REVERSE MATHEMATICS

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In addition to Anil being the most effective advocate for the substantial representation of mathematical logic in the

American mathematics departments, very much including his promotion of applied logic, he was a pioneer, along with major colleagues like Ershov, in what has become known as Recursive Mathematics. That is the part of his work that is closest to part of mine, as the structure and techniques from Recursive Mathematics are closely allied to Reverse Mathematics. We can think of RM as a bold faced version of Recursive Mathematics, or if you like, Recursive Mathematics as a light faced version of RM.

But RM has an additional intriguing feature. Recursive Mathematics is grounded on the usual nonnegative integers, whereas RM is usually but by no

means necessarily grounded in the usual nonnegative integers. This surfaces in RM in reversals to various levels of arithmetic induction. For instance

I. The pigeonhole principle with arbitrarily many colors is equivalent to Δ^0_2 induction

from J. L. Hirst, Combinatorics in Subsystems of Second Order Arithmetic, PhD Dissertation, The Pennsylvania State University, 1987 combined with T. A. Slaman, Σ_n-bounding and Δ_n-induction. Proc. Amer. Math. Soc. 132 (2004) 2449-2456.

II. Ramsey's Theorem for pairs and arbitrarily many colors implies Δ^0_3 induction

from Cholak, Jockusch, Slaman, On the strength of Ramsey's theorem for pairs, JSL 66 (2001), no. 1, 1-55.

My original conception of Reverse Mathematics was fundamentally more ambitious, and I now call it Strict Reverse Mathematics. The idea is that it is just like RM, but the base theory, the target theories, and the strictly mathematical statements being reversed, are required to be strictly mathematical - in the sense that absolutely no coding is allowed.

The rationale for SRM is twofold.

1. I have always cared about this HUGE CRITICAL STRAW MAN:

LOGICAL STRENGTH IS A MYTH AND LOGICIANS ARE FAKES. YOU CAN PROBABLY EASILY PROVE THE CONSISTENCY OF ALL OF MATHEMATICS BY FORMALIZING IT PROPERLY AND WITHOUT THESE SILLY LOGICAL CONSTRUCTIONS LOGICIANS PUT IN THEIR FORMALIZATIONS. THEN PROVE THE CONSISTENCY DOING STUFF WE MATHEMATICIANS WILL READILY ACCEPT, USING EVEN CONVENTIONAL ARITHMETIC REASONING.

How to best destroy this straw man?

2. In RM, coding pretty much from the start is required because almost no mathematics lies in $L(Z_2)$. And in the mathematics as it is written and conceived by the professional mathematician, the coding isn't

really done. Maybe in a full blown undergrad course where the mathematician is trying to be really careful, one goes back to basics and spells out all codings. But not in the normal course of doing research mathematics.

With regard to coding, much of this is so second nature and seemingly totally innocent that it seems silly to even talk about it. For instance, in RCA_0 we have only the semigroup of nonnegative integers, and of course the full group of integers quickly plays an essential role in almost all of mathematical thought. So normally this is handled by routine coding or even actually defining the integers in $L(Z_2)$

as certain nonnegative integers. It is obviously safer and more honest to say that, e.g., the integer 3 is coded by (1,3) which is coded by $2^1 3^3 = 2(9) = 18$, rather than say, e.g., that the integer 3 *is* the nonnegative integer 18.

But even in this most baby case, actually treating the group of integers when we start only with the semigroup of integers takes a bit more care than you would expect. You want to have a new sort for integers (in addition to the original sort for nonnegative integers) with new +,-,•,<,0,1. We also need a function symbol from sort ω to this new sort Z and say that we have an ordered semigroup isomorphism onto the nonnegative

part of the integer sort. But we also need to have not only subsets of ω but also subsets of Z and some interaction axioms between these new sorts. Then you can easily prove "the obvious things". And of course there is the challenge of saying something definitive about what "the obvious things" are.

But SRM really comes into its own when we come to the real numbers. Actually treating the real numbers in isolation, as a field, already has an issue. The coding in classical RCA_0 has the reals as 2^{-n} Cauchy sequences of rationals, and so the reals do not form an ordered field of real numbers but rather an ordered pseudo field. Actually I do not find this fully

satisfactory, and of course I like even less using equivalence classes.

I won't go further into this as a lot of it is work in progress. The issues that arise from this strict approach grow almost exponentially as we get to continuous real functions, Polish spaces, and so forth.

Now I would like to come to the approach to SRM that I had in mind in my 1976 JSL abstracts. It was already clear to me going back to the 1960s, how to do research in RM. You look at large numbers of varied strictly mathematical theorems S that lie in the language of second order arithmetic, or at least lend themselves to principled

systematic codings that put them into $L(Z_2)$, and investigate the axiomatic system RCA + S generally showing that RCA + S is logically equivalent to a well existing formal system from classical f.o.m.

By 1976 I had second thoughts about going with Reverse Mathematics instead of Strict Reverse Mathematics, and my whole move to replacing full induction in RCA and the other systems, by set induction was motivated by my going for the SRM point of view. I recognized that although set induction works well as a substitute for full induction for systems with at least arithmetic comprehension, it was too weak for RCA (and WKL). So I found

the right way to preserve the SRM point of view and to also have very elemental induction even for RCA and WKL by moving from subsets of ω to functions on ω.

So in my 1976 JSL abstracts, I proposed a system ETF + Δ^0_1-CA as the base theory for Reverse Mathematics. ETF was a carefully framed system using functions on ω rather than subsets of ω. And then I made the claim that ETF proves Δ^0_1-CA, without proof, intending to publish this nontrivial result elsewhere. The significance of this is that ETF is a strictly mathematical system, unlike any system with Δ^0_1-CA. ETF is read "elementary theory of functions".

With the easy result from 1976, that ETF + Δ^0_1-CA proves $\sum^0_1$-IND, all in the language of functions, people took my ETF + Δ^0_1-CA and put it back from functions into sets, arriving at the RCA_0 we know today in $L(Z_2)$ based on Δ^0_1-CA, $\sum^0_1$-IND, and basic axioms.

But from the viewpoint of SRM, this move back to $L(Z_2)$ was very backward. Especially, after a proof of Δ^0_1-CA from ETF was promised, so that ETF would then serve as the base theory of SRM.

This is ancient history and ETF went into the dustbin of history, superseded by the present RCA_0. So I never made good on the promise to publish

ETF proves Δ^0_1-CA. I did make good on this in

H. Friedman, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/ 117. The Emergence of (Strict) Reverse Mathematics, December 29, 2021, 110 pages.

I now want to present this strictly mathematical ETF. It's language is L[fcn], with four sorts, ω, 1-ary, 2-ary, 3-ary functions. We have variables over each sort, constant 0 of sort ω, 1-ary function symbol S on ω, and = between terms (of sort ω), with the usual connectives and quantifiers over all four sorts. Here terms are

all of sort ω. Axioms for ETF there were as follows:

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axiom.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

Here are the details, and don't worry, I'll just point out the interesting features.

1. Successor Axioms.

i. $S(n) \neq 0$

ii. $S(n) = S(m) \rightarrow n = m$

iii. $n \neq 0 \rightarrow (\exists m)(S(m) = n)$

2. Initial Function Axioms.

i. There exists 1-ary, 2-ary, 3-ary functions that are constantly any n. Here n is a variable of sort ω.
ii. The three 3-ary projection functions exist. The two 2-ary projection functions exist. The 1-ary identity function exists.
iii. S(n) defines a 1-ary function.

Since equality between function sorts is not allowed, i,ii,iii are existence statements. Of course, extensional uniqueness is immediate.

3. Composition Axioms.

i. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n,m))$
ii. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n))$

iii. $(\exists f)(\forall n,m)(f(n,m) = g(n,m,m))$

iv. $(\exists f)(\forall n)(f(n) = g(n,n,n))$

v. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(h_1(n,m,r),h_2(n,m,r),h_3(n,m,r)))$

This is a careful simplification of full composition.

4. Primitive Recursion Axiom.

$(\exists f)(f(n,0) = g(n) \wedge (\forall m)(f(n,S(m)) = h(n,m,f(n,m))))$.

Also a simplification of full primitive recursion for 1,2,3-ary functions.

5. Permutation Axiom.

Every 1-ary function that maps ω one-one onto ω has an inverse.

6. Rudimentary Induction Axiom.

$f(0) = g(0) \wedge (\forall n)(f(n) = g(n) \rightarrow f(S(n)) = g(S(n))) \rightarrow f(n) = g(n)$.

So this system ETF is clearly strictly mathematical and is shown in my 2021 manuscript to be synonymous with the usual RCA_0.

I want to close with my destruction of that straw man arguing that classical f.o.m. is BS. This work is in

[Fr09] H. Friedman, The Inevitability of Logical Strength: strict reverse mathematics, Logic Colloquium

'06, ASL, ed. Cooper, Geuvers, Pillay, Vaananen, 2009, 373 pages, Cambridge University Press, pp. 135-183.

In this paper I give several versions of strictly mathematical base theories for Finite Reverse Mathematics. They are strongly related, and they can even be combined nicely into one. Here we just talk about one of these in isolation, L(Z,fsq), which can serve as the base theory for Finite SRM.

L(Z,fsq) is two sorted. Integers and finite sequences of integers. Ring operations on Z and on finite sequences of integers. Order on Z. Length of finite sequences. Value of finite sequences at a place.

The signature of FSQZ is L(Z,fsq). The nonlogical axioms of FSQZ are stated informally as follows.

1. Linearly ordered integral domain axioms.

2. $\mathrm{lth}(\alpha) \geq 0$.

3. $\mathrm{val}(\alpha,n)\downarrow \rightarrow 1 \leq n \leq \mathrm{lth}(\alpha)$.

4. The finite sequence (0,...,n) exists.

5. $\mathrm{lth}(\alpha) = \mathrm{lth}(\beta) \rightarrow (\exists\gamma,\delta,\rho)(\gamma = -\alpha \wedge \delta = \alpha+\beta \wedge \rho = \alpha\bullet\beta)$.

6. The concatenation of α,β exists.

7. For all $n \geq 1$, the concatenation of α, n times, exists.

8. There is a finite sequence enumerating the terms of α that are not terms of β.

9. Every nonempty finite sequence has a least term.

(In 5 above, those equalities are expanded out quantifying over coordinates. This amounts to a clearer way of stating 5 than in [Fr09].)

We stay well within the SRM paradigm if we add some extra symbols with obvious axioms in order make 1-9 conveniently formally perfect.

THEOREM 1. FSQZ logically implies $I\Delta_0$(Z,fsq).

I claimed equivalence in Theorem 1 but now I think maybe we only get an interpretation of FSQZ in there, and therefore in Q. In any case, clearly FSQZ is provable in $I\Delta_0$(exp;Z,fsq).

We now have some crucial reversals.

POWERS. For all n there exists a finite sequence of length n starting with 1, where each successive term is double the preceding term.

POWERS′. There are arbitrary long finite geometric progressions with any starting term and any ratio.

CM. Every 1,...,n has a nonzero common multiple.

THEOREM 2. $I\Delta_0$(exp;Z,fsq) is logically equivalent to any of Powers, Powers′, CM over FSQZ.

Experience shows that the EFA = $I\Delta_0$(exp) level is where logical strength and Goedel phenomena really get going. The Straw Man

is essentially dead. We can go further and make it deader than dead by obvious Finite SRM extensions.

HAPPY BIRTHDAY ANIL!