**ADVENTURES IN FOUNDATIONS OF MATHEMATICS**
**Ross Program 2022**
**MWF during June 27 - July 8, 2022**
**1. Logical reasoning**
**2. Finite set theory**
**3. Infinite sets and ZFC**
**4. Order equivalence, emulators, and stability**
**5. Stability of emulators of two pairs**
**6. Stability of emulators of three pairs**

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## 3. INFINITE SETS AND ZFC

From the second lecture, we have a completely self contained basement for Finite Mathematics, with the axioms 1-5:

1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.
2. Emptyset. $\emptyset$ exists. $(\exists x)(\forall y)(y \notin x)$.
3. Insertion. Given x,y, we can insert y into x. I.e., $x \cup \{y\}$ exists. $(\exists z)(\forall w)(w \in z \leftrightarrow (w \in x \vee w = y))$.
4. Every expressible property that holds of some x holds of some $\in$ minimal x. $P(x) \rightarrow (\exists x)(P(x) \wedge (\forall y \in x)(\neg P(y)))$.
5. Every nonempty set has an $\in$ maximal element.

In 4, expressible is by formulas in predicate logic from the first lecture.

Also we take the axioms and rules of predicate logic as given. In other words, the full self contained basement has

1. Axioms and rules of inference for predicate logic as discussed in lecture 1.
2. Axioms 1-5 treated as premises.

A proof in this basement for Finite Mathematics consists of a finite sequence of formulas in predicate logic with $\in$,=, where each entry is either one of the logical axioms from lecture 1, or one of the statements in 1-5, or follows from one or two previous entries by one or more of the rules of inference in predicate logic from lecture 1.

Of course, this formulation of the basement for Finite Mathematics can be made more aligned with the way the Finite Mathematician thinks, as an engineering project, as is done in the area in computer science called Proof Assistants.

**We now want to rework this basement for Finite Mathematics into a closely related basement for Infinite Mathematics, or really just Mathematics.**

**If you recall, we presented the basement for Finite Mathematics by first identifying the universe of finite objects using common mathematical thinking, and then proved various fundamental facts about this universe of finite objects using common mathematical thinking. This was preparation for 1-5 and the claim that all of this can be done from inside. I.e., that 1-5 is enough to do all that work, arriving at a self contained basement for Finite Mathematics.**

**Here we try to follow the same pattern to create the basement for all of Mathematics.**

**Recall the way that we handle nonnegative integers in the Finite basement. We used $\in$ connected transitive sets. These are surprisingly convenient.**

**DEFINITION. An ordinal is an $\in$ connected transitive set. I.e., x is an ordinal if and only if**
**i. $(\forall y,z)(y \in z \wedge z \in x \rightarrow y \in x)$.**
**ii. Let $y \in x$ and $z \in x$. Then exactly one of the following holds: $y \in z$, $z \in y$, $y = z$.**
**These are also called the von Neumann ordinals after John von Neumann.**

**(That the trichotomy in ii is required was not made clear in lecture 2 and should have been. However, in that context one can prove the trichotomy because of the finiteness we assumed there).**

**In Lecture 2, we were only concerned with finite ordinals. We (you) proved that the finite ordinals are themselves well ordered by $\in$, using some grubby work and induction. So < is defined to be $\in$. Also $\emptyset$ is the least finite ordinal, and the immediate successor of finite ordinal x is the finite ordinal $x \cup \{x\}$. Also**

**#) if a property holds of $\emptyset$ and whenever it holds of finite ordinal x, it holds of finite ordinal $x \cup \{x\}$, then it holds of all finite ordinals.**

**thus justifying induction in the basement for Finite mathematics.**

**In fact, we make the following clarifying definition.**

**DEFINITION. 0 = ∅. n+1 = n ∪ {n}.**

**The above is the set theoretic treatment of nonnegative integers.**

**THEOREM 1. Every n is a finite ordinal. Every finite ordinal is some n.**

**But, perhaps surprisingly, ordinals are a great idea even if we don't assume that they are finite! We get the same information without being finite, EXCEPT**

**##) if a property holds of ∅ and whenever it holds of ordinal x, it holds of ordinal x ∪ {x}, then it holds of all ordinals.**

**Well how do we know that there is an infinite ordinal at all? Well, right now we are behaving like mathematicians trying to get familiar with ordinals in general.**

**DEFINITION. ω = {0,1,2,...}.**

**But why is it even legal to make this definition? Two remarks. Firstly, if we can't do something like this, then we are basically throwing away all Infinite Mathematics. Secondly, when we get to the self contained basement for Mathematics, we will derive this a weaker statement.**

**THEOREM 2. ω is an ordinal. ω has no ∈ maximal element.**

**THEOREM 3. All of the standard facts we proved about finite Ordinals also works for all Ordinals, except ##). ##) is refuted using ω.**

**Since I said that ordinals are a great idea, without requiring that they be finite, we should have a replacement for ##).**

***) if a property holds of some ordinal then there is a least ordinal for which it holds.**

**The above is called the least element principle for ordinals. Also**

****) if a property holds of an ordinal x whenever it holds of all ordinals less than x, then it holds of all ordinals.**

The above is called transfinite induction on ordinals.

THEOREM 4. *) and **) are straightforwardly equivalent.

Now just how high do the ordinals really go? The glib answer is "forever".

But coming back to the finite ordinals. You can ask just how high to the finite ordinals go? Well, forever for the finite Mathematician. But for the infinite Mathematician, they really don't go very far.

DEFINITION. We generally use Greek letters for ordinals. $\alpha+1$ is defined to be $\alpha \cup \{\alpha\}$. More generally, $\alpha+(n+1) = (\alpha+n)+1$.

In Lecture 2, we used $\alpha+1 = \alpha \cup \{\alpha\}$ for finite ordinals $\alpha$. We knew this exists by insertion. We continue to use insertion.

THEOREM 5. For ordinals $\alpha$, $\alpha+n$ is an ordinal.

DEFINITION. Let A be a set of sets. $\cup A$ = union of A, is $\{x:(\exists y \in A)(x \in y)\}$.

THEOREM 6. The union of any finite set of finite sets is a finite set. The union of any set of ordinals is an ordinal.

DEFINITION. The union of $\{\omega,\omega+1,\omega+2,...\}$ is written $\omega+\omega$.

Why is this legal? Well let's step back a bit and consider this.

BURALI FORTI PARADOX. The set W of all ordinals is an ordinal. $W \in W$. W+1 is an ordinal. But W+1 is not an ordinal because the trichotomy fails using W+1.

THEOREM 7. The set W of all ordinals does not exist.

This is something you have to get used to. Obviously it is seductive to think that you can form the set of all sets that satisfy any property given in advance. But this leads to a contradiction even in situations much simpler than the ordinal situation.

RUSEELL's PARADOX. Let U be the set of all sets x such that $x \notin x$. If $U \notin U$ then obviously $U \in U$. If $U \in U$ then obviously $U \notin U$.

**THEOREM 8.** The set U of all sets which are not members of themselves, does not exist.

We really do want to adhere to the idea that we can form $\{x \in A: P(x)\}$, for any property P, as things are limited by the set A. Thus using this, we have

**THEOREM 9.** There is no set of all sets.

So coming back to: is $\{\omega,\omega+1,\omega+2,...\}$ legal?

The same kind of thinking that led us to accept $\omega$ as a set will lead us to accepting $\{\omega,\omega+1,\omega+2,...\}$ as a set. But can we be more general about this?

**PRINCIPAL.** Suppose you have a property such that for all $n \in \omega$, there is exactly one set y such that $P(n,y)$. Then we can make all of these y's (corresponding to the $n \in \omega$) into a set.

This will actually take us quite a ways up the ordinals. And we will ultimately strengthen this Principal.

OK, we have said enough intuitively as mathematicians about ordinals. Now we are ready to extend the finite cumulative hierarchy

$$V(0) = \emptyset$$
$$V(n+1) = \wp V(n)$$

that we used in lecture 2 for our Finite Universe, to

$$V(0) = \emptyset$$
$$V(\alpha+1) = \wp V(\alpha)$$
$$V(\lambda) = \cup_{\alpha<\lambda} V(\alpha)$$

This is the full Cumulative hierarchy of sets. Here $\alpha$ is any ordinal and $\lambda$ is any limit ordinal.

**DEFINITION.** A limit ordinal is a nonzero ordinal with no greatest element.

**THEOREM 10.** Every ordinal is either zero ($\emptyset$) or a successor (of the form $\beta+1$) or a limit.

As a mathematician, we can feel reasonably confident that we haven't done something bad like the Burali Forti or Russell's Paradox. That this cumulative hierarchy definition is OK.

Of course, one thing stands out very clearly here. We are using the "fact" that the power set of any set exists. Recall that we proved by induction that the power set of any finite set exists and is finite.

It is truly amazing just how powerful the power set operation really is. We saw that V(6) has cardinality $2^{65,536}$. In the finite world, that is a rather impressive number coming from 6.

THEOREM 11. For any set A, there is no function from A onto $\wp A$, and there is no one-one function from $\wp A$ into A.

Now let's try to write down fundamental axioms that we see hold in this cumulative hierarchy universe. Actually, we can put the entire thing together as

$$V(0) = \emptyset$$
$$V(\alpha+1) = \wp V(\alpha)$$
$$V(\lambda) = \cup_{\alpha<\lambda} V(\alpha)$$
$$V = \cup_{\alpha} V(\alpha)$$

with the Entire Mathematical Universe as V. In lecture 2, we formed $(V(\omega),\in)$, and here we form $(V,\in)$, with the caution that V is not a set.

CAUTION: The first three lines and the right side of the fourth line involve only sets.

THEOREM 12. V is not a set.

Now we are ready to make this basement for Mathematics self contained. Recall how we made the basement for Finite Mathematics self contained with

1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.
2. Emptyset. $\emptyset$ exists. $(\exists x)(\forall y)(y \notin x)$.
3. Insertion. Given x,y, we can insert y into x. I.e., $x \cup \{y\}$ exists. $(\exists z)(\forall w)(w \in z \leftrightarrow (w \in x \vee w = y))$.
4. Every expressible property that holds of some x holds of some $\in$ minimal x. $P(x) \rightarrow (\exists x)(P(x) \wedge (\forall y \in x)(\neg P(y)))$.
5. Every nonempty set has an $\in$ maximal element.

**We must reject 5 because of $\omega$, which has no $\in$ maximal element. Also $V(\omega)$ has no maximal element. We certainly want to keep 1. And obviously 2,3 hold in $(V,\in)$. We can replace 5 with the Power Set Axiom and the denial of 5.**

**1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.**
**2. Emptyset. $\varnothing$ exists. $(\exists x)(\forall y)(y \notin x)$.**
**3. Insertion. Given x,y, we can insert y into x. I.e., $x \cup \{y\}$ exists. $(\exists z)(\forall w)(w \in z \leftrightarrow (w \in x \vee w = y))$.**
**4. Every expressible property that holds of some x holds of some $\in$ minimal x. $P(x) \rightarrow (\exists x)(P(x) \wedge (\forall y \in x)(\neg P(y)))$.**
**5. Power Set. $(\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in y \rightarrow z \in A))$.**
**6. There is a nonempty set without an $\in$ maximal element.**

**THEOREM 13. 1-6 hold in $(V,\in)$.**

**Recall that from Lecture 2, you were able to prove $\{x \in A: P(x)\}$ exists if A is finite.**

**But how to prove this for every A? Doesn't seem promising.**

**THEOREM 14. 1-6 above does not prove that $\{x \in A: P(x)\}$ exists for all sets A.**

**So we better add it. It goes under the name of Separation.**

**Also experience with formalizing theorems reveals some elementary facts that are easy to see holds in $(V,\in)$. Namely the axioms of Pairing and Union.**

**1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.**
**2. Pairing. $(\exists w)(\forall z)(z \in w \leftrightarrow (z = x \vee z = y))$.**
**3. Union. $(\exists w)(\forall z)(z \in w \leftrightarrow (\exists y)(y \in x \wedge z \in y))$.**
**4. Separation. For all expressible properties P, $(\exists x)(\forall y)(y \in x \leftrightarrow P(y))$.**
**5. Every expressible property that holds of some x holds of some $\in$ minimal x. $P(x) \rightarrow (\exists x)(P(x) \wedge (\forall y \in x)(\neg P(y)))$.**
**6. Power Set. $(\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in y \rightarrow z \in A))$.**
**7. There is a nonempty set without an $\in$ maximal element.**

**THEOREM 14. This 1-7 proves the old Emptyset and Insertion.**

**We (you) have already proved the axiom of choice in $(V(\omega),\in)$. The basic idea was to use induction, but that isn't going to work in $(V,\in)$. Better add it.**

**1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.**
**2. Pairing. $(\exists w)(\forall z)(z \in w \leftrightarrow (z = x \vee z = y))$.**
**3. Union. $(\exists w)(\forall z)(z \in w \leftrightarrow (\exists y)(y \in x \wedge z \in y))$.**
**4. Separation. For all expressible properties P, $(\exists x)(\forall y)(y \in x \leftrightarrow P(y))$.**
**5. Foundation. Every nonempty set has an $\in$ minimal element.**
**6. Power Set. $(\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in y \rightarrow z \in A))$.**
**7. Infinity. There is a nonempty set without an $\in$ maximal element.**
**8. Axiom of Choice. For every set of pairwise disjoint sets, some set has exactly one element in common with each of these sets.**

**Note that we have simplified 5. But clearly the previous 5 follows from this 5 and Separation.**

**THEOREM 15. Let $\lambda$ be a limit ordinal $> \omega$. This 1-7 holds in $(V(\lambda),\in)$. This 1-8 holds in $(V(\lambda),\in)$ assuming you are a believer in the Axiom of Choice. Also the same thing holds for $(V,\in)$.**

**So clearly we don't have enough axioms to prove that for all $\alpha$, $V(\alpha)$ exists, or for that matter to develop the entire cumulative hierarchy solely within our axioms.**

**What are we missing here?**

**Recall that we came across the following principal when we wanted to prove the existence of $V(\omega+\omega)$.**

**PRINCIPAL. Suppose you have a property such that for all $n \in \omega$, there is exactly one set y such that $P(n,y)$. Then we can make all of these y's (corresponding to the $n \in \omega$) into a set.**

**There is an obvious much more general form. Instead of "indexing" by $\omega$, why don't we index along any set whatsoever?**

**1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.**
**2. Pairing. $(\exists w)(\forall z)(z \in w \leftrightarrow (z = x \vee z = y))$.**
**3. Union. $(\exists w)(\forall z)(z \in w \leftrightarrow (\exists y)(y \in x \wedge z \in y))$.**

4. Separation. For all expressible properties P, $(\exists x)(\forall y)(y \in x \leftrightarrow P(y))$.
5. Foundation. Every nonempty set has an $\in$ minimal element.
6. Power Set. $(\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in y \rightarrow z \in A))$.
7. Infinity. There is a nonempty set without an $\in$ maximal element.
8. Axiom of Choice. For every set of pairwise disjoint sets, some set has exactly one element in common with each of these sets.
9. Replacement. Let P be a two place expressible property. Suppose $(\forall x \in A)(\exists! y)(P(x,y))$. Then $(\exists B)(\forall x \in A)(\exists y)(P(x,y) \wedge y \in B)$.

THEOREM 16. This 1-9 allows us to define and prove the existence and basic properties of the cumulative hierarchy

$$V(0) = \emptyset$$
$$V(\alpha+1) = \wp V(\alpha)$$
$$V(\lambda) = \cup_{\alpha<\lambda} V(\alpha)$$

THEOREM 17. This 1-9 proves that every set lies in some $V(\alpha)$.

Now let's add 1-9 on as premises to the axioms of predicate logic and the rules of inference of predicate logic from lecture 1. The result is called Zermelo Frankel set theory with the Axiom of Choice. This is the official basement of Mathematics.

The de facto standard for a claimed mathematical theorem is that the offered proof be readily formalizable in ZFC (sugared to allow hierarchical definitions).

But a wide swath of classical statements have been shown to be neither provable nor refutable from ZFC. They always are based on unrestricted or lightly restricted subsets of an uncountable set. Heavily restricted subsets of $\Re$, for example, would include countable, or closed, or even Borel measurable subsets of $\Re$. I don't have time here to go into more detail.

The most well known and basic of these statements based on unrestricted subsets of an uncountable set is the Cantor Continuum Hypothesis.

CH. Let $S \subseteq \Re$. There is a one-one function from S into N or a one-one function from $\Re$ into S.
CH. Let $S \subseteq \Re$. There is a function from N onto S or a function from S onto $\Re$.

**THEOREM 18. There is a one-one function from A into B if and only if there is a function from B onto A.**

**Let's look again.**

**1. Extensionality. $x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$.**
**2. Pairing. $(\exists w)(\forall z)(z \in w \leftrightarrow (z = x \vee z = y))$.**
**3. Union. $(\exists w)(\forall z)(z \in w \leftrightarrow (\exists y)(y \in x \wedge z \in y))$.**
**4. Separation. For all expressible properties P, $(\exists x)(\forall y)(y \in x \leftrightarrow P(y))$.**
**5. Foundation. Every nonempty set has an $\in$ minimal element.**
**6. Power Set. $(\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in y \rightarrow z \in A))$.**
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**8. Axiom of Choice. For every set of pairwise disjoint sets, some set has exactly one element in common with each of these sets.**
**9. Replacement. Let P be a two place expressible property. Suppose $(\forall x \in A)(\exists! y)(P(x,y))$. Then $(\exists B)(\forall x \in A)(\exists y)(P(x,y) \wedge y \in B)$.**

**These go on top as premises of the axioms of predicate logic and the rules of inference of predicate logic from lecture 1.**

**It is almost known that all 512 subsets of these axiom groups are different - not logically equivalent to each other. The most well studied subsets are**

**1. Just drop AxC. Called ZF.**
**2. Just drop Power Set. ZFC\POW.**
**3. Just drop Infinity. ZFC\INF.**
**4. Just drop Replacement. ZFC\REP.**
**5. Just drop Foundation. ZFC\FND.**
**6. Just drop Extensionality. ZFC\EXT.**

**ZFC does not prove that ZFC is consistent, unless ZFC is inconsistent. (Goedel's Second Incompleteness Theorem for ZFC). Con(ZFC) is considered to be referring to arbitrary subsets of uncountable sets.**

**NEXT WEEK - you will learn about a totally new kind of statement that is neither provable nor refutable in ZFC. Statement lives in $Q[-n,n]^k$ for all dimensions k. Here $Q[-n,n] = Q \cap [-n,n]$.**

**It is already deep and rich in dimension 2 where it is provable in ZFC. But there are lots of open challenges. We focus on $Q[-1,1]^2$.**