STRICT REVERSE MATHEMATICS

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I'll be using these four references. Two old, and two this century.

[Fr75] Some Systems of Second Order Arithmetic and Their Use, Proceedings of the 1974 International Congress of Mathematicians, Vol. 1, (1975), pp. 235-242.

[Fr76] Subsystems of Second Order Arithmetic with Restricted Induction I,II, abstracts, J. of Symbolic Logic, Vol. 1, No. 2 (1976), pp. 557-559.

[Fr09] H. Friedman, The Inevitability of Logical Strength: strict reverse mathematics, Logic Colloquium '06, ASL, ed. Cooper, Geuvers, Pillay, Vaananen, 2009, 373 pages, Cambridge University Press, pp. 135-183.

[Fr21] H. Friedman, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/
#117. The Emergence of (Strict) Reverse Mathematics, December 29, 2021, 110 pages.

RM and SRM (Reverse Mathematics, Strict Reverse Mathematics) were born out of my attempts to impress Stanford mathematicians at afternoon teas, of the robustness and fundamental mathematical relevance of various formal systems arising in mathematical logic. This occurred during my presence there in

1967-69, 1971, where I felt compelled to give the more ambitious SRM idea the most emphasis in those early days.

As detailed in [Fr21], the roots of SRM/RM, in this sense, go back to the late 1960’s, with some key reversals already before 1970.

My SRM point of view had a great influence on the way I rolled out RM. In publications I started off with [Fr75] with systems based on full induction. But I saw full induction as incompatible with the SRM viewpoint and became determined to do something about this. This motivated me to switch to systems with restricted induction in [Fr76]. In fact, I went all the way down to induction for equations f(x) = g(x) in [Fr76]. To preserve the SRM point of view, I also moved from $L(Z_2)$ with its sets, to L[fcn] with its functions. Logicians quickly accepted this move to restricted induction but chose to return to $L(Z_2)$, giving up on the SRM point of view. Thus the RCA_0 base theory formulation in [Fr76] was moved back to $L(Z_2)$, and became the standard base theory for conventional RM.

In this talk we will return back to functions starting with the L[fcn] of [Fr76], so convenient for SRM purposes.

REMARKS ON “STRICTLY MATHEMATICAL” AND “CODING”

What is a strictly mathematical statement? This is already the bread and butter of conventional RM when mathematical statements are selected for reversals. Of course, in RM, these statements are then modified by coding to fit into $L(Z_2)$ and then reversed. But the coded versions are rarely strictly mathematical in the sense that the original statement before coding is strictly mathematical.

The highest level of strictly mathematical is of course: literally present and featured in a highly regarded published core mathematical paper. Lower levels of strictly mathematical include statements that are obviously publishable but weren’t actually published for various reasons - reasons including that they weren’t thought of but clearly should have been or needed to be.

Ultimately, “strictly mathematical” is a living breathing thing, and one may have a statement that is strictly mathematical, but another statement is yet more strongly strictly mathematical. So we are not claiming any single hard and fast “yes or no” notion of strictly mathematical. The presence of even a hint of ad hoc features in a mathematical statement generally rules it out as not strictly mathematical.

In RM, coding is used for the very formulation of the strictly mathematical theorems being reversed. As explained in the next section, this is emphatically not the case with SRM.

Normally, the coding in RM is so second nature to recursion theorists that it is barely noticed, yet alone discussed. This is particularly clear in the case of countable combinatorial mathematics. However, there comes a point in the RM development where coding becomes noticeable and discussable. This starts with the introduction of real numbers, but goes far beyond that. Already RM has chosen between Dedekind cuts, Cauchy sequences, and explicitly Cauchy sequences. RM has, of course chosen explicitly Cauchy sequences. There are a number of good reasons to make this choice, which are particularly compelling from the SRM point of view. But we will discuss some fundamental issues surrounding the treatment of $\mathfrak{R}$ and $\mathfrak{R}^\omega$ in RM and SRM.

We always use many sorted free logic with function variables. This is a known very convenient version of predicate logic with a standard completeness theorem. In models, the sorts it simplifies matters to require sorts to be pairwise disjoint.

COUNTABLE AND FINITE SRM
BASE THEORIES
SRM REVERSALS

SRM begins with a strictly mathematical base theory. I.e., a finite set of strictly mathematical theorems. In Countable SRM, the objects referred to in the base theory are countable. In Finite SRM, the objects referred to in the base theory are finite. We have chosen a base theory for each and will discuss both in this talk.

SRM can be viewed as

RM with an *attitude*

The attitude is that we do reversals on strictly mathematical theorems as they come, unadulterated, from mathematics, where

codings are merely a technical tool and
not a part of any statements to be reversed

In conventional RM, strictly mathematical theorems φ to be reversed are either actually converted to (approximately) $L(Z_2)$, weakening or destroying the strictly mathematical character of φ, or sometimes coding is viewed in passing for abbreviations, before the reversal takes place. SRM (generally) forbids this conventional RM approach.

Thus we start an SRM development by choosing a strictly mathematical base theory B. We start by entering a finite list $\varphi_1,\ldots,\varphi_n$ of strictly mathematical theorems (no coding allowed!) which are intended to be reversed. This means that we do a logical analysis of the formal system

$B,\varphi_1,\ldots,\varphi_n$

which uses pure logic. I.e., reversals like

$B,\varphi_1,\ldots,\varphi_n$ logically implies K

where K is a target theory. Generally, K is an obvious synonymous adaptation of a familiar target theory from conventional RM, in the language of B.

This is just the first step in the SRM development over B. Then we might treat another finite set of strictly mathematical theorems in the same way, $\varphi_1',\ldots,\varphi_m'$, and establish

$B,\varphi_1',\ldots,\varphi_m'$ logically implies K′

Next we may want to reverse some strictly mathematical theorem that naturally builds on $B,T_1,\ldots,T_n,T_1',\ldots,T_m'$, say $T_1^*,\ldots,T_r^*$. We reverse by establishing

$B,\varphi_1,\ldots,\varphi_n,\varphi_1',\ldots,\varphi_m',\varphi_1^*,\ldots,\varphi_r^*$ logically implies K*

In this way we build an ever expanding series of finite posets, with vertices labeled by nonempty finite lists of strictly mathematical theorems, where each new vertex lies just above a maximal existing vertex.

The codings are completely essential tools for accomplishing the SRM reversals. But

codings only lie under the hood

and are really just interpretations ultimately going back to the base theory, adapted to many sorted free logic, which preserve the primitives of the base theory B. It is this preservation of the primitives by the codings that allows us to make reversals outright to target theories in the language of B.

CODINGS = ADMISSIBLE INTERPRETATIONS

Model theoretically, an admissible interpretation of T + φ in T consists of a mapping H from models of T into models of T + φ, H given by a definition in the language of T, where each M,H(M) agree on the language of T. (Here φ can be a finite list of sentences, which of course can be conjuncted).

This notion is completely standard when T + φ has the same sorts as T. Otherwise, this notion needs some explanation. Let M be a model of T and let α be a new sort in φ. What does H(M) look like on α?

We can't use a defined set of elements of M (i.e., elements of the union of the sorts of M) for this purpose. Instead we want to use a defined set of tuples of length ≥ 2 from the union of the sorts of M, where there is an explicit bound on the length of the tuples involved. With multiple new sorts in φ, we require that these defined sets of tuples are pairwise disjoint (and disjoint from the elements of M).

Note that "H is an admissible interpretation of T + φ in T" is a statement φ* in the language of T. We then reverse φ* in the usual sense of conventional RM.

In the course of SRM, when we choose a new strictly mathematical theorem φ to reverse, we generally find that φ alone, together with its predecessors in the finite poset discussed earlier, is just not powerful enough to accomplish much of a reversal. We will need some basic strictly mathematical theorems to be added on to form a coherent list. So technically speaking, it isn't just φ that is reversed, but rather the conjunction of the rest of the new list together with all of the items on the lists that are predecessors in the finite poset.

However, generally speaking, we can still refer to "the SRM reversal of φ" where it is understood that one is including all of the strictly mathematical theorems building up to and surrounding φ.

We emphasize that although we require that φ be a strictly mathematical theorem, there is no requirement that these eminently useful admissible interpretations be strictly mathematical. However, the more mathematical they are the better, as the SRM researcher must use them and think about them easily. In most cases, when introducing φ for SRM, we will also be adding new sorts and new symbols. These new items must all be strictly mathematical. It's just that the admissible interpretations don't have to be, although it is a big advantage that they be easy to work with.

COUNTABLE SRM BASE THEORY ETF = ELEMENTARY THEORY OF FUNCTIONS

For the base theory for Countable SRM, we now take ETF. Its language is L[fcn], which has the four sorts ω,FCN[1],FCN[2],FCN[3], the constant 0 of sort ω, and the function symbol S from sort ω into sort ω. We use the standard many sorted free logic with function variables, and equality only for sort ω. Terms of sort ω are defined inductively in the obvious way. The atomic formulas are s = t, where s,t are terms of sort ω.

The intended interpretation of FCN[i] is the family of all i-ary functions from ω into ω, $1 \leq I \leq 3$, and S is the successor function from ω inro ω. We use variables $n_1,n_2,\ldots$, ranging over sort ω, and variables F^1_n, F^2_n. F^3_n. $n \geq 1$, ranging over sorts FCN[1], FCN[2], FCN[3], respectively.

The nonlogical axioms of ETF are as follows.

1. Successor Axioms.
2. Initial Function Axioms.
3. Composition Axioms.
4. Primitive Recursion Axiom.
5. Permutation Axiom.
6. Rudimentary Induction Axiom.

QUESTION: There are 64 subsets of these 6 axioms. What are they logically, and how are they to be classified in terms of provability and interpretability?

Note that one removal is particularly natural, namely removing the Permutation Axiom. You just get bold faced primitive recursive arithmetic. Further weakenings involve weakening the Primitive Recursion Axiom. Reversals to ETF will be interesting and important. Such rival developments have a lot of major consequences that we haven't yet systematically explored.

Here are the axioms of ETF spelled out in full detail.

Successor Axioms.
i. $S(n) \neq 0$
ii. $S(n) = S(m) \rightarrow n = m$
iii. $n \neq 0 \rightarrow (\exists m)(S(m) = n)$.

Initial Functions Axioms.
i. There exists 1-ary functions that are constantly any n. Here n is a variable of sort ω.
ii. The three 3-ary projection functions exist.
iii. S(n) defines a 1-ary function.

Composition Axioms.
i. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n,m))$
ii. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(n))$
iii. $(\exists f)(\forall n,m)(f(n,m) = g(n,m,r))$
iv. $(\exists f)(\forall n)(f(n) = g(n,m,r))$
v. $(\exists f)(\forall n,m,r)(f(n,m,r) = g(h_1(n,m,r),h_2(n,m,r),h_3(n,m,r)))$

Primitive Recursion Axiom.
$(\exists f)(\forall n)(f(n,0) = g(n) \wedge (\forall m)(f(n,S(m)) = h(n,m,f(n,m))))$

Permutation Axiom.
Every 1-ary function that maps ω one-one onto ω has an inverse. I.e., let f be a permutation of ω. There exists g such that $(\forall n)(f(g(n)) = n)$.

Rudimentary Induction.
$f(0) = g(0) \wedge (\forall n)(f(n) = g(n) \rightarrow f(S(n)) = g(S(n))) \rightarrow f(n) = g(n)$

ETF is logically equivalent to RCA_0 when RCA_0 is put into L[fcn] in the obvious way. ETF and RCA_0 are synonymous using obvious interpretations back and forth. This is spelled out in [Fr21].

I defined my RCA_0 there as ETF plus $\Delta^0{}_1$-CA, all in L[fcn]. I intended to separately publish ETF = my RCA_0. But everybody went back to $L(Z_2)$ from L[fcn] with the now usual RCA_0, and left ETF and L[fcn] in the dustbin of history.

Never too late?? A complete proof of ETF = my RCA_0 is in [Fr21] 45 years later. Also the natural synonymy between ETF and (standard) RCA_0.

A FIRST SRM EXTENSION

ETF does not have 1,+,•,<. We start the SRM development by introducing them as follows.

1. 1 = S(0).
2. n+0 = n.
3. n+Sm = S(m+n).
4. n•0 = 0.
5. n•Sm = n•m + n.
6. ¬n < 0.
7. n < Sm ↔ n < m ∨ n = m.

These are certainly strictly mathematical theorems. An admissible interpretation in ETF is

n = 1 ↔ n = S(0).
n+m = r ↔ (∃f)(∀n,m)(f(n,0) = n ∧ f(n,Sm) = S(f(n,m))).
n•m = r ↔ (∃f,g)(∀n,m)(f(n,0) = n ∧ f(n,Sm) = S(f(n,m)) ∧ g(n,0) = 0 ∧ g(n,Sm) = f(g(n,m),n)).
n < m ↔ (∃f,g)(∀n,m)(f(n,0) = n ∧ f(n,Sm) = S(f(n,m)) ∧ g(n,0) = 0 ∧ g(n,Sm) = 1 ↔ g(n,m) = 1 ∨ n = m).

Armed with this admissible interpretation, We can now do ordinary reversals of strictly mathematical theorems in this extension of ETF and L[fcn], as long as we stay within this extended language.

A SECOND SRM EXTENSION

So far we don't have any relations or sets. We introduce sorts REL[1], REL[2], REL[3], SET[ω], with relation application and ∈ as primitives. We use these strictly mathematical theorems.

1. (∃f)(∀n)(n ∈ A ↔ f(n) = 0).
2. (∃A)(∀n)(n ∈ A ↔ f(n) = 0).
3. (∃R)(∀n)(R(n) ↔ f(n) = 0).
4. (∃f)(∀n)(R(n) ↔ f(n) = 0).
5. (∃R)(∀n,m)(R(n,m) ↔ f(n,m) = 0).

6. $(\exists f)(\forall n,m)(R(n,m) \leftrightarrow f(n,m) = 0)$.
7. $(\exists R)(\forall n,m,r)(R(n,m,r) \leftrightarrow f(n,m,r) = 0)$.
8. $(\exists f)(\forall n,m,r)(R(n,m,r) \leftrightarrow f(n,m,r) = 0)$,

We can either view this SRM extension as over ETF, or over the first SRM extension over ETF in the previous section.

We interpret the unary relations as the (1,f), f 1-ary characteristic function; the binary relations as the (2,f), f 2-ary characteristic function; the ternary relations as the (3,f), f 3-ary characteristic function; the subsets of ω; the sets as the (0,f), f 1-ary characteristic function. 0,1,2,3 are written with S,0, and so this is parameter free. Relation application and membership defined in the obvious way. No need for additional axioms.

COMPOSITIONAL RICHNESS

In SRM, we often introduce sorts that are obviously denumerable. The only such sort we have so far is the ω already in ETF.

Suppose we have obviously denumerable sorts $\omega,\alpha_1,\ldots,\alpha_k$. We obviously have the strictly mathematical theorems that there are bijections between any two of these sorts. We also have the strictly mathematical theorems that all bijections have inverses (recall Perm in ETF), and that compositions exist. We can obviously use these in the SRM development as needed.

We can introduce sorts for the 1,2,3-ary functions and relations, and for the subsets, as in the previous section, only this time instead of just using ω, we can use any of $\omega,\alpha_1,\ldots,\alpha_k$ for domains of functions and relations and any of $\omega,\alpha_1,\ldots,\alpha_k$ for codomains and for taking subsets.

In practice, we lean on all this compositional richness as we need it. Obviously it is rather long to completely list all of this.

A THIRD SRM EXTENSION

So far we do not have the ordered ring of integers. We add a new "declared to be countable" sort Z, and $0,1,<,+,-,\bullet$ on sort Z, and use the axioms

1. $(Z,0,1,<,+,-,\bullet)$ is a commutative ordered ring.

2. There is an isomorphism from the semiring ω into the nonnegative part of the ordered ring Z.

Here to make use of 2 properly, we are leveraging off of compositional richness, which we are applying to the two countable sorts ω,Z.

We leave it to you to give the obvious admissible interpretation, and see that all of the fundamental facts about N,Z are now statable and provable.

A FOURTH SRM EXTENSION

We now introduce the ordered field of rationals. We add a new "declared to be countable" sort Q, and 0,1,<,+,-,•,÷ on W. We use the following axioms.

1. (Q,0,1,<,+,-,•,÷) is an ordered field.
2. There is an isomorphism h from the ordered ring Z into the ordered field Q, such that every element of Q is the ratio of some two values of h.

We leverage off of compositional richness for countable sorts ω,Z,Q to see that all of the fundamental facts about ω,Z,Q are statable and provable.

MORE COMPOSITIONAL RICHNESS

We have introduced compositional richness with sorts for the functions from explicitly countable sorts into explicitly countable sorts.

Now we go further and introduce compositional richness with sorts for the functions from explicitly countable sorts into all sorts. Most notably, for each sort α, we have the new sort for all functions from ω into α. These are of course just the infinite sequences from α. We can of course iterate this process of taking the infinite sequences, or all functions from explicitly countable sorts into sorts. But we just state what we need.

So all of the sorts are still countable or power c, with the latter analogous to proper classes and the former analogous to sets.

REAL NUMBERS IN RM

Before we discuss real numbers in SRM, let's look at real numbers in conventional RM. The crucial definitions there are the definition of a real number as a convergent sequence of rationals with prescribed convergence, and the infinite sequences of reals as a convergent double sequence of rations also with prescribed convergence. Equality is taken to be the obvious associated equivalence relations.

We know that treatments of the reals and infinite sequences of reals are extremely robust, allowing many provably equivalent variants, if we are working in RM over ACA_0. But we want to work over RCA_0.

Note that this approach is not extensional, which can arguably be criticized from some points of view. One vivid way of saying this is that the following is not only unprovable in RCA_0, but is actually refutable:

strictly between any two distinct real numbers lies a rational

So one important question is this: is there an extensional treatment of the reals and infinite sequences of reals appropriate in conventional RM? I.e., where the above is provable? The obvious approach is to define reals as Dedekind cuts, but then natural extensions of that to infinite sequences of reals via Dedekind cuts fails on several counts to work over RCA_0. It can instead be made to work over ACA_0. See Brown and Simpson, which set existence axioms are needed to prove the separable Hahn-Banach theorem?, Annals of Pure and Applied Logic 31 (1986), 123-144 for details.

This matter justifies some further research even for conventional RM. But we expect SRM to provide different kinds of insights and issues.

FURTHER SRM EXTENSIONS FOR REAL NUMBERS

We start introducing the field of real numbers with sort $\Re$, and $0,1,<,+,-,\bullet,\div,|\ |$ on $\Re$, with the following.

1. $(\Re,0,1,<,+,-,\bullet,\div)$ is an ordered field, with $|x| = x$ if $0 < x$; $-x$ otherwise.
2. There is an isomorphism from the ordered field Q into the ordered field $\Re$ whose values have no upper bound.

So far, $\Re$ could simply be Q. We must now come to the matter of completeness.

We say that $f:\omega \to Q$ is g-Cauchy if and only if $g:N \to N$ and $(\forall k > 0)(\forall n,m > g(k))(|f(n)-f(m)| < 1/k)$. We say that $f:\omega \to Q$ is explicitly Cauchy if and only if it is g-Cauchy for some g.

We say that $f:\omega \to Q$ g-converges to x if and only if $g:N \to N$ and $x \in \Re$ and $(\forall k > 0)(\forall n > g(k))(|f(n)-x| < 1/k)$. We say that $f:\omega \to Q$ explicitly converges if and only if there exists g,x such that it g-converges to x.

We make the same definitions above with Q replaced by $\Re$. For completeness of $\Re$ over Q, we have

3. Every explicitly Cauchy $f:\omega \to Q$ explicitly converges.

For internal completeness, we have

4. Every explicitly Cauchy $f:\omega \to \Re$ explicitly converges.

We will not automatically get completeness for $\Re^\omega$, but we can proceed analogously using double sequences from Q (use compositional richness above to support this).

Now consider the following:

5. For real numbers x, some sequence of rationals explicitly converges to x.
6. For infinite sequences of real numbers x*, some double sequence of rationals explicitly converges to x*.

Presumably 5 does not follow from 1-4 and 6 does not follow from 1-5.

Let's not forget extensionality for reals:

EXT/$\Re$. If x,y are distinct real numbers then there exists rational p such that $x < p < y$ or $y < p < x$.

In this discussion, we may or may not use EXT/$\Re$.

To reduce this SRM development to essentially coincide with the conventional RM development of real analysis over RCA_0, we would

NOT have a separate sort $\Re$, but instead define real numbers as certain infinite sequences of rationals, and define infinite sequences of real numbers as certain double sequences of rationals.

Coming back to this SRM development, what additional strictly mathematical theorems do we need to have, if any, in order to get some, most, or all of the real analysis results obtained in conventional RM over RCA_0?

There are too many issues here even with just $\Re$ and $\Re^\omega$ to talk about. Let alone when we bring in such essentials as Polish Spaces.

We now move to Finite SRM.

HUGE CRITICAL STRAW MAN*: LOGICAL STRENGTH IS A MYTH AND LOGICIANS ARE FAKES.YOU CAN PROBABLY EASILY PROVE THE CONSISTENCY OF ALL OF MATHEMATICS BY FORMALIZING IT PROPERLY AND WITHOUT THESE SILLY LOGICAL CONSTRUCTIONS LOGICIANS PUT IN THEIR FORMALIZATIONS. THEN PROVE THE CONSISTENCY DOING STUFF WE MATHEMATICIANS WILL READILY ACCEPT, USING EVEN CONVENTIONAL ARITHMETIC REASONING.*

This straw man motivated my development of SRM. Although in a sense SRM was officially founded in [Fr76] (modified from the earlier [Fr75]) with the presentation of ETF and the undocumented nonobvious claim that ETF proves $\Delta^0{}_1$-CA/f, and the obvious claim that ETF prove $\sum^0{}_1$-IND/f, it wasan’t until [Fr09] that I refocused on SRM and back to basics with the highly motivating Straw Man above.

The most convincing defeat of this Straw Man should come from Finite Mathematics and so Finite SRM was explicitly founded in [Fr09]. Recall the title The inevitability of Logical Strength: strict reverse mathematics”.

FINITE SEQUENCES OF INTEGERS
FSQZ

There are several interesting base theories for Finite SRM presented in [Fr09]. They are strongly related, and they can even be combined nicely into one. Here we just talk about one of these in isolation, L(Z,fsq), which can serve as the base theory for Finite SRM.

L(Z,fsq) is two sorted. Integers and finite sequences of integers. Ring operations on integers and on finite sequences of integers. length of finite sequences. value of finite sequences at a place.

The signature of FSQZ is L(Z,fsq). The nonlogical axioms of FSQZ are stated informally as follows.

1. Linearly ordered integral domain axioms.
2. $\mathrm{lth}(\alpha) \geq 0$.
3. $\mathrm{val}(\alpha,n)\downarrow \leftrightarrow 1 \leq n \leq \mathrm{lth}(\alpha)$.
4. The finite sequence (0,...,n) exists.
5. $\mathrm{lth}(\alpha) = \mathrm{lth}(\beta) \rightarrow (\exists\gamma,\delta,\rho)(\gamma = -\alpha \wedge \delta = \alpha+\beta \wedge \rho = \alpha\bullet\beta$ coodinatewise).
6. $(\exists\gamma)(\gamma$ is the concatenation of $\alpha,\beta)$.
7. For all $n \geq 1$, the concatenation of α, n times, exists.
8. There is a finite sequence enumerating the terms of α that are not terms of β.
9. Every nonempty finite sequence has a least term.

Axioms 4,5,6,7 are formalized using lth and val. We could alternatively add additional primitives to simplify the actual formalizations. As these additional primitives are in constant use in actual mathematics, we would still be within the SRM paradigm.

THEOREM 1. FSQZ logically implies $I\Delta_0$(Z,fsq).

I claimed equivalence in Theorem 1 but now I think maybe we only get an interpretation of FSQZ in $I\Delta_0$. In any case, clearly FSQZ is provable in $I\Delta_0$(exp;Z,fsq).

We now have some important reversals.

POWERS. For all n there exists a finite sequence of length n starting with 1, where each successive term is double the preceding term.

POWERS'. There are arbitrary long finite geometric progressions with any starting term and any ratio.

CM. Every 1,...,n has a nonzero common multiple.

THEOREM 2. $I\Delta_0$(exp;Z,fsq) is logically equivalent to any of Powers, Powers', CM over FSQZ.

Experience shows that the EFA = $I\Delta_0$(exp) level is where logical strength and Goedel phenomena really get going. The Straw Man is essentially dead. We can go further and make it deader than dead by obvious Finite SRM extensions, say in the next secions.

A FINITE SRM EXTENSION

Our base theory FSQZ has integers, arithmetic, and finite sequences of integers. It is natural to add the sorts.

A. Finite sets of integers.
B. Finite sequences of (integers and finite sets of integers and finite sequences of integers).
C. Finite sets of (integers and finite sets of integers and finite sequences of integers).

Use extensionality, and obvious axioms relating finite sets of integers and finite sequences of integers. For finite sequences of finite sequences of integers, say that they correspond to a single finite sequence of integers with a special integer used to separate blocks. Relate finite sets of finite sequences of integers to finite sequences of finite sequences of integers.

Note that everything really comes back naturally and clearly to integers and finite sequence of integers, and so we use the power of the base theory FSQZ to derive all of the basic facts. Actually we have Theorem 1 from the previous section.

We can now state strictly mathematical theorems about finite graph theory and finite Ramsey theory (FRT) up to strict SRM standards. In particular our finite Kruskal Theorem (FKT).

THEOREM 1. FSQZ + FRT and $I\Delta_0$(superexp;Z,fsq) are logically equivalent.

THEOREM 2. FSQZ + FKT and 1-Con(TI($<\theta_{\Omega^\wedge\omega}(0)$)) are logically equivalent.

IN CONCLUSION

We have seen 46 official years of a steady growth of RM. We look forward to 46 official years of a steady growth of SRM, and many of you will be around to see that.