ADVENTURES IN FOUNDATIONS OF MATHEMATICS
Ross Program 2022
MWF during June 27 - July 8, 2022
1. Logical reasoning
2. Finite set theory
3. Infinite sets and ZFC
4. Order equivalence, emulators, and stability
5. Stability of emulators of two pairs
6. Stability of emulators of three pairs

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7. FINAL TALK
July 22, 2022

In this farewell talk I won't give any proofs but just summarize what we have done on Negatively Stable Maximal Emulators, and mention several research challenges.

I also want to explain the higher dimensional form of Negatively Stable Maximal Emulators.

Recall the crucial definition of emulator.

DEFINITION 1. S is an emulator of $E \subseteq Q[-1,1]^2$ if and only if the concatenation of any two elements of S is order equivalent to the concatenation of some two elements of E.

This immediately leads to some counting problems. To make these problems more doable, you can restrict the cardinality of the $E \subseteq Q[-1,1]^2$. Especially relevant is ≤ 3 for 1,2,4 below.

1. How many equivalence classes are there for the equivalence relation "E_1 and $E_2 \subseteq Q[-1,1]^2$ have the same emulators"? The same maximal emulators? The same unique maximal emulators?

2. How many emulators can $E \subseteq Q[-1,1]^2$ have? Maximal emulators? Unique maximal emulators?

3. What is the smallest size of a set $E \subseteq Q[-1,1]^2$ such that every $S \subseteq Q[-1,1]^2$ is an emulator of E?

4. What is the smallest number n such that every $E \subseteq Q[-1,1]^2$ has a subset of cardinality $\leq n$ with the same emulators? The same maximal emulators? The same unique maximal emulators?

Here is the crucial definition of negatively stable.

DEFINITION 2. $S \subseteq Q[-1,1]^2$ is negatively stable if and only if for all $p \in Q[-1,0)$,
$(0,0) \in S \leftrightarrow (1,1) \in S$
$(p,0) \in S \leftrightarrow (p,1) \in S$
$(0,p) \in S \leftrightarrow (1,p) \in S$

We proved this in lecture 5:

NEGATIVELY STABLE MAXIMAL EMULATOR/2. NSME/2. Every ≤ 2 element subset of $Q[-1,1]^2$ has an algorithmic negatively stable maximal emulator.

We proved this in lecture 6 (I had to revise those notes to its present form in order to debug the proof).

NEGATIVELY STABLE MAXIMAL EMULATOR/3. NSME/3. Every ≤ 3 element subset of $Q[-1,1]^2$ has an algorithmic negatively stable maximal emulator.

How "good" can we make these negatively stable maximal emulators? Good means simple. Algorithmic is kind of good but still is very very broad. The normal situation in concrete mathematics is that countable subsets of simple objects are much much better than merely algorithmic.

We talked about these levels, better and better. Typical spaces to discuss these are the Q^k, but it makes perfect sense using our $Q[-1,1]^2$ or even $Q[-1,1]^k$.

a. Algorithmic. Formalized by models of computation like originally Turing machines. Famous old vague issue: Church's Thesis.
b. Various complexity classes studied intensively in theoretical computer science. Most famous ones are P (polynomial time), NP (nondeterministic polynomial time), PSPACE (deterministic polynomial space). Believed that P $\subseteq\neq$ NP $\subseteq\neq$ PSPACE, and known without $\neq$. It is known that deterministic and nondeterministic polynomial space are the same.
c. Propositional combinations of polynomial inequalities with rational coefficients. Often called semi algebraic. Most often done in the $\Re^k$ with real coefficients.
d. Propositional combinations of linear inequalities with rational coefficients. Usually called piecewise linear. Can also be done in $\Re^k$ with real coefficients.

e. Propositional combinations of pure inequalities x < y, c < x, x < c, where c's are constants. I call these order theoretic. Makes good sense in any linear ordering.
f. Propositional combinations of pure inequalities x < y (no constants allowed). Makes good sense in any linear ordering. Given a linear ordering, there are only finitely many of these in each dimension. These are called order invariant.

5. In NSME/2, how good (scale from a-f) can we get the negatively stable maximal emulator to be? What about in NSME/3?

6. Revisit the counting problems 1-4 above with emulator modified to include various combinations of:

maximal
uniquely maximal
negatively stable maximal
any of a,b,c,d,e,f

I have proved using transfinite machinery:

NEGATIVELY STABLE MAXIMAL EMULATOR DIM 2. NSME(dim 2). Every subset of $Q[-1,1]^2$ has a negatively stable maximal emulator.

But nowhere near being able to put "algorithmic" there. So can you add algorithmic in NSME(dim 2)?

How about |E| = 4? Is this true?

NEGATIVELY STABLE MAXIMAL EMULATOR/4. NSME/4. Every ≤4 element subset of $Q[-1,1]^2$ has an algorithmic negatively stable maximal emulator.

The idea is to extend the ideas in the proof of NSME/3 (lecture 6) and add new ideas, to go from 3 to 4.

If you look at lecture 6, the case < < < was to me much simpler than I expected. So I would start looking at the case < < < < and see if that can be made to work in a similar way. I haven't looked at this.

Even before that, there must be some low hanging fruit here with obviously
= = = =
= = = <

I have no idea how hard = = < < is. Or = = < >. I would think that the ones starting with = = could be manageable.

All of this is completely untouched. I haven't the slightest idea how difficult all of this will prove to be.

Now staying in two dimensions, we are about to mix things up drastically in two ways.

A. We are first going to use $Q[-n,n]^2$ instead of $Q[-1,1]^2$.
B. We are going to use stronger kinds of Emulators.

We now use $Q[-n,n]^2$. The notion of emulator is unchanged. But we modify negatively stable S as follows.

DEFINITION 3. $S \subseteq Q[-n,n]^2$ is negatively stable if and only if for all $p \in Q[-n,0)$ and for all $i,j \in \{0,...,n-1\}$,
$(i,j) \in S \leftrightarrow (i+1,j+1) \in S$
$(p,i) \in S \leftrightarrow (p,i+1) \in S$
$(j,p) \in S \leftrightarrow (j+1,p) \in S$

Of course I am going to suggest that we start by revisiting everything we have done and talked about for $Q[-1,1]^2$, with $Q[-2,2]^2$ instead. Of course only the stuff involving negatively stable need be revisited.

I'm not done mixing this up even with dimension 2. Recall that the notion of emulator involves looking at concatenations of pairs of pairs. Why not look at concatenations of d-tuples of pairs?

DEFINITION 4. S is a d-emulator of $E \subseteq Q[-1,1]^2$ if and only if the concatenation of any d elements of S is order equivalent to the concatenation of some d elements of E.

Obviously an emulator is just a 2-emulator. So

A. Revisit everything with $Q[-2,2]^2$ and 2-emulators (emulators).
B. Revisit everything with $Q[-1,1]^2$ and 3-emulators.
C. Revisit everything with $Q[-2,2]^2$ and 3-emulators.

Again, my transfinite machinery is good enough to prove this:

NEGATIVELY STABLE MAXIMAL EMULATOR/dim 2/full. NSME/dim 2/full. Every subset of $Q[-n,n]^2$ has a negatively stable maximal r-emulator.

But of course without algorithmic. I believe that NSME/dim2/full cannot be made algorithmic, but haven't yet looked at this. Also I expect that NSME/dim2/full cannot be proved without use of uncountable sets.

Now we throw the ultimate monkey wrench into all of this. Raising the dimension 2 to dimension k. First let's rewrite negatively stable in 2 dimensions as follows.

DEFINITION 5. $S \subseteq Q[-n,n]^2$ is negatively stable if and only if for all $p,q \in Q[-n,0)$ and for all $i,j \in \{0,...,n-1\}$, the following four equivalences hold:

$$(p \text{ or } i, q \text{ or } j) \in S \leftrightarrow (p \text{ or } i+1, q \text{ or } j+1) \in S$$

We first extend this to 3 dimensions.

DEFINITION 6. $S \subseteq Q[-n,n]^3$ is negatively stable if and only if for all $p,q,r \in Q[-n,0)$ and for all $a,b,c \in \{0,...,n-1\}$, the following 8 equivalences hold:

$$(p \text{ or } a, q \text{ or } b, r \text{ or } c) \in S \leftrightarrow$$
$$(p \text{ or } a+1, q \text{ or } b+1, r \text{ or } c+1) \in S$$

Now consider

NEGATIVELY STABLE MAXIMAL EMULATOR/dim 3/full. NSME/dim 3/full. Every subset of $Q[-n,n]^3$ has a negatively stable maximal d-emulator.

We have been able to prove this but only by going way beyond ZFC. But what about for very small n,d? For instance,

NEGATIVELY STABLE MAXIMAL EMULATOR/dim 3/special. NSME/dim 3/special. Every ≤ 2 element subset of $Q[-1,1]^3$ has a negatively stable maximal emulator.

NEGATIVELY STABLE MAXIMAL EMULATOR/dim 3/special. NSME/dim 3/special. Every ≤ 3 element subset of $Q[-2,2]^3$ has a negatively stable maximal emulator.

and other variants.

Now for the full higher dimensional case.

DEFINITION 7. $S \subseteq Q[-n,n]^k$ is negatively stable if and only if for all $p_1,...,p_k \in Q[-n,0)$ and for all $m_1,...,m_k \in \{0,...,n-1\}$, the 2^k equivalences

$$(p_1 \text{ or } m_1, \ldots ,p_k \text{ or } m_k) \in S \leftrightarrow (p_1 \text{ or } m_1+1, \ldots ,p_k \text{ or } m_k+1) \in S$$

NEGATIVELY STABLE MAXIMAL EMULATOR/full. NSME/full. Every subset of $Q[-n,n]^k$ has a negatively stable maximal d-emulator.

With just emulator:

NEGATIVELY STABLE MAXIMAL EMULATOR. NSME. Every subset of $Q[-n,n]^k$ has a negatively stable maximal emulator.

THEOREM. These last two statements cannot be proved in ZFC. They can be proved in ZFC extended by certain well studied large cardinal hypotheses.

This Theorem will be the planned focus for a lecture series on Zoom hosted by Gent in the fall of 2022.

There are plans to develop a general theory of order theoretic stability, in which everything we have talked about would be a special case.