ADVENTURES IN FOUNDATIONS OF MATHEMATICS
Ross Program 2022
MWF during June 27 - July 8, 2022
1. Logical reasoning
2. Finite set theory
3. Infinite sets and ZFC
4. Order equivalence, emulators, and stability
5. Stability of emulators of two pairs
6. Stability of emulators of three pairs

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6. STABILITY OF EMULATORS OF THREE PAIRS

6.1. SOME USEFUL THEORY

DEFINITION 6.1.1. Let $A,B \subseteq Q[-1,1]^2$. An isomorphism from A onto B is a bijection $f:[-1,1] \to [-1,b]$, $fld(B) \subseteq [-1,b]$, such that for all $p,q \in [-1,1]$, $p < q \leftrightarrow f(p) < f(q)$ and $(p,q) \in A \leftrightarrow (f(p),f(q)) \in B$. A global isomorphism A onto B is an isomorphism from A onto B whose domain and range are Q[-1,1]. A,B ⊆ Q[-1,1]2 are isomorphic (globally isomorphic) if and only if there is a (global) isomorphism from A onto B. We say that f sends (p,q) to (f(p),f(q)).

LEMAM 6.1.1. Every global isomorphism of A onto B maps -1 to -1 and 1 to 1. We can compose isomorphisms and global isomorphisms and take inverses of global isomorphisms.

LEMMA 6.1.2. Every isomorphism from A onto B sends every pair to an order equivalent pair.

LEMMA 6.1.3. Let $E,S,S' \subseteq Q[-1,1]^2$. If S,S′ are isomorphic then S is an emulator of E if and only if S′ is an emulator of E. If S,S′ are globally isomorphic then S is a maximal emulator of E if and only if S′ is a maximal emulator of E. The latter fails if we remove "globally".

Proof: The first claim is easily proved using Lemma 6.1.2. For the second claim, we can't just work with E,S,S′ but rather need to work with supersets of S,S′. Hence the need for a global isomorphism. QED

CAUTION: Thus isomorphisms preserve emulators but not necessarily maximal emulators. Global isomorphisms do preserve maximal emulators. But global isomorphisms have the limitation that they must be the identity on -1 and 1.

DEFINITION 6.1.2. We say that $\{(p,q),(r,s),(t,u)\}$ has unique stray if and only if $(p,q),(r,s)$ are order equivalent, but not order equivalent to (t,u). We use $\cup.$ for disjoint union, meaning whenever we see $X \cup. Y$ we mean $X \cup Y$ and are asserting that X,Y are disjoint.

LEMMA 6.1.4. Let $E = \{(p,q),(r,s),(t,u)\}$ with unique stray. Let $S \subseteq Q[-1,1]^2$ be a maximal emulator of $\{(p,q),(r,s)\}$. All maximal emulators of E extending S have at most one element outside S. In addition,
i. S is a maximal emulator of $\{(p,q),(r,s)\}$; or
ii. There exists (b,c) order equivalent to (t,u) such that $S \cup. \{(b,c)\}$ is a maximal emulator.

Proof: Let E,S be as given. We first claim that there is no emulator $S \cup. \{(b,c)\} \cup. \{(d,e)\}$ of E. Suppose $S \cup. \{(b,c)\} \cup. \{(d,e)\}$ is an emulator of E. If (b,c) and (p,q) are order equivalent then $S \cup \{(b,c)\}$ is an emulator of $\{(p,q),(r,s)\}$, which is impossible. Hence (b,c) and (p,q) are not order equivalent and therefore (b,c) and (t,u) are order equivalent. By the same token, $\{(d,e)\}$ is also order equivalent to (t,u). Hence the pair $(b,c),(d,e)$ is not reflected in E, and so we have arrived at a contradiction.

Let $S \subseteq S'$ where S' is a maximal emulator of E. By the first claim, $S' \setminus S$ has at most 1 element. If no elements then S is already a maximal emulator of E, and we are done. if one element then write $S' = S \cup \{(b,c)\}$. By the same reasoning above, (b,c) and (t,u) are order equivalent. QED

DEFINITION 6.1.3. We say that $S \subseteq Q[-1,1]^2$ is bounded below 1 if and only if all of the coordinates of its elements are $< p$ for some fixed $p < 1$.

LEMMA 6.1.5. Let $S \subseteq Q[-1,1]^2$ be bounded below 1 and algorithmic. There is an algorithmic global isomorphism from S onto an algorithmic subset of $Q[-1,0)^2$.

Proof: Let $S \subseteq Q[-1,b]2$, $b < 1$, and let $f:Q[-1,1] \to Q[-1,1]$ be piecewise linear with two pieces, one piece from $Q[-1,b]$ one-one onto $Q[-1,-1/2]$ and the other piece from $Q[b,1]$ one-one onto $Q[-1/2,1]$, both sending b to $-1/2$. Then f is a global isomorphism from S onto $f[S]$, where $f[S] = \{(f(p),f(q)): p,q \in Q[-1,b]\}$. $f[S]$ is algorithmic. QED

LEMMA 6.1.6. If $E \subseteq Q[-1,1]^2$ has an algorithmic maximal emulator bounded below 1 then E has an algorithmic negatively stable maximal emulator.

Proof: Use a global isomorphism that maps E onto an algorithmic subset of $Q[-1,0)^2$ given by Lemma 6.1.5. Every subset of Q[-1,0) is trivially negatively stable. QED

LEMMA 6.1.7. Let E = {(p,q),(r,s),(t,u)} with unique stray, and {(p,q),(r,s)} have an algorithmic maximal emulator bounded below 1. Then E has an algorithmic negatively stable maximal emulator.

Proof: Let E be as given and let S be an algorithmic maximal emulator of {(p,q),(r,s)} that is bounded below 1. Let E ⊆ Q[-1,a]2, a < 1. By Lemma 6.1.4, S or some S ∪. {(b,c)} is a maximal emulator of E,, (b,c) order equivalent to (t,u). If S is a maximal emulator of E then apply Lemma 6.1.6. Otherwise, let (b′,c′) be an adjustment of (b,c} so that S ∪. {(b,c)} remains unchanged on Q[-1,a]2 but b′,c′ < 1 and (b,c),(b′,c′) are order equivalent. Then S ∪. {(b,c)} is isomorphic to S ∪. {(b′,c′)}, and so S ∪. {(b′,c′)} is an emulator of E. By Lemma 6.1.4, S ∪. {(B′,c′)} is an algorithmic maximal emulator of E which is bounded < 1. Now apply Lemma 6.1.5. QED

LEMMA 6.1.8. Let E = {(p,q),(r,s),(t,u)} with unique stray, where max(t,u) ≥ p,q,r,s. Then E has an algorithmic negatively stable maximal emulator.

Proof: Let E,t,u be as given. Let S be an algorithmic negatively stable maximal emulator of {(p,q),(r,s)} which is either bounded below 1 or in which 1 appears as a coordinate (from the previous lecture). By Lemma 6.1.7 we are done if S is bounded below 1. So we assume that 1 appears as a coordinate of S. We can also assume that S is not a maximal emulator of E. By Lemma 6.1.4, fix (b,c) order equivalent to (t,u), where S ∪. {(b,c)} is a maximal emulator of E. Obviously max(b,c) = 1.

Now let f:Q[-1,1] → Q[-1,-1/2] be the linear bijection fixing -1. It is clear that f maps S ∪. {(b,c)} onto an emulator S′ ∪. {(b′,c′)} of E, where max(b′,c′) = -1/2. We now claim that S′ ∪. {(b′,c′)} is a maximal emulator of E. To see this, we cannot add any (b*,c*) with max(b*,c*) > -1/2 because of max(t,u) ≥ p,q,r,s. Therefore the only possibility is that we can add some (b*,c*) with max(b*,c*) ≤ -1/2, But then we can use f to transfer this back to adding $(f^{-1}(b^*),f^{-1}(c^*))$ to S ∪. {(b,c)}, directly contradicting that S ∪. {(b,c)} is a maximal emulator of E. Obviously S′ ∪. {(b′,c′)} is negatively stable. QED

LEMMA 6.1.9. Let E = {(p,q),(r,s),(t,u)} with unique stray. Let S be a maximal emulator of {(p,q),(r,s)}. Let (b,c) be order equivalent to (t,u) and there exists x,y,z ∈ S\{(b,c)} such that (b,c),x and (b,c),y and (b,c),z are pairwise not order equivalent. Then S ∪. {(b,c)} is not an emulator of E.

Proof: Let E,b,c,x,y,z be as given. (b,c),x and (b,c),y and (b,c)z are each order equivalent to (b,c),(p,q) or (b,c),(r,s). This is impossible. QED

DEFINITION 6.1.4. Let $x,y,z,w \in Q[-1,1]^2$. We say that (x,y) and (z,w) are order/switch equivalent if and only if
i. xy and zw are order equivalent; or
ii. xy and wz are order equivalent.
Here xy is concatenation.

LEMMA 6.1.10. Let $E,S \subseteq Q[-1,1]^2$. S is an emulator of E if and only if every element of S^2 is order/switch equivalent to an element of E^2.

DEFINITION 6.1.5. Let $E \subseteq Q[-1,1]^2$. The profile of E is the set of all elements of E^2 up to order/switch equivalence.

Profiles are used only in the following way. We work with convenient sets of representatives of the equivalence relation order/switch equivalence on E^2. We require that the representatives all be not order/switch equivalent. We also require that the x,x be left off the list assuming x appears elsewhere in the list.

EXAMPLE: Let E = {(-1,p): -1 ≤ p ≤ 1}. The profile of E is given by the following list (among other lists).
(0,0),(0,1)
(0,1),(0,2)
Note that we left off (0,0),(0,0) and (0,1),(0,1) and (0,2),(0,2).

LEMMA 6.1.11. Let $E,E',S \subseteq Q[-1,1]^2$. S is an emulator of E if and only if the profile of S is contained in the profile of E (up to order/switch equivalence). E,E′ have the same emulators if and only if they have the same profile (up to order/switch equivalence).

LEMMA 6.1.12. Let S be an emulator of $E \subseteq Q[-1,1]^2$, $|E| \leq 3$. Then there does not exist four elements of S^2 which are pairwise order/switch inequivalent.

Proof: By Lemma 6.1.11. QED

LEMMA 6.1.13. Let S be a rational piecewise linear emulator of E $\subseteq Q[-1,1]^2$. There is an algorithmic negatively stable maximal emulator of E extending S.

Proof: This is by the usual construction using the well known decision procedure for the structure (Q,<). QED

We give a convenient complete set of representatives for order/switch equivalence. We give friendly names to the ones which friendly names will help. We will omit the x,x.

Notice the 6 headers ==, =<, =>, <<, >>, <>.

==

(0,0),(1,1)

=<

(0,0),(0,1) Low Equal/=<
(1,1),(0,1) High Equal/=<
(1,1),(0,2) Linker/<<
(0,0),(1,2) LowHigh/=<
(2,2),(0,1) HighLow/=<

=>

(0,0),(1,0) Low Equal/=>
(1,1),(1,0) High Equal/=>
(1,1),(2,0) Linker
(0,0),(2,1) LowHigh/=>
(2,2),(1,0) HighLow/=>

<<

(0,1),(0,2) Low Equal/<<
(0,2),(1,2) High Equal/<<
(0,1),(1,2) Toucher/<<
(0,2),(1,3) Linker/<<
(0,3),(1,2) Surrounder/<<
(0,1),(2,3) Disjointer/<<

>>

(1,0),(2,0) Low Equal/>>
(2,0),(2,1) High Equal/>>
(1,0),(2,1) Toucher/>>
(2,0),(3,1) Linker/>>
(3,0),(2,1) Surrounder/>>
(1,0),(3,2) Disjointer/>>

<>

(0,1),(2,0) Low Equal/<>
(0,2),(2,1) High Equal/<>
(0,1),(2,1) Toucher/<>
(0,2),(3,1) Linker/<<
(0,1),(3,2) Disjointer/<>
(0,3),(2,1) Surrounder/<>

We now give the following crude classification of the 3 element E ⊆ Q[-1,1]2. Here E = {(p,q),(r,s),(t,u)}.

= = =
= = <
= < <
= < >
< < <
< < >

Because of switching, we need not consider, e.g., = = >.

Three of these six above can be dealt with rather easily.

LEMMA 6.1.14. = = =. Every 3 element E = {(p,p),(q,q),(r,r)} has an algorithmic negatively stable maximal emulator.

Proof: {(p,p): -1 ≤ p ≤ 1} is an algorithmic negatively stable maximal emulator. QED

LEMMA 6.1.15. = < >. Every 3 element E = {(p,p),(r<s),(t>u)} has an algorithmic negatively stable maximal emulator.

Proof: We claim that every emulator of E has at most three elements. If there are more then some two elements are order equivalent. But (a,b),(c,d) cannot reflected in E if (a,b),(c,d) are distinct and order equivalent. Now choose p′,r′,s′,t′,u′ < 0 such that p,r,s,t,u and p′,r′,s′,t′,u′ are order equivalent.

Then $\{(p',p'),(r'.s),(t',u')\}$ is a maximal emulator of E that is obviously algorithmic and negatively stable. QED

LEMMA 6.1.16. $= = <$. Every 3 element $E = \{(p,p),(r,r),(t<u)\}$ has an algorithmic negatively stable maximal emulator.

Proof: Let $S = \{(p,p): -1 \leq p \leq 1\}$. Let $(b<c)$. Look at $(b<c),(b,b)$ and $(b<c),(c,c)$ and $(b<c),((b+c)/2,(b+c)/2)$. Each of these have to be order equivalent to some pair from E. Hence each of these have to be each order equivalent to at least one of $(t<u),(t<u)$, $(t<u),(p,p)$, $(t<u),(q,q)$, as the $(t<u)$ must come first. Hence $(t<u),(t<u)$ must be used, but that is impossible. QED

In the final three sections we respectively handle $< < >$, $< < =$, $< < <$.

6.2. $< < >$

We divide this by the profile of $\{(p<q),(r<s)\}$ using what we did in Lecture 4 when we treated $|E| = 2$.

(0,1),(0,2) Low Equal/<<
(0,2),(1,2) High Equal/<<
(0,1),(1,2) Toucher/<<
(0,2),(1,3) Linker/<<
(0,1),(2,3) Disjointer/<<
(0,3),(1,2) Surrounder/<<

LEMMA 6.2.1. Low Equal/<<. Let $E = \{(p,q),(p,r),(t,u)\}$, $p < q < r$, and $t > u$. E has an algorithmic negatively stable maximal emulator.

Proof: $S = \{(-1,a): -1 < a \leq 1\}$ is a maximal emulator of $\{(p,q),(p,r)\}$. We claim that S is a maximal emulator of E. Let $S \cup. \{(b,c)\}$ be an emulator of E. Then $b > c$ and $(b,c),(-1,c)$ and $(b,c),(-1,b)$ and $(b,c),(-1,(b+c)/2)$ are pairwise not order equivalent. By Lemma 6.1.9, $S \cup \{(b,c)\}$ is not an emulator of E. This is a contradiction. QED

LEMMA 6.2.2. High Equal/<<. Let $E = \{(p,r),(q,r),(t,u)\}$, $p < q < r$, and $t > u$. E has an algorithmic negatively stable maximal emulator.

Proof: $S = \{(a,-1/2): -1 \leq a < -1/2\}$ is a maximal emulator of $\{(p,q),(p,r)\}$ which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.2.3. Toucher/<<. Let E = {(p,r),(q,r),(t,u)} where p < q < r, and t > u. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(-1,-1/2),(-1/2,-1/3)} is a maximal emulator of {(p,r),(q,r)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.2.4. Linker/<<. Let E = {(p,q),(r,s),(t,u)} where p < r < q < s, and t > u. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(a,a+.4): -1 ≤ a < -.6} is a maximal emulator of {(p,q),(r,s)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.2.5. Surrounder/<<. Let E = {(p,q),(r,s),(t,u)} where p < r < s < q, and t > u. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(-1+a,-.5-a): 0 ≤ a < .25} is a maximal emulator of {(p,q),(r,s)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.2.6. Disjointer/<<. Let E = {(p,q),(r,s),(t,u)} where p < r < s < q, and t > u. E has an algorithmic negatively stable maximal emulator.

Proof: Let S be an algorithmic maximal emulator of {(p,q),(r,s)} containing (-1,-1/2),(0,1), as in Lecture 4 (there we didn't use (-1,-1/2)). Suppose t = 1. Apply Lemma 6.1.8. Now suppose t < 1. Then -1 ≤ u < t < 1. By Lemma 6.1.4, S is a maximal emulator of E or some S ∪. {b,c} is a maximal emulator of E, b > c. Both of these are negatively stable unless c < b = 0. In that case, (0,c),(-1,-1/2) and (0,c),(d,e) and (0,c),(0,1) are pairwise not order equivalent, with (d,e) chosen to lie in S, c < d < e < 0. This is impossible by Lemma 6.1.9. QED

6.3. < < =

We again divide this by the profile of {(p<q),(r<s)} using what we did in Lecture 4 when we treated |E| = 2.

(0,1),(0,2) Low Equal/<<
(0,2),(1,2) High Equal/<<
(0,1),(1,2) Toucher/<<
(0,2),(1,3) Linker/<<
(0,1),(2,3) Disjointer/<<
(0,3),(1,2) Surrounder/<<

LEMMA 6.3.1. Low Equal/<<. Let E = {(p,q),(p,r),(t,t)}, p < q < r. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(-1,a): -1 < a ≤ 1} is a maximal emulator of {(p,q),(p,r)}. If t ≥ r then we are done by Lemma 6.1.8. Now suppose t < r. By Lemma 6.2.4, S is a maximal emulator of E or some S ∪. (b,c) is a maximal emulator of E. We assume the latter. Then b = c < 1. Now if b ≠ -1 then (b,b),(-1,(b-1)/2) and (b,b),(-1,b) and (b,b),(-1,(b+1)/2) are pairwise not order equivalent. By Lemma 6.1.7, S ∪ {(b,b)} is not an emulator of E. Hence b = -1. So S ∪. (-1,-1) is a negatively stable maximal emulator of E. QED

LEMMA 6.3.2. High Equal/<<. Let E = {(p,q),(p,r),(t,t)}, p < q < r. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(a,-1/2): -1 ≤ a < -1/2} is a maximal emulator of {(p,q),(p,r)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.3.3. Toucher/<<. Let E = {(p,r),(q,r),(t,t)} where p < q < r. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(-1,-1/2),(-1/2,-1/3)} is a maximal emulator of {(p,r),(q,r)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.3.4. Linker/<<. Let E = {(p,q),(r,s),(t,t)} where p < r < q < s. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(a,a+.4): -1 ≤ a < -.6} is a maximal emulator of {(p,q),(r,s)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.3.5. Surrounder/<<. Let E = {(p,q),(r,s),(t,t)} where p < r < s < q. E has an algorithmic negatively stable maximal emulator.

Proof: S = {(-1+a,-.5-a): 0 ≤ a < .25} is a maximal emulator of {(p,q),(r,s)} which is bounded below 1. Apply Lemma 6.1.7. QED

LEMMA 6.3.5. Disjointer/<<. Let E = {(p,q),(r,s),(t,t)} where p < q < r < s. E has an algorithmic negatively stable maximal emulator.

Proof: Let S be an algorithmic maximal emulator of {(p,q),(r,s)} containing (-1,-1/2),(0,1/3),(1/2,1) as in Lecture 4 (there we

didn't use (-1,-1/2),(1/2,1)). Suppose t = 1. Apply Lemma 6.1.8. Now suppose $-1 \leq t < 1$. By Lemma 6.1.2, S is a maximal emulator of E or S ∪. {t,t} is a maximal emulator of E. Both of these are negatively stable unless t = 0. In that case, (0,0),(-1,-1/2) and (0,0),(0,1/3) and (0,0),(1/2,1) are pairwise not order equivalent. This is impossible by Lemma 6.1.9. QED

6.4. < < <

We don't have strays as we did in sections 6.2, 6.3. We will only need to work with these:

(0,1),(0,2) Low Equal/<<
(0,2),(1,2) High Equal/<<
(0,1),(1,2) Toucher/<<

In this final section, we only use $(p,q) \in Q[-1,1]^2$ with $p < q$. E always is a 3 element subset of $Q[-1,1]^2$ with $(p,q) \in E \rightarrow p < q$.

All profiles of the E must be among the 3 element subsets of these six above. We have to handle all of these profiles.

LEMMA 6.4.1. Let E have a Low Equal/<< but no High Equal/<<. Then E has an algorithmic negatively stable maximal emulator.

Proof: Let $S = \{(-1,p): -1 < p \leq 1\}$. By Lemma 6.? let S' be an algorithmic negatively stable emulator of E extending S. Clearly there are no (p,0) or (p,1) ∈ S' with p < 0, and (-1,0),(-1,1) ∈ S. Therefore S' is negatively stable. QED

LEMMA 6.4.2. Let E have a Low Equal/<< and a High Equal/<<. Then E has an algorithmic negatively stable maximal emulator.

Proof: Write E = {(p,q),(p,r),(s,r)}, where p ≠ q,r,s. Let $S = \{(-1,p): -1 < p \leq 1\}$. Suppose S ∪. {(b,c)} is an emulator of E. Obviously b ≠ -1. Then (-1,(b+c)/2) and (b,c) have no coordinates in common. Therefore S ∪ {(b,c)} is not an emulator of E. Hence S is a maximal emulator of E. QED

LEMMA 6.4.3. Let E have a Low Equal/<<. Then E has an algorithmic negatively stable maximal emulator.

Proof: By Lemmas 6.4.1 and 6.4.2. QED

LEMMA 6.4.4. Let E have a High Equal/<<. Then E has an algorithmic negatively stable maximal emulator.

Proof: If E has a Low Equal/<< then we are done by Lemma 6.4.3. Assume E has no Low Equal/<<. Let $S = \{(p,1/2): -1 \leq p \leq ½\}$. Let S′ be an algorithmic maximal emulator of E extending S. Clearly no $(p,0)$, $p < 0$, lies in S, and no $(p,1)$, $p < 0$, lies in S. Hence S′ is negatively stable. QED

Henceforth we assume E has no Low Equal/<< and no High Equal/<<.

LEMMA 6.4.5. Let E have no Toucher/<<. Then E has an algorithmic negatively stable maximal emulator.

Proof: S be an algorithmic maximal emulator extending $(0,1)$. There is no $(p,0) \in S$, $p < 0$ (no Toucher/<<), and there is no $(p,1) \in S$, $p < 0$ (no High Equal/<<). Hence S is negatively stable. QED

LEMMA 6.4.6. Let E have a Toucher/<<.

Proof: Let S be an algorithmic maximal emulator extending $(1/2,0),(0,1)$. There is no $(p,0) \in S$, $p < 0$ (no High equal/<<), and there is no $(p,1) \in S$, $p < 1$, $p < 0$ (no High equal/<<). Hence S is negatively stable. QED

STUDENT THEOREM. Every ≤ 3 element subset of $Q[-1,1]^2$ has an algorithmic negatively stable maximal emulator.

Proof: Lemmas 6.4.1 – 6.4.6 cover the < < < cases. The < < > cases is handled by section 6.2. The < < = cases is handled by section 6.3.