ADVENTURES IN FOUNDATIONS OF MATHEMATICS
Ross Program 2022
MWF during June 27 - July 8, 2022
1. Logical reasoning
2. Finite set theory
3. Infinite sets and ZFC
4. Order equivalence, emulators, and stability
5. Stability of emulators of two pairs
6. Stability of emulators of three pairs

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2. FINITE SET THEORY

The first lecture on Logical Reasoning was necessarily a pretty complex mixture of mathematics, philosophy, and culture. You may be happy to hear that we will now be acting like real mathematicians today - at least for the most part.

We will start by freely using the natural number system N of nonnegative integers, <,= on N, finite subsets of N, ordered pairs from N, finite sequences from N, ∈.

But at some point, towards the end of today, we are going to get fussy again, and say things like "what are natural numbers, anyways?", "what really is an ordered pair?", and so forth.

But I promise you we won't get so fussy that we question what = means, what is a set, what does it mean to be an element of a set, and so forth. Otherwise we would find ourselves below the basement, helpless, and begging the philosophers for help!

ORDERED PAIRS, SETS, FUNCTIONS

We will assume without question, ordered pairs of any two objects, and this is written <x,y>.

Crucial fact about ordered pairs is this.

$$(\forall x,y)(\langle x,y\rangle = \langle z,w\rangle \leftrightarrow (x = z \wedge y = w))$$

axiom

We will assume without question, sets. Actually until Friday, we only care about finite sets, but we really aren't quite ready to define "finite" yet. There is the crucial x ∈ A, meaning the object x is an element of the set A.

The crucial fact about sets is that membership of objects is the only thing that counts. I.e., we have

$$(\forall A,B)((\forall x)(x \in A \leftrightarrow x \in B) \rightarrow A = B)$$

axiom

$$(\forall A,B)(A \subseteq B \leftrightarrow (\forall x)(x \in A \rightarrow x \in B))$$

definition

$$(\forall A,B)(A \supseteq B \leftrightarrow (\forall x)(x \in B \rightarrow x \in A))$$

definition

THEOREM 1. $(\forall A,B)(A = B \leftrightarrow (A \subseteq B \wedge A \supseteq B))$.

There are three very important binary operations on sets. These are $\cup$, $\cap$, $\backslash$.

$$A \cup B = \{x\colon x \in A \vee x \in B\}$$

axiom/definition

$$A \cap B = \{x\colon x \in A \vee x \in B\}$$

axiom/definition

$$A\backslash B = \{x\colon x \in A \wedge x \notin B\}$$

axiom/definition

$$\wp A = \{x\colon x \subseteq A\}$$

axiom/definition

THEOREM 2. $A \cup B = B \cup A$. $A \cup (B \cup C) = (A \cup B) \cup C$. $A \cap B = B \cap A$. $A \cap (B \cap C) = (A \cap B) \cap C$.

THEOREM 3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

THEOREM 4. $(A\backslash B)\backslash C = A\backslash(B \cup C)$. $(A \cup B)\backslash C = (A\backslash C) \cup (B\backslash C)$.

A function is a set A of ordered pairs which is univalent. I.e.,

$$(\forall x,y)(\langle x,y\rangle,\langle x,z\rangle \in A \rightarrow y = z)$$

definition

We use letters f,g,h,F,G,H, with or without subscripts, for functions.

The domain of a function f is

$$\mathrm{dom}(f) = \{x: (\exists y)(\langle x,y\rangle \in f)\}$$
definition

The range of f is

$$\mathrm{rng}(f) = \{x: (\exists y)(\langle y,x\rangle \in f)\}.$$
definition

Nobody writes $\langle x,y\rangle \in f$ even though it is technically correct. We instead write

$$f(x) = y \leftrightarrow f \text{ is a function} \wedge (x,y) \in f$$
definition

$$f:A \to B \leftrightarrow (f \text{ is a function} \wedge \mathrm{dom}(f) = A \wedge \mathrm{rng}(f) \subseteq B)$$
definition

$$f \text{ is one-one} \leftrightarrow (f \text{ is a function} \wedge (\forall x,y)(f(x) = f(y) \to x = y))$$
definition

$$f:A \to B \text{ is surjective if and only if } \mathrm{rng}(f) = B$$
definition

$$f:A \to B \text{ is a bijection if and only if } f \text{ is one-one and surjective}$$
definition

Function composition is important.

THEOREM 5. Let $f:A \to B$ and $g:B' \to C$, where $B \subseteq B'$. There is a unique function $h:A \to C$ such that for all x, $h(x) = g(f(x))$. h is written gof.

THEOREM 6. Let $f:A \to B$, $g:B \to C$, $h:C \to D$. Then $ho(gof) = (hog)of$.

NATURAL NUMBERS, FINITE SETS, <,+,•

We don't attempt to define the nonnegative integers 0,1,2,... with its $<,\leq,>,\geq,=,\neq$ as well as +,•. We don't attempt to define the sets. However we will define the finite sets.

We use i,j,k,n,m,r,s,t with and without subscripts for nonnegative integers.

We will write N for the set of all nonnegative integers.

1) Every nonempty A ⊆ N has a least element
axiom

2) If 0 ∈ A and (∀n ∈ A)(n+1 ∈ A), then N ⊆ A
axiom

Here is the induction principle used constantly in proofs.

3) If a property holds of 0 and whenever it holds of n it holds of n+1,
then it holds of all of N
axiom

There is another form that is very useful.

4) If a property holds of any n whenever it holds of any m < n,
then it holds of all n.
axiom

THEOREM 7. 1),2),3),4) are all equivalent.

There is an even more fundamental axiom lurking around here that is "second nature".

5) If P is a property of nonnegative integers then {n: P(n)} is a set
(∃A)(∀n)(n ∈ A ↔ P(n))
axiom

We are now ready to define finite sets.

DEFINITION. {0,...,n} = {i: 0 ≤ i ≤ n}. A is finite if and only if (∃n)(∃f)(f:{0,...,n} → A is a bijection). card(A) is the least n such that this holds.

THEOREM 8. If A,B are finite then A ∪ B is finite.

THEOREM 9. If A is finite then ℘A is finite.

Best way to prove Theorem 9 is by induction on card(A).

THEOREM 10. Let A,B be finite. $A \subseteq B \rightarrow \text{card}(A) \leq \text{card}(B)$. $\text{card}(A \cup B) \leq \text{card}(A) + \text{card}(B)$. If $A \cap B = \emptyset$ then $\text{card}(A \cup B) = \text{card}(A) + \text{card}(B)$. $\text{card}(\wp A) = 2^{\text{card}(A)}$.

THE FINITE CUMULATIVE HIERARCHY

We make the following crucial definition by induction on $n \geq 0$. When we get fussy again, we will stop accepting that we can make such a definition.

$$V(0) = \emptyset.$$
$$V(n+1) = \wp(V(n)).$$
$$V(\omega) = \{x:(\exists n)(x \in V(n))\}$$

$V(0) = \emptyset$
$V(1) = \wp\emptyset = \{\emptyset\}$.
$V(2) = \wp V(1) = \{\emptyset,\{\emptyset\}\}$.
$V(3) = \wp V(2) = \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$.
$V(4) = \wp V(3) =$
$\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\},\{\{\{\emptyset\}\}\},\{\{\emptyset,\{\emptyset\}\}\},$
$\{\emptyset,\{\{\emptyset\}\}\},\{\emptyset,\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\emptyset\},$
$\{\emptyset,\{\emptyset\}\}\},\{\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\},$
$\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\},\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\{\emptyset,\{\{\emptyset\}\},$
$\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\},\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}\}$.

$2^{[n]}$ defined by induction on n. $2^{[0]} = 1$, $2^{[n+1]} = 2^{2\wedge[n]}$.

THEOREM 11. $\text{card}(V(0)) = 0$, $\text{card}(V(1)) = 1$, $\text{card}(V(2)) = 2$, $\text{card}(V(3)) = 4$, $\text{card}(V(4)) = 16$, $\text{card}(V(5)) = 2^{16} = 65{,}536$. $\text{card}(V(6)) = 2^{65{,}536}$. For $n \geq 1$, $\text{card}(V(n)) = 2^{[n-1]}$.

CONJECTURE. Everything interesting in mathematics can be "seen" in V(6).

We can think of V(ω) as

the UNIVERSE of FINITARY MATHEMATICAL OBJECTS
officially: the family of HEREDITARILY FINITE SETS

Now let's behave like mathematicians and prove some facts about this V(ω).

DEFINITION. Let $x \in V(\omega)$. rk(x) is the least n such that $x \in V(n)$.

DEIFNITION. A is transitive if and only if every element of every element of A is an element of A.

DEFINITION. An $\in$-minimal element of a set A is an element of A no element of which belongs to A.

DEFINITION. An $\in$-maximal element of a set A is an element of A which is not an element of any element of A.

THEOREM 12.
1. $V(n) \subseteq V(n+1)$.
2. $n \leq m \rightarrow V(n) \subseteq V(m)$.
3. $V(n) \notin V(n)$.
4. $V(n) = \{x \in V(\omega): rk(x) \leq n\}$.
5. $\emptyset$ is the unique set with rank 1.
6. Each V(n) is transitive. $V(\omega)$ is transitive.
7. $x \subseteq y \rightarrow rk(x) \leq rk(y)$.
8. $x \in y \rightarrow rk(x) < rk(y)$.
9. $V(n) = \{y: rk(y) \leq n\}$.
10. In V(n), log many elements have rank < n and the rest have rank n.
11. Let $x \in V(\omega)$. rk(x) is the maximum of the rk(y), $y \in x$, plus 1.
12. Every element of an element of V(n) is an element of V(n).
13. $V(\omega)$ is infinite. Every element of $V(\omega)$ is finite.
14. Every finite subset of $V(\omega)$ is an element of $V(\omega)$.
15. Every nonempty subset of $V(\omega)$ has an $\in$-minimal element.
16. Every nonempty element of $V(\omega)$ has an $\in$-minimal element and an $\in$-maximal element.
17. Any finite union of finite sets is finite.
18. Any finite union of elements of $V(\omega)$ is an element of $V(\omega)$.
19. The finite axiom of choice. Let A be a finite set of pairwise disjoint nonempty finite sets. There is a finite set that has a unique point of intersection with each of these nonempty finite sets.

V(ω) AS THE BASEMENT OF FINITE MATHEMATICS

You can already see how rich $V(\omega)$ in terms of finite combinatorial mathematics. (Yes, it is really no good for any normal treatment of continuous mathematics with its real numbers.) We proved a lot of facts about $V(\omega)$ using standard accepted mathematical methods.

But this is not fully satisfactory. How do we justify the methods that we used to prove facts about V(ω)? Obviously, it is preferable for us to LIVE ENTIRELY in V(ω) and take V(ω) to be the entire basement for FINITE mathematics. And actually PROVE that the standard mathematical methods we used are correct. (INFINITE mathematics will be discussed on Friday.)

FIVE INCREDIBLY SIMPLE AND INCREDIBLY STRONG AXIOMS FOR FINITE MATHEMATICS

The universe consists entirely of finite sets. The so called "intended structure" is (V(ω),∈), which links the end of this lecture with the first lecture.

1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.
2. Emptyset. ∅ exists. $(\exists x)(\forall y)(y \notin x)$.
3. Insertion. Given x,y, we can insert y into x. I.e., $x \cup \{y\}$ exists. $(\exists z)(\forall w)(w \in z \leftrightarrow (w \in x \vee w = y))$.
4. Every expressible property that holds of some x holds of some ∈ minimal x. $P(x) \rightarrow (\exists x)(P(x) \wedge (\forall y \in x)(\neg P(y)))$.
5. Every nonempty set has an ∈ maximal element.

In 4, expressible is by formulas in predicate logic from the first lecture.
Also we take the axioms and rules of predicate logic as given here.

You can derive nearly all of finite mathematics from just 1-5. You can do this:

i. Define nonnegative integers, <, and S (successor).
ii. Define and justify unordered pairs and ordered pairs.
iii. Define functions.
iv. Define finite sets and cardinalities.
v. Prove standard facts about these notions.
vi. Prove that every set is finite.
vii. Prove that $\{x \in y: P(x)\}$ exists.
viii. Prove that definitions by induction are valid.
ix. Prove that the power set of any set exists.
x. Define and justify the finite cumulative hierarchy, i.e., the V(n)'s.
xi. Define ranks and prove basic facts.
xii. Prove that every set is an element of some V(n).
xiii. Prove all of the basic facts about the V(n)'s.

Regarding i. Nonnegative integers nicely treated as transitive ∈ connected sets. x is ∈ connected if and only if for any distinct y,z ∈ x, we have y ∈ z ∨ z ∈ y. Take < to be ∈.

Regarding ii. {x,y} is the unordered pair of x,y. {{x},{x,y}} is the ordered pair of x,y.

EXERCISES

Prove Theorems 1-12 as indicated before the last section.

Then go back and carry out 1-xiii in detail making sure you are only using axioms 1-5.