a constant symbol of sort SET(FIN). So FIN is of sort SET[FIN].
4. SET(FIN) comes with ∈. The other sorts in 2 above use their own natural application (...).
5. Free many sorted logic with its usual syntax and semantics and complete Hilbert style axiomatization based on free logic.

Now let's look inside the autonomous 1 above. Here I will talk less formally. We want

a. ω as a constant set. Like FIN, its elements are elements of FIN. But of course ω itself is not an element of FIN.
b. Finite partial functions from FIN into FIN of arities 1,2,3.
c. Finite subsets of FIN.
d. Finite relations on FIN of arities 1,2,3.
e. Tuples are treated as finite partial functions from FIN into FIN whose domains are finite initial segments of ω\{0}.

f. All entities in b,c,d,e are themselves elements of FIN.

This is similar to the overall architecture of L[FSRA], with $(\omega,0,S,+,\bullet,<)$ greatly subsumed under FIN with all of its bells and whistles.

We shall see how CM supports a fairly substantial RMI/nocode. Not only is Countable Mathematics itself quite rich and extensive, fitting nicely into L[CM], but a great deal of analysis have corresponding statements in CM preserving mathematical meaning and which are also strictly mathematical. Furthermore, focusing on rational numbers is not a bad idea when it comes to fleshing out computational issues. We will close with a discussion of some strictly mathematical approach to reals/infinite sequences of reals that go just beyond CM.

Here is a list of statements from [Si99,09] that are featured by Simpson in Chapter 1, and reversed later in the book. Since we are examining all of these statements, we have given them a double label. E.g., the BW theorem carries the single label 2 when it appears in Chapter 1, and here carries the double label 1.2. We comment on how this would fit into the strictly mathematical framework proposed above. By "supported in CM" we mean that it properly fits into the language L[CM] where background facts are proved in CM. The statement itself will not be provable in CM.

[Si99.09], p. 33:

ACA_0 is equivalent over RCA_0 to any of the following.
1.1. Every bounded, or bounded increasing, sequence of real numbers has a least upper bound. Supported in FCM as: In every bounded, or bounded increasing, sequence of rationals, the set of all rational numbers less than some term exists.
1.2. The BW theorem: Every bounded sequence of real numbers, or of points in Rn, has a convergent subsequence. Supported in CM as: Every bounded sequence of rational numbers, or of points in

Q^n, has a subsequence which is Cauchy (with definite or indefinite estimate). We should distinguish several different notions of convergence here
1.3. Every sequence of points in a compact metric space has a convergent subsequence. Supported in FCM as for 1.2 using totally bounded rational metric spaces. See 1) below. No completion is made. Again distinguish several notions of convergence here.
1.4. The Ascoli Lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence. Probably supported in CM through the use of rationally continuous functions. See 2) below.
1.5. Every countable commutative ring has a maximal ideal. Supported in CM with ring elements from FIN.
1.6. Every countable vector space over Q, or over any countable field, has a basis. Supported in CM with vectors from FIN and finite sequence apparatus in FIN.
1.7. Every countable field (of characteristic zero) has a transcendence basis. Supported in CM with field elements from FIN and finite sequence apparatus in FIN.
1.8. Every countable Abelian group has a unique divisible closure. Supported in CM with group elements from FIN.
1.9. Konig's Lemma: Every infinite finitely branching tree has an infinite path. Supported in CM via relations on FIN or via finite sequences (from FIN) trees.
1.10. Ramsey's theorem for colorings of $[N]^3$, or of $[N]^4$, $[N]^5$, ... Supported in CM with functions of several integer variables, as unary functions on FIN via finite sequence apparatus in FIN.

1) Rational metric spaces are metric spaces on a subset of FIN with rational distances. Use total boundedness instead of compactness.
2) Rationally continuous functions from Q into Q are partial functions from Q into Q satisfying various uniform continuity conditions.

[Si99,09], p. 34:

Π^1_1-CA_0 is equivalent over RCA_0 to any of the following.
2.1. Every tree has a largest perfect subtree. Supported in CM as for 1.8.
2.2. The Cantor/Bendixson theorem: Every closed subset of $\Re$, or any complete separable metric space, is the union of a countable set and a perfect set. Not readily supported in CM.
2.3. Every countable Abelian group is the direct sum of a divisible group and a reduced group. Supported in CM with group elements from FIN.
2.4. Every difference of two open sets in the Baire space N^N is determined. Supported in CM with syntactic descriptions.
2.5. Every G_δ set in [N]N has the Ramsey property. Supported in CM with syntactic descriptions.
2.6. Silver's theorem: For every Borel (or coanalytic, or F_σ) equivalence relation with uncountably many equivalence classes, there exists a nonempty perfect set of inequivalent elements. Supported in CM with syntactic descriptions.

Full blown Borel descriptive set theory can be supported via syntactic descriptions. Σ^1_1 and Π^1_1 formulas.

[Si99,09], p. 36:

WKL_0 is equivalent over RCA_0 to any of the following.
3.1. The Heine/Borel covering lemma: Every covering of the closed unit interval [0,1] by a sequence of open intervals has a finite subcovering. Not readily supported in CM.
3.2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering. Not readily Supported in CM.
3.3. Every continuous real valued function on [0,1] or any compact metric space, is bounded. Supported in CM through 1),2).
3.4. Every continuous real valued function on [0,1], or any compact metric space, is uniformly continuous. Probably supported in CM through 1),2).

3.5. Every continuous real valued function on [0,1] is Riemann integrable. Supported in CM through 1),2).
3.6. The maximum principle: Every continuous real-valued function on [0,1], or on any compact metric space, has, or attains, a supremum. Supported in CM through 1),2).
3.7. The local existence theorem for solutions of (finite systems of) ordinary differential equations. Not readily supported in CM.
3.8. Gödel's completeness theorem: every finite, or countable, set of sentences in the predicate calculus has a countable model. Supported in CM with syntax in FIN.
3.9. Every countable commutative ring has a prime ideal. Supported in CM with ring elements from FIN.
3.10. Every countable field (of characteristic zero) has a unique algebraic closure. Supported in CM with field elements from FIN and finite sequence facility in FIN.
3.11. Every countable formally real field is orderable. Supported in FCM with field elements from FIN.
3.12. Every countable formally real field has a (unique) real closure. Supported in CM with field elements from FIN and finite sequence facility in FIN.
3.13. Brouwer's fixed point theorem: Every uniformly continuous function $F:[0,1]^n \to [0,1]^n$ has a fixed point. Supported in CM using 2). Fixed point is replaced by a Cauchy sequence of rationals.
3.14. The separable Hahn/Banach theorem: If f is a bounded linear functional on a subspace of a separable Banach space, and if $||f|| \leq 1$, then f has an extension f* to the whole space such that $||f^*|| \leq 1$. Not readily supported in CM.

[Si99,09], p. 39:

ATR_0 is equivalent over RCA_0 to any of the following.
4.1. Any two countable well orderings are comparable. Supported by CM with points in orderings from FIN.
4.2. Ulm's theorem: Any two countable reduced Abelian p-groups which have the same Ulm invariants are isomorphic. Supported in

CM with group elements from FIN and with the use of well orderings instead of ordinals for Ulm invariants.

4.3. The perfect set theorem: Every uncountable closed, or analytic, set has a perfect subset. Closed version supported in CM using sets of rationals, and calling for a self dense rational subset whose Cauchy sequences correspond to Cauchy sequences in the given set of rationals.

4.4. Lusin's separation theorem: Any two disjoint analytic sets can be separated by a Borel set. Not readily supported in CM.

4.5. The domain of any single-valued Borel set in the plane is a Borel set. Supported using syntactic descriptions, Δ^1_1, and using N^N instead of the reals.

4.6. Every open or clopen, subset of N^N is determined. Supported using syntactic descriptions.

4.7. Every open, or clopen, subset of $[N]^N$ has the Ramsey property. Supported using syntactic descriptions.

There are many additional reversals in [Si99,09] not included in the above list. Here are most of these.

CHAPTER III. Arithmetical Comprehension.

1. Every (one-one) $f:\omega \rightarrow \omega$ has a range. Supported in CM.

3. Every bounded increasing sequence of real numbers is convergent. Supported in CM for rationals and "convergent" replaced by multiple notions of "Cauchy".

5. In any complete separable metric space, every Cauchy sequence is convergent. Not readily supported in CM.

6. Every countable field has a strong algebraic closure. Supported in CM with field elements from FIN and finite sequence facility in FIN.

7. Every countable field is isomorphic to a subfield of a countable algebraically closed field. Supported in CM with field elements from FIN and finite sequence facility in FIN.

8. Every countable ordered field has a strong real closure. Supported in CM with field elements from FIN and finite sequence facility in FIN.

9. Every countable ordered field is isomorphic to a subfield of a countable real closed ordered field. Supported in CM with field elements from FIN and finite sequence facility in FIN.
10. Every countable vector space over Q either is finite dimensional or contains an infinite linearly independent set. Supported in CM with vectors from FIN and finite sequence facility in FIN.
11. For every pair of countable fields $K \subseteq L$ there exists a transcendence base for L over K. Supported in CM with field elements from FIN and finite sequence facility in FIN.
12. Let L be any countable field of characteristic zero with no finite transcendence base. Then L contains an infinite algebraically independent set. Supported in CM with field elements from FIN and finite sequence facility in FIN.
13. Every countable Abelian group has a subgroup consistent of the torsion elements. Supported in CM with group elements from FIN and finite sequence facility in FIN.
14. Every countable divisible Abelian group is injective. Supported in CM with group elements from FIN and finite sequence facility in FIN.
15. Rado selection lemma, weak and strong forms. Supported in CM without issue.

CHAPTER IV. Weak Konig's Lemma.

1. Every uniformly continuous real-valued function on [0,1] is a uniform limit of polynomials over the rationals. Supported in CM by 2).
2. Variants of 1. Supported in CM by 2).
3. Every countable consistent set of sentences has a completion. Supported in CM using syntax in FIN.
4. If every finite subset of an infinite set of sentences has a model then the infinite set has a model. Supported in CM using syntax in FIN.
5. Completeness and compactness for propositional logic with countably many atoms. Supported in CM using syntax in FIN.
6. Σ^0_1 separation. Supported in CM using syntax in FIN.

7. Every countable commutative ring contains a radical ideal. Supported in CM using ring elements from FIN.
8. Let C be a nonempty closed convex set in $[-1,1]^\omega$. Then every continuous function $f:C \to C$ has a fixed point. Supported in CM using 2).
9. Peano's existence theorem for solutions of ordinary differential equations. Not readily supported in CM.

CHAPTER V. Arithmetical Transfinite Recursion.

1. Σ^1_1 separation. Supported in CM using syntax in FIN.
2. Any Σ^1_1 set of well orderings has a countable ordinal bound on their lengths. Supported in CM using syntax in FIN.

CHAPRTER VI. Π^1_1 Comprehension.
1. For any infinite sequence of finite sequence trees, the set of indices of those that have an infinite path, exists. Supported in CM using finite sequence facility in FIN.
2. The Σ^0_∞ Ramsey theorem. Supported in CM using syntax in FIN.

So it remains to flesh out the system CM and its language L[CM] in full detail, and give a strictly mathematical axiomatization of CM along with its extensions to the levels of WKL_0/s, ACA_0/s, ATR_0/s, Π^1_1-CA_0/s along with uniform synonymy results. We can presumably still use CM, CM + P_1, CM + P_2, CM + P_3, CM + P_4, so the real issue for CM remains to give a strictly mathematical axiomatizations of CM. Also verify that the reversals from [Si99,09] that are within the purview of CM still go through as expected. This is beyond the scope of this paper.

We now come to where some of the truly radical effects of the SRM perspective kick in. This aspect, at its most basic and critical level, surrounds the real number system and infinite sequences of real numbers as discussed much earlier in this section featuring Simpson's experience with this.

My current thinking, which is exploratory, is that from the SRM perspective, we should look at a very natural succession of increasingly powerful portions of mathematics along the following lines.

1. Finite Mathematics. Here all objects are finitary. This is where [Fr09] lives. With our strictly mathematical ETF, we get logical strength with ETF\PERM. But finite mathematics does not use the variables over functions on ω that we see in ETF. To put ETF\PERM in Finite Mathematics, we must use something like PRA, where function symbols are built up by schemes, and the induction uses the various function symbols introduced. To get logical strength, a rather involved long series of function introductions would be needed, which would be just the opposite of strictly mathematical. [Fr09] proceeds rather differently and uses variables over integers and variables of finite sets of integers and/or over finite sequences of integers. This language is still too finite oriented in order to go the ETF\PERM route with single sentence (two sentence, including initial clause) primitive recursion, but instead uses construction like the difference set A-A, finite A, and achieves far less strength than PRA. It achieves the strengths of bounded arithmetic and of EFA (with say finite geometric progressions added). [Fr09] needs to be extended to higher logical strengths within Finite Mathematics.

We do recommend some streamlining here. I like finite sequences to be handled as functions with domains {1,...,n}. Ordered pairs are sequences of length 2 which are functions from the set {1,2}. Sets are not functions and functions are not graphs. We want functions of arities just 1,2,3, as they arise so commonly. But higher arity is reflected in the action of functions on finite sequences.

Binary relations are supremely important, and are not sets and not functions. Once one has binary relations, one might as well have arities 1,2,3 just like for functions.

Integers and rationals are ordered pairs, and ω,Z,Q are incomparable under inclusion. But they get consolidated by ω' ⊆ Z' ⊆ Q'. Later, in 2, we have the number system ℜ, and a further consolidation ω* ⊆ Z* ⊆ Q* ⊆ ℜ, where ℜ is virtual. Yet another consolidation is after C in 3.1, arriving at ω# ⊆ Z# ⊆ Q# ⊆ ℜ# ⊆ C, where ℜ# and C are virtual. But these consolidations are really minor details.

2. Countable Mathematics. As discussed above beginning with ETF[FSRA]. Here the finite mathematics is obtained strictly mathematically using the formalization of countable mathematics, and so the issues [Fr09] successfully grappled with are treated with the extra power of the countably infinite with variables over functions, set, and relations on all of ω. When applying this to actual RM we are led to the finite mathematics part being formalized through the richer FIN, which is naturally closed under power set (of finite sets). This does suggest a reworking of [Fr09] based on a considerably richer finite framework, where the logical power would have to be derived internally in FIN. However, [Fr09] would still have great advantages in terms of economy and simplicity. But in CM, the power of FIN comes to light through the use of function, set, relation variables on FIN. In this way, in the move to this 2, we don't view the actual details of the FIN part as crucial at this preliminary stage. We have seen the Countable Mathematics in action as we have so many countable versions of so much RM that was strictly mathematical and preserved at least the essence of the mathematical content.

Also within Countable Mathematics is the real number system. The number system ℜ is defined in terms of Dedekind cuts in the rationals. This is preferred by many but certainly not all mathematicians who care about foundations of classical analysis. There are important alternatives, but this has the advantage of not needing any story about equality of real numbers, and having to return to that story continually in order to be thoroughly honest. Notice that the complex number system is, in a strict sense, out of range here as it consists of ordered pairs of Dedekind cuts. Thus we don't yet have a strictly mathematical treatment of $\Re^n$, or even $\Re^2$ and the complex number system C. $\Re$, as opposed to ω,Z,Q, $\Re$ and C do not form sets, just like SET[FIN] does not form a set object. See 3.1 below.

3. Extended Countable Mathematics. Here is where real numbers and variants are brought in for a literal reading of classical analysis. We are thinking of a series of principled extensions.

3.1. Closing up the partial functions, sets, relations on FIN, under finite partial functions, finite sets, and finite relations. This supports the complex number system C through $\Re^2$, and also $\Re^n$ and C^n. But that only needs the first stage of this closing up. We clearly have, say, finite sequences from $\Re^n$ here. Of course, this step is rather tame, and doesn't really break any new ground.

3.2. Infinite sequences from $\Re$ and the like. Here is where the essence of the SRM really starts kicking in. We take infinite sequences from $\Re$, and much more generally, partial functions from FIN into what we have built in 3.1. Domains of partial functions from FIN are required to be subsets of FIN. **DO NOT EVEN ATTEMPT TO CODE INFINITE SEQUENCES FROM $\Re$ BY PARTIAL FUNCTIONS, SETS, RELATIONS ON FIN.** Instead formalize the classical analysis of infinite sequences from $\Re^n$ strictly as it is written. Then analyze logically the status of the resulting statements in terms of logical equivalences among them, interpretation powers, consistency strengths, and so forth.

The most general form of this is partial functions from FIN into what we have in 3.1. Domains are required to be subsets of FIN.

Here we also have $f:\Re \rightarrow \Re$ and the like by ACTION. This is much weaker than having general $f:\Re \rightarrow \Re$. Very importantly, we have a treatment of Polish Spaces by action. See below for discussion of action.

3.3. General $f:\Re \rightarrow \Re$ and the like. The overwhelming concern of classical analysis is of course those $f:\Re \rightarrow \Re$ resulting from actions. But there is the real suspicion that when one sees actual use of general $f:\Re \rightarrow \Re$, even with basic function infrastructure, no unusual interpretation power or consistency strength arises. This remains to be investigated.

Let's close with a discussion of the important notion of action. Let $f:A \rightarrow \Re$, where $A \subseteq Q$. ($Q \subseteq \Re$ needs to be discussed). We say that f acts on $x \in \Re$ just in case there is a limit as you approach x from both sides at once. And the action is to be that necessarily unique limit. We write f* for this resulting action. Then f* is a virtual partial function from $\Re$ into $\Re$. These limits can be taken to be with or without estimates. Thus we have built in pointwise continuity of f* with or without estimates. There are statements relating continuity and uniform continuity notions for f and f*.

We now come to the use of actions to develop Polish spaces. We start with a metric space on some set $A \subseteq$ FIN which puts us in 3.2 above. We treated real numbers as sets with standard equality. Here we also do this. We use pairs (x,p), where $x \in A$ and p is a positive rational. The idea is to think of these (x,p) as closed balls in the metric space on A. We identify certain sets of such closed balls. These are the points in the Polish space generated by this metric space. There are various natural conditions to place on these sets of closed balls of positive rational radius, which need to be applied and compared with the $\Re$ case and the metric space on Q. Infinite sequences from such Polish spaces need to be taken as primitive and not coded.

13. SUBSYSTEMS OF ETF[FSRA] AND SRM CHALLENGES

Early in section 12 we discussed four strictly mathematical statements P_1,P_2,P_3,P_4 in the L[FSRA] of section 11 which reverse to WKL_0/FSRA, ACA_0/FSRA, ATR_0/FSRA, Π^1_1-CA_0/FSRA over the strictly mathematical ETF/FSRA. These were put forth as examples of an adventure in strict reverse mathematics giving strictly mathematical axiomatizations of these four systems as ETF[FSRA] + P_1-P_4. Later parts of section 12 were rather expansive, and referring to work in progress.

Here I want to return to matters more in the style of P_1-P_4 in order to perhaps generate some specific research interest in Strict Reverse Mathematics.

The subsystems of the strictly mathematical system ETF/FSRA are rather interesting from many points of view. There are a handful of important ones that naturally arise particularly in section 10. Aside from standard metamathematical investigations into these systems, there is the project of giving strictly mathematical axiomatizations of these systems, sometimes requiring the relevant statements to be within certain language fragments of L[FSRA].

To first simplify the discussion, let us work entirely in L[fcn]. Recall the axiomatization of ETF given in detail at the beginning of section 10.

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axiom.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

For our purposes, it does seem natural to organize these axioms as follows.

1. Successor, Initial Function, Composition.

2. Primitive Recursion.
3. Permutation.
4. Rudimentary Induction.

Here is an enlarged list of relevant axioms and axiom schemes. They are all in L[fcn] as indicated by "f".

1. Successor, Initial Function, Composition/f.
2. Primitive Recursion/f.
3. Permutation/f.
4. Rudimentary Induction/f.
5. Standard existence of functions for +,•,</f.
6. Bounded Primitive Recursion/f. Any 2-ary f defined by primitive recursion exists provided $f(n,m) \leq g(n,m)$ for some 2-ary g given in advance.
7. Δ^0_1-CA/f. See Definition 10.8.
8. WΔ^0_1-CA/f. Let φ be Σ^0_1/f, ψ be Π^0_1/f, variable f not in φ,ψ. $(\forall n)(\varphi \leftrightarrow \psi) \rightarrow (\exists f)(\forall n)(\varphi \leftrightarrow f(n) = 0)$.
9. Σ^0_1-IND/f.

PROBLEM 1. There are 29 = 512 direct subsystems of these 9 axiom (schemes). Determine all of the logical implications between them. Order them according to their provable ordinals, interpretation power, and consistency strength.

PROBLEM 2. Give a strictly mathematical axiomatization in L[fcn] of all 512 direct subsystems.

Next we consider this situation for the wider language L[fcn] $\cup$ L[set]. We list these again, this time identifying what language each axiom is using.

1. Successor, Initial Function, Composition/f.
2. Primitive Recursion/f.
3. Permutation/f.
4. Rudimentary Induction/f.
5. Standard existence of functions for +,•,</f.

6. Bounded Primitive Recursion/f. Any 2-ary f defined by primitive recursion exists provided f(n,m) ≤ g(n,m) for some 2-ary g given in advance.
7. Δ^0_1-CA/fs. See Definition 10.8 using Σ^0_1/fs, Π^0_1/fs.
8. WΔ^0_1-CA/fs. Let φ be Σ^0_1/fs, ψ be Π^0_1/fs, variable f not in φ,ψ. (∀n)(φ ↔ ψ) → (∃f)(∀n)(φ ↔ f(n) = 0).
9. ∑01-IND/fs.

We give the same problems for these. This time, for problem 2, we require that the axioms used for the strictly mathematical axiomatizations be in a specified fragment of L[fcn] ∪ L[set]. These fragments include:

i. L[fcn].
ii. L[set].
iii. L[fcn] ∪ L[set].
iv. Any of the above where we throw out any specified subset of {0,S,+,•,<,=}.

Also there are the analogous problems for ETF/FSRA of section 11.

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