ADVENTURES IN FOUNDATIONS OF MATHEMATICS
Ross Program 2022
MWF during June 27 - July 8, 2022
1. Logical reasoning
2. Finite set theory
3. Infinite sets and ZFC
4. Order equivalence, emulators, and stability
5. Stability of emulators of two pairs
6. Stability of emulators of three pairs

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revised July 5, 2022

4. ORDER EQUIVALENCE, EMULATORS, AND STABILITY

"But a wide swath of classical statements have been shown to be neither provable nor refutable from ZFC. They always are based on unrestricted or lightly restricted subsets of an uncountable set. Heavily restricted subsets of $\Re$, for example, would include countable, or closed, or even Borel measurable subsets of $\Re$. I don't have time here to go into more detail."

I'm finishing up work to be presented in final form in Fall, 2022, claiming to give statements of a radically new kind that are neither provable nor refutable from ZFC, which only involves sets of tuples of rational numbers. For impact, it is essential that such new statements resonate with a wide range of mathematicians. Preliminary reactions from famous mathematicians in core areas of mathematics seem encouraging.

Furthermore, I claim that these new statements can be proved in certain well studied strong extensions of ZFC based on so called "large cardinal hypotheses". So there is a positive side to this development. It shows the kind of new mathematics that can be proved if we go beyond the usual ZFC axioms using large cardinal hypotheses that we can't prove if we stay within ZFC.

These new statements involve sets of k-tuples of rational numbers. We claim that the statements are not provable in ZFC when stating them for all dimensions k. Even for dimension 3, we do not know how to prove them in ZFC. But for dimension 2, we do know how to prove them in ZFC. So we will focus on dimension 2 for this second week.

The proof for dimension 2 uses some heavy machinery and I won't try to present this here. And furthermore, there is something not ideal about the proof for dimension 2. The statement calls

for a subset of $Q[-1,1]^2$ with certain nice properties. My proof does not yield an algorithmically computable subset of $Q[-1,1]^2$.

However, we will prove that under certain restrictions (so called size), we do get the required subset of $Q[-1,1]^2$ that is obviously algorithmically computable. I.e., size ≤ 3. There is an obvious important challenge here. Prove that we get an algorithmically computable subset of $Q[-1]^2$ for size ≤ 4 and beyond. Maybe even one for any size. (For us, always dimension 2).

4.1. PRELIMINARIES

We often use ∧ for "and" and ∨ for "or" and → for "if then" and ↔ for "if and only if".

Q is the set of all rational numbers.
Z is the set of all integers.
N is the set of all nonnegative integers.
< is the usual comparison between rationals.
Q[-1,1] is the set of all rationals p such that -1 ≤ p ≤ 1.

We will use a,b,c,d,e,p,q,r,s,t,u,v,w for rational numbers in Q[-1,1] unless indicated otherwise. We use n,m,i,j,k for positive integers unless otherwise indicated.

We will always use the term "pair" for ordered pair. Thus (1,1/3) is not the same as (1/3,1). The unordered pair of 1 and 1/3 is {1,1/3} = {1/3,1}.

We use A,B,C,D,E,S,T,U,V for subsets of $Q[-1,1]^2$ unless indicated otherwise. We use ⊆ for subset.

We use |E| for the cardinality of E, or the number of elements of E. All sets that we encounter here are of cardinality 0,1,2,3,..., or of cardinality ω.

A set of tuples of rationals is ALGORITHMIC if and only if there is a computer algorithm which tells us whether or not a given tuple of rationals is in the set. We will use this important concept without getting into its well known mathematical treatments.

THEOREM 4.1.1. Every countable dense linear ordering (D,<) with a left and right endpoints, both different, is isomorphic to (Q[-1,1],<).

THEOREM 4.1.2. Every countable linear ordering (D,<) where every element has an immediate successor and an immediate predecessor, and where there are only finitely many elements strictly between any two elements, is isomorphic to (Z,<).

THEOREM 4.1.3. Every countable linear ordering (D,<) with a least element, where every element has an immediate successor, and where there are finitely many elements less than any given element, is isomorphic to (N,<).

4.2. ORDER EQUIVALENCE AMONG TUPLES

A^k is the set of all k-tuples of elements of A. They are written $(x_1,...,x_k)$, where $x_1,...,x_k \in A$. Usually, we focus on small k, say k = 1,2,3,4.

DEFINITION 4.2.1. Let $x,y \in Q^k$. x,y are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

THEOREM 4.2.1. Order equivalence is an equivalence relation on Q^k. I.e., for all $x,y,z \in Q^k$,
i. x,x are order equivalent.
ii. x,y are order equivalent if and only if y,x are order equivalent.
iii. If x,y are order equivalent and y,z are order equivalent, then x,z are order equivalent.

Equivalence relations on a nonempty set classify the elements of the set into one or more "equivalence classes", and these equivalence classes are pairwise disjoint. Normally, the number of equivalence classes is generally much smaller than the number of elements.

GEOGRAPHY: "born in the same country" is an equivalence relation on people. It classifies people by their countries of origin. The number of countries is far smaller than the number of people.

Let's see how this order equivalence definition works for small k.

Any two elements of Q^1 are order equivalent. So there is exactly one kind of element of Q^1 according to order equivalence. In general, it is customary to write A^1 as simply A.

Let $x \in Q^2$. There are three possibilities.
a. $x_1 = x_2$.
b. $x_1 < x_2$.
c. $x_1 > x_2$.

Two elements of Q^2 are order equivalent if and only if they lie in the same one of these three categories.

Let $x \in Q^3$. Given $x = (x_1,x_2,x_3)$, there are 13 possibilities. First of all, there are the above three possibilities for x_1,x_2. Under a, x_3 can go in one of three positions - equal to $x_1 = x_2$, below $x_1 = x_2$, or above $x_1 = x_2$. Under b, x_3 can go in one of five positions - below x_1, at x_1, strictly between x_1,x_2, at x_2, and above x_2. Under c, x_3 can go in one of five positions - below x_3, at x_3, strictly between x_3,x_1, at x_1, and above x_1. So there are 3+5+5 = 13 possibilities. So x,y are order equivalent if and only if they are in the same one of these 13 categories.

Let $x \in Q^4$. Given $x = (x_1,x_2,x_3,x_4)$, there are 75 possibilities. Verify this by considering all 13 possibilities of the x_1,x_2,x_3, and for each one of the 13, considering all of the varying number of possibilities for the placement of x_4. You can save yourself a lot of work if you realize how to organize the 13 possibilities of the x_1,x_2,x_3 into three groups, where the varying number of possibilities for the placement of x4 is the same for any x_1,x_2,x_3 in that same one of the three groups. Then it all adds up to 75 (verify).

In our main development, we will not need to consider any tuples of higher length than 4.

THEOREM 4.2.2. The number of equivalence classes under order equivalence from Q^k is the same as the number of equivalence classes under order equivalence from A^k, where $A \subseteq Q$ has at least k elements. In particular, the number of equivalence classes under order equivalent from Q^k is the same as the number of equivalence classes under order equivalence from $\{1,2,...,k\}^k$. It is greater than the number of equivalence classes under order equivalence from $\{1,2,...,i\}^k$, no matter what integer $0 \leq i \leq k-1$ we choose.

The number of equivalence classes under order equivalence on Q^k (or $\{1,...,k\}^k$) is written ot(k), for "order types of dimension k". Thus we have seen that ot(1) = 1, ot(2) = 3, ot(3) = 13, ot(4) = 75.

The name preferential arrangement is used in the literature. E.g., see https://www.jstor.org/stable/2312725?seq=1 The preferential arrangements of 1,2,...,k are in nice one-one correspondence with the equivalence classes of k-tuples from {1,...,k} under order equivalence, which has ot(k) equivalence classes.

ot(1) = 1, ot(2) = 3, ot(3) = 13, ot(4) = 75, ot(5) = 541, ot(6) = 4,683, ot(7) = 47,293, ot(8) = 545,835, ot(9) = 7,087,261, ot(10) = 102,247,563, ot(11) = 1,622,632,573, ot(12) = 28,091,567,595, ot(13) = 526,858,348,381, ot(14) = 10,641,342,970,443.

The above is from O.A. Gross, Preferential Arrangements, American Mathematical Monthly, 1962.

ot(k) is also the number of ways a horse race with k horses can end with ties allowed. How many with no ties allowed? k!. Prove this. See Elliott Mendelson (1982) Races with Ties, Mathematics Magazine, 55:3, 170-175, DOI: 10.1080/0025570X.1982.11976977

There is a great resource you should know about for numerical sequences. It is the Online Encyclopedia of Integer Sequences. There is an entry on these ot(k), and they are called Fubini Numbers. See https://oeis.org/A000670 about these Fubini numbers, ot(k).

Also see Elliott Mendelson (1982) Races with Ties, Mathematics Magazine, 55:3, 170-175, DOI: 10.1080/0025570X.1982.11976977

4.3. EMULATORS

DEFINITION 4.3.1. Let x,y be two finite sequences. xy is the concatenation of x with y, and is obtained by taking x and continuing with y. Thus the length of xy is the sum of the lengths of s and y.

DEFINITION 4.3.2. An emulator of $E \subseteq Q[-1,1]^2$ is an $S \subseteq Q[-1,1]^2$ such that any pair of elements of S "look like" some pair of elements of E. I.e., for all $x,y \in S$ there exists $z,w \in E$ such that xy is order equivalent to zw. Note that we are allowing E to be finite or infinite, or even $\emptyset$.

Bear in mind here that the $x,y \in S$ and the $z,w \in E$ are each a pair of rationals, (a,b). So we are using xy and zw which are both 4-tuples of rational numbers. Hence we are using order

equivalence on $Q[-1,1]^4$. We will not generally need to consider order equivalence for longer tuples. This is because we will be focusing on the space $Q[-1,1]^2$, which is rich enough to challenge us.

We don't have to worry about large $E \subseteq Q[-1,1]^2$ because of the following.

THEOREM 4.3.1. Every $E \subseteq Q[-1,1]^2$ has the same emulators as some $E' \subseteq E$ of cardinality ≤ 150.

Proof: We find $E' \subseteq E$ such that every xy, $x,y \in E$, is order equivalent to some zw, $z,w \in E'$. (Why would such an E' have the same emulators as E?) We can construct E' with at most ot(4) pairs of elements of E thrown into E'. (How?) So $|E'| \leq 150 = 2ot(4)$. QED

OPEN PROBLEM A. What is the least n such that we can replace 150 with n in Theorem 4.3.1? Or at least (substantially) reduce 150.

EXERCISES. Prove the following. Assume each $E \subseteq Q[-1,1]^2$. I.e., we are working in $Q[-1,1]^2$.
1. Every subset of E is an emulator of E.
2. If S is an emulator of E and T is an emulator of S then T is an emulator of E.
3. Every emulator of $\emptyset$ is $\emptyset$.
4. If $|E| \leq 1$ then every emulator of E has at most one element.
5. There exists E with $|E| = 2$ with an emulator of cardinality 2 but no greater.
6. There exists E with $|E| = 2$ with an infinite emulator.
7. Any E with $|E| = 2$ that has an emulator of cardinality 3 has an infinite emulator.
8. There exists E with $|E| = 3$ whose emulators all have at most 3 elements.
9. Any E with $|E| = 3$ that has an emulator of cardinality 4 has an infinite emulator.
10. Every E with $|E| = 4$ has an infinite emulator.
11. There is a finite set E such that every S is an emulator of E.
12. We can strengthen "infinite emulator" in 6,7,9,10 to "algorithmic infinite emulator".

OPEN PROBLEM B. What is the smallest cardinality m of such an E in 11? What is the relationship between the n in Open Problem A and this m here?

We now define an equivalence relation (on subsets of $Q[-1,1]^2$).

E is related to S if and only if E is an emulator of S and S is an emulator of E.

OPEN PROBLEM. How many equivalence classes does this equivalence relation have?

OPEN PROBLEM. What are the relationships between the four Open Problems above in this section?

NOTE: Since Emulation Theory is so new, coming into clear and concise form only in 2020, Open Problem really means, at the moment, that either I have not found the time to look seriously at it, or I have looked at it with some seriousness but see some significant obstacles.

4.4. ORDER INVARIANT, ORDER THEORETIC, SEMI LINEAR, ALGORITHMIC SUBSETS OF $Q[-1,1]^2$

An arbitrary subset of $Q[-1,1]^2$ is generally a rather complicated, unruly, wild, impenetrable mathematical object. Essentially nothing mathematically interesting can be said about them in general. Over time mathematicians have gotten very interested in certain naturally defined categories of subsets of $Q[-1,1]^2$, and have uncovered interesting properties that they have. They have developed techniques for showing that certain sets are not present in certain categories.

These explorations are normally conducted for subsets of X^k, where X is a mathematical domain with natural operations defined on it. Four very common spaces are the Z^k, Q^k, $\Re^k$, C^k, where $Z,Q,\Re,C$ are the integers, rationals, reals, and complex numbers, incorporating addition for the first two, and both addition and multiplication for the last two.

TINY INTRO. The sets definable using predicate logic on the ordered groups $Z,Q,\Re,C$ are very well behaved, whereas on the ordered ring Z and the ordered field Q, they are horribly behaved. They are also very well behaved on the ordered fields $\Re,C$.

Here we use our $Q[-1,1]^2$ which is very much like Q^2 for present purposes. It turns out that the "nice" subsets of $Q[-1,1]^2$ are the intersections of the "nice" subsets of Q^2 with $Q[-1,1]^2$.

ORDER INVARIANT. $S \subseteq Q[-1,1]^2$ is order invariant if and only if for all order equivalent $x,y \in Q[-1,1]^2$, $x \in S \leftrightarrow y \in S$.

Order Invariance is extremely restrictive.

THEOREM 4.4.1. The order invariant $S \subseteq Q[-1,1]^2$ are the following $8 = 2^3$.
i. $\emptyset$.
ii. $Q[-1,1]^2$.
iii. $\{(p,q) \in Q[-1,1]^2: p = q\}$.
iv. $\{(p,q) \in Q[-1,1]^2: p < q\}$.
v. $\{(p,q) \in Q[-1,1]^2: q < p\}$.
vi. $\{(p,q) \in Q[-1,1]^2: p \leq q\}$.
vii. $\{(p,q) \in Q[-1,1]^2: q \leq p\}$.
viii. $\{(p,q) \in Q[-1,1]^2: p \neq q\}$.

There is another way to look at order invariant $S \subseteq Q[-1,1]^2$. We can relate them to the first lecture a week ago today.

Consider the statements $p < q$, $q < p$. We can build up more statements by using the five common logical connections "not", "and", "or", "if then", "if and only if". They are written as $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. For example,

$$(q \wedge (p \rightarrow (\neg q))) \leftrightarrow (p \vee (q \rightarrow (\neg p)))$$

just a nasty uninteresting mess for illustrative purposes only. So we consider what we call the $S \subseteq Q[-1,1]^2$ given by propositional combinations of $p < q$ and $q < p$.

THEOREM 4.4.2. Write $\wedge,\rightarrow,\leftrightarrow$ in terms of $\neg,\vee$. Write $\vee,\rightarrow,\leftrightarrow$ in terms of $\neg,\wedge$. Write $\wedge,\vee,\leftrightarrow$ in terms of $\neg,\rightarrow$. None of $\wedge,\vee,\rightarrow$ be written in terms of $\neg,\leftrightarrow$.

Proof: Prove this. Precise statements of these results and proofs are standard in math logic courses. QED

THEOREM 4.4.3. Let $S \subseteq Q[-1,1]^2$. The following are equivalent.
i. S is order invariant.
ii. S is the set of all $(p,q) \in Q[-1,1]^2$ satisfying a propositional combination of clauses $p < q$, $q < p$.

ORDER THEORETIC. $S \subseteq Q[-1,1]^2$ is order theoretic if and only if S is the set of all $(p,q) \in Q[-1,1]^2$ satisfying a propositional

combination of clauses $p < q$, $q < p$, $c < p$, $p < c$, $c < q$, $q < c$, where c is various constants from Q[-1,1].

For instance, we get vertical and horizontal line segments in $Q[-1,1]^2$ is $\{(p,q) \in Q[-1,1]^2: p = c\}$, $\{(p,q) \in Q[-1,1]^2: q = c\}$. They are order theoretic.

THEOREM 4.4.4. For order theoretic subsets of Q[-1,1], we can allow the constants c to be from Q, and not just from Q[-1,1].

THEOREM 4.4.5. Every infinite order theoretic $S \subseteq Q[-1,1]^2$ contains a vertical line segment, a horizontal line segment, or a line segment in the line $y = x$.

The next level that we discuss is piecewise linear.

PIECEWISE LINEAR. $S \subseteq Q^n$ is piecewise linear if and only if S is the set of all $(p_1,...,p_n)$ such that $p_1,...,p_n$ satisfies a propositional combination of linear inequalities in n variables with coefficients (including constant coefficients) from Q.

THEOREM 4.4.6. $\{(x,y) \in Q[-1,1]^2: y = -x\}$ is piecewise linear but not order theoretic.

THEOREM 4.4.7. Let $S \subseteq Q^n$ be piecewise linear. The set of first (second) coordinates of elements of S is a finite union of intervals whose endpoints are from $Q \cup \{-\infty,\infty\}$.

In fact, the method in Model Theory of Quantifier Elimination actually proves something much stronger. Any subset of Q that is definable in predicate logic (lecture 1) over the structure $(Q,+,<)$ is a finite union of intervals in Q whose endpoints are from $Q \cup \{-\infty,\infty\}$.

Next is semi algebraic, a more common name than piecewise algebraic.

CAUTION: Semi algebraic is normally discussed using bigger fields than Q.

SEMI ALGEBRAIC. $S \subseteq Q[-1,1]^n$ is semi algebraic if and only if S is the set of all $(p_1,...,p_n)$ such that $p_1,...,p_n$ satisfies a propositional combination of polynomial inequalities in n variables with coefficients (including constant coefficients) from Q.

THEOREM 4.4.8. $\{p: p^2 < 2\}$ is semi algebraic but not piecewise linear.

THEOREM 4.4.9. Let $S \subseteq Q^n$ be semi algebraic. The set of first (second) coordinates of elements of S is a finite union of intervals in Q. However, the endpoints may not be in $Q \cup \{-\infty,\infty\}$.

Quantifier Elimination for $(Q,+,\bullet,<)$ fails. In fact, there are subsets of Q definable in predicate logic (lecture 1) over $(Q,+,\bullet,<)$ which are not a finite union of intervals in Q.

Semi algebraic sets are wonderful just like the piecewise linear sets. The trouble here is that the right setting for the algebraic sets is a larger field than Q. $\Re$ works well, and there is QE for $(\Re,<,+,-,\bullet,0,1)$, and Theorem 4.4.7 holds. The real algebraic numbers form the smallest subfield of $\Re$ where semi algebraic sets work just as well.

Now we go all the way up to algorithmic.

ALGORITHMIC $S \subseteq Q[-1,1]^2$. There is an algorithm for determining whether a given element of $Q[-1,1]^2$ is or is not an element of S.

THEOREM 4.4.10. There is an algorithmic subset of $Q[-1,1]^2$ which is not a subset or a superset of any infinite semi algebraic subset of $Q[-1,1]^2$.

There is an enormous jump from semi algebraic $S \subseteq Q[-1,1]^2$ to algorithmic $S \subseteq Q[-1,1]^2$. The usual intermediate steps are treated in terms of another subject, computational complexity, in theoretical computer science. There are interesting computational complexity issues that arise in our Emulation Theory.

Perhaps the most famous category except for algorithmic is "polynomial time computable", but we won't work with this here.

Of course the jump from algorithmic $S \subseteq Q[-1,1]^2$ to arbitrary $S \subseteq Q[-1,1]^2$ is also gigantic. There are only countably many of the former, but there are uncountably many, and in fact continuumly, many of the latter.

4.5. MAXIMAL EMULATORS

DEFINITION 4.5.1. S is a maximal emulator of $E \subseteq Q[-1,1]^2$ if and only if S is an emulator of $E \subseteq Q[-1,1]^2$ and no proper superset of S is an emulator of $E \subseteq Q[-1,1]^2$.

THEOREM 4.5.1. S is a maximal emulator of $E \subseteq Q[-1,1]^2$ if and only if S is an emulator of $E \subseteq Q[-1,1]^2$ where no new element can be added to S and S remain an emulator of $E \subseteq Q[-1,1]^2$.

It is convenient to write $\cup.$ for disjoint union. I.e., when we write A $\cup.$ B we mean $A \cup B$ with the understanding that A,B are disjoint (i.e., $A \cap B = \emptyset$).

THEOREM 4.5.2. Every $E \subseteq Q[-1,1]^2$ has a maximal emulator.

Proof: List the elements of $Q[-1,1]^2$ without repetition, $x_1,x_2,\ldots$. Create the maximal emulator S in stages, $\emptyset = S_0 \subseteq S_1 \subseteq \ldots$, taking $S = \cup_i S_i$. Suppose S_i, $i \geq 0$, has been constructed. Take $S_{i+1} = S_i \cup \{x_1\}$ if this is an emulator of E; S_i otherwise. Prove that S is an emulator of E. Prove that S is a maximal emulator of E by assuming that some S $\cup.$ $\{x_i\}$ is an emulator of E, and obtaining a contradiction. QED

There is an important generalization.

THEOREM 4.5.3. For all $E \subseteq Q[-1,1]^2$, every emulator of E is contained in a maximal emulator of $E \subseteq Q[-1,1]^2$.

Proof: Let X be an emulator of $E \subseteq Q[-1,1]^2$. List the elements of $Q[-1,1]^2$ without repetition, $x_1,x_2,\ldots$. Create the maximal emulator $S \supseteq X$ in stages, $X = S_0 \subseteq S_1 \subseteq \ldots$, taking $S = \cup_i S_i$. Suppose S_i has been constructed, $i \geq 0$. Take $S_{i+1} = S_i \cup \{x_i\}$ if this is an emulator of E; S_i otherwise. Prove that S is a maximal emulator of E by assuming that some S $\cup.$ $\{x_i\}$ is an emulator of E, and obtaining a contradiction. QED

THEOREM 4.5.4. There is an algorithm that determines, for two finite sets $S,E \subseteq Q[-1,1]^2$, whether S is an emulator of $E \subseteq Q[-1,1]^2$.

THEOREM 4.5.5. Every $E \subseteq Q[-1,1]^2$ has an algorithmic maximal emulator.

Proof: By Theorem 4.3.1 we can assume that $|E| \leq 150$. Follow the proof of Theorem 4.5.2, and see that the construction there can be carried out algorithmically using Theorem 4.5.4. QED

THEOREM 4.5.6. E = {(-1,1),(0,1/2)} has no order theoretic maximal emulator. However, it has a piecewise linear maximal emulator.

Proof: Let S be an order theoretic maximal emulator of E. Prove that S is infinite. By Theorem 3.1.4.6, S contains a vertical line segment, a horizontal line segment, or a line segment in the line y = x. Derive a contradiction. The second claim will be proved in lecture 5. QED

THEOREM 4.5.7. E = {(-1,0),(1/2,1)) has no semi algebraic maximal emulator except {(-1,1)}.

Proof: Let S be such other than {(-1,1)}. Then S must be infinite. Furthermore the set of first coordinates of its elements cannot contain a nontrivial interval. Now apply Theorem 4.4.9 to obtain a contradiction. QED

Again there is an important generalization of Theorem 4.5.5s.

THEOREM 4.5.7. For all $E \subseteq Q[-1,1]^2$, every finite emulator of $E \subseteq Q[-1,1]^2$ is contained in an algorithmic maximal emulator of E.

Proof: Follow the proof of Theorem 4.5.3 and use Theorem 4.5.4. QED

THEOREM 4.5.8. If S is a maximal emulator of $E \subseteq Q[-1,1]^2$ then S is a maximal emulator of S. There exists an emulator S of some $E \subseteq Q[-1,1]^2$ such that S is a maximal emulator of S but S is not a maximal emulator of E.

Determine which of the following statements is true for all $E,S \subseteq Q[-1,1]^2$:

If S is a (maximal) emulator of E and S' is a (maximal) emulator of S then S' is a (maximal) emulator of E.

In the above, there are 8 statements according to whether you choose "maximal" or not, with three independent choices. Prove or give counterexamples to all 8 statements.

Below all sets are subsets of $Q[-1,1]^2$.

OPEN PROBLEM. How many different maximal emulators can a set E have?
OPEN PROBLEM. How many different maximal emulators of E containing E can a set E have?
OPEN PROBLEM. What can the set of cardinalities of the maximal emulators of a set be?
OPEN PROBLEM. What can the set of cardinalities of the maximal emulators containing E can a set E have?
OPEN PROBLEM. What are the cardinalities of the sets S such that S is a maximal emulator of S?

How can we strengthen Theorem 4.5.5?

4.6. PURELY STABLE AND NEGATIVELY STABLE SUBSETS OF $Q[-1,1]^2$

DEFINITION 4.6.1. $S \subseteq Q[-1,1]^2$ is purely stable if and only if the following holds.
$(0,0) \in S$ if and only if $(1,1) \in S$.

DEFINITION 4.6.2. $S \subseteq Q[-1,1]^2$ is negatively stable if and only if the following holds.
1. $(0,0) \in S$ if and only if $(1,1) \in S$.
2. For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.
3. For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$.

Typical cases of clauses 2,3 are

$(0,-1/2) \in S \leftrightarrow (1,-1/2) \in S$.
$(-1/2,0) \in S \leftrightarrow (-1/2,1) \in S$.

These definitions of stable and negatively stable are obtained from taking the official definitions of stable and negatively stable for $S \subseteq Q[-n,n]^k$ from the professional manuscript, and specializing them to $S \subseteq Q[-1,1]^2$ and simplifying them (without changing their meaning). Our two dimensions (ordered pairs) is simpler than dimension k, and also -1,1 is simpler than -n,n.

On Friday, I will say more about the higher dimensional version which we claim cannot be proved or refuted in ZFC.

THEOREM 4.6.1. There are continuumly many negatively stable $S \subseteq Q[-1,1]^2$.

Order invariance is much stronger than negative stability.

THEOREM 4.6.2. There are exactly 8 order invariant $S \subseteq Q[-1,1]^2$ (Theorem 4.4.1). They are all negatively stable. Any subset of $Q[-1,0)^2$ is negatively stable.

Here is a sometimes useful sufficient condition for negative stability.

THEOREM 4.6.3. Suppose $S \subseteq Q[-1,1]^2$, where fld(S\{(0,1)} is disjoint from {0,1}. Then S is negatively stable.

I.e., assume that in S, if we throw away the single pair (0,1), then we do not encounter 0 or 1. Then S is negatively stable.

4.7. PURELY STABLE, NEGATIVELY STABLE MAXIMAL EMULATORS

If we just want pure stability, things are very straightforward in $Q[-1,1]^2$:

THEOREM 4.7.1. Every $E \subseteq Q[-1,1]^2$ has an algorithmic purely stable maximal emulator.

Proof: Let $E \subseteq Q[-1,1]^2$. Argue by three cases according to no $(p,p) \in E$, exactly one $(p,p) \in E$, and at least two $(p,p) \in E$. QED

Can we improve Theorem 4.7.1 by sharpening "algorithmic"? Order theoretic is out of the question. This is already an issue

THEOREM 4.7.2. $\{(0,0)\} \subseteq Q[-1,1]^2$ does not have an order invariant maximal emulator. All of its maximal emulators have cardinality 1.

THEOREM 4.7.3. {(-1,1),(0,1/2)} does not have a semi algebraic maximal emulator.

But we do have this:

THEOREM 4.7.3. (Friedman) Every $E \subseteq Q[-1,1]^2$ has a negatively stable maximal emulator.

The only proof that we have of Theorem 4.7.3 uses some advanced machinery involving uncountable length transfinite recursion, and certainly does not establish that the negatively stable maximal emulator can be made algorithmic.

OPEN PROBLEM. Does every $E \subseteq Q[-1,1]^2$ have an algorithmic negatively stable maximal emulator?

THEOREM. Not every $E \subseteq Q[-1,1]^2$, $|E| = 2$, has a piecewise linear negatively stable maximal emulator.

We will prove the above in lecture 5 using that the set of first (second) coordinates of elements of a piecewise linear set is a finite union of intervals (endpoint $\infty,-\infty$ allowed).

It is with this Open Problem that you have the clearest path to new advances in Emulation Theory. Let me explain.

On Friday, we are going to prove the following.

MAIN STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, has an algorithmic negatively stable maximal emulator.

CONJECTURE. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 4$, has an algorithmic negatively stable maximal emulator.

Can you prove this conjecture, or at least for some significant portion of these E's?

4.8. ORDER ISOMORPHISM AND COORDINATE SWITCHING

DEFINITION 4.8.1. f is an order isomorphism if and only if there exists $b > -1$ such that $f:Q[-1,1] \to Q[-1,b]$ is strictly increasing and has range $Q[-1,b]$. For $S \subseteq Q[-1,1]^2$, fld(S) is the field of S which is the set of all coordinates of elements of S.

DEFINITION 4.8.2. f is an order isomorphism from S onto S' if and only if f is an order isomorphism which maps fld(S) onto fld(S'), where for all $p,q \in$ fld(S), $(p,q) \in S \leftrightarrow (f(p),f(q)) \in S'$. $S,S' \subseteq Q[-1,1]^2$ are order isomorphic if and only if there is an order isomorphism from S onto S'.

THEOREM 4.8.1. Let $E,E' \subseteq Q[-1,1]^2$ be order isomorphic. Then E,E' have the same emulators. If S,S' are order isomorphic then S is an emulator of E if and only if S' is an emulator of E.

THEOREM 4.8.2. If $E,E' \subseteq Q[-1,1]^2$ have the same emulators then they have the same maximal emulators and the same negatively stable maximal emulators. There are $E \subseteq Q[-1,1]^2$ and order isomorphic $S,S' \subseteq Q[-1,1]^2$ such that S is a negatively stable maximal emulator of E and S' is not a maximal emulator of E.

So order isomorphisms of emulators are not enough to guarantee that we preserve emulator maximality. Order isomorphisms of sets are not enough to preserve negative stability (prove). We will fix this by using "global order isomorphisms" below. But first we show that we get a limited kind of maximality.

THEOREM 4.8.3. Let S be a maximal emulator of $E \subseteq Q[0,1]^2$ and $f:Q[-1,1] \to Q[-1,b]$ be an order isomorphism from S onto S'. Then there is no emulator $S' \cup. (c,d)$ of E with $c,d \leq b$.

Proof: Prove this. QED

DEFINITION 4.8.3. $S,S' \subseteq Q[0,1]^2$ are globally isomorphic if and only if there is an order isomorphism f from S onto S' such that $f(1) = 1$.

THEOREM 4.8.4. Let $E,S,S' \subseteq Q[-1,1]^2$. If S and S' are globally isomorphic then S is a maximal emulator of E if and only if S' is a maximal emulator of E.

Proof: Prove using Theorem 4.8.3. QED

DEFINITION 4.8.4. The coordinate switch of $(a,b) \subseteq Q^2$ is (b,a). The coordinate switch of $S \subseteq Q[-1,1]^2$ is the set of coordinate switches of its elements. We write csw((a,b)) = (b,a), and $csw(S) = \{csw(x): x \in S\}$.

THEOREM 4.8.5. Let $E,S \subseteq Q[-1,1]^2$. S is an emulator of E if and only if csw(S) is an emulator of csw(E). S is a maximal emulator if and only if csw(S) is a maximal emulator of csw(E). S is stable if and only if csw(S) is stable. S is negatively stable if and only if csw(S) is negatively stable. S is order theoretic, semi linear, algorithmic if and only if csw(S) is order theoretic, semi linear, algorithmic, respectively.

Now for some important application of order isomorphisms and global order isomorphisms.

THEOREM 4.8.6. Suppose $E \subseteq Q[-1,1]^2$ has a finite maximal emulator S whose field omits 1. Then E has a negatively stable maximal emulator of the same cardinality.

Proof: Make a global isomorphism whose image has field omitting both 0 and 1. QED

THEOREM 4.8.7. Suppose that among the finite emulators of $E \subseteq Q[-1,1]^2$ there is one of largest finite size. Then one of these of largest finite size is a negatively stable maximal emulator of $E \subseteq Q[-1,1]^2$.

Proof: Let S be a finite emulator of $E \subseteq Q[-1,1]^2$ of largest finite size $n \geq 0$. Make an order isomorphism of S onto $S' \subseteq Q[-1,1]^2$ whose field excludes 0,1. Then E' is a negatively stable maximal emulator of E of cardinality n (why?). QED

Using a topic in mathematical logic, we can easily sharpen Theorem 4.8.5 as follows.

THEOREM 4.8.8. Suppose every emulator of $E \subseteq Q[-1,1]^2$ is finite. There is a finite negatively stable maximal emulator of E.

Proof: Let E be as given. We claim that there is a largest size among the finite emulators of E. Suppose this is false. Then there are arbitrarily large finite emulators of $E \subseteq Q[-1,1]^2$. We can convert this to the context where the classical compactness theorem for countable models in first order predicate calculus with equality applies. In this way we get an infinite emulator of $E \subseteq Q[-1,1]^2$, contradicting the hypotheses. From the claim we are done by Theorem 4.8.7. QED