ADVENTURES IN FOUNDATIONS OF MATHEMATICS
Ross Program 2022
MWF during June 27 - July 8, 2022
1. Logical reasoning
2. Finite set theory
3. Infinite sets and ZFC
4. Order equivalence, emulators, and stability
5. Stability of emulators of two pairs
6. Stability of emulators of three pairs

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5. STABILITY OF EMULATORS OF TWO PAIRS

Today we prove the Baby Student Theorem and the Easy Student Theorem and the Little Student Theorem. The Main Student Theorem is proved Friday.

We will rely some on section 4.8 from Lecture 4 on Monday (order isomorphism and coordinate switching). I didn't get to that last section 4.8 but the notes from Lecture 4 are on the Ross site.

5.1. EMULATING ZERO OR ONE PAIR - BABY STUDENT THEOREM

BABY STUDENT THEOREM. Let $E \subseteq Q[-1,1]^2$, $|E| \leq 1$. E has a $|E|$ element negatively stable maximal emulator. If $|E| = 0$ then E has exactly one emulator, $\emptyset$. If $|E| = 1$ then E has infinitely many emulators, one being $\emptyset$ and the rest being maximal emulators of cardinality 1. Some but not all of these maximal emulators are negatively stable maximal emulators.

EXERCISE. Let $|E| = \{(p,q)\}$. Determine exactly what the maximal emulators of E are in terms of p,q. Determine exactly the purely stable maximal emulators of E. Determine the negatively stable maximal emulators of E.

5.2. EMULATING TWO PAIRS - LITTLE STUDENT THEOREM

We now prove the following.

LITTLE STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| = 2$, has an algorithmic negatively stable maximal emulator.

If the two distinct elements are not order equivalent, then we have a particularly easy situation.

EASY STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| = 2$, whose two elements are not order equivalent, has a two element negatively stable maximal emulator. E may or may not have a one element negatively stable maximal emulator.

Proof: Let E be as given. Map E isomorphically into $Q[-1,0)^2$ (via any order isomorphism from Q^2 into $Q[-1,0)^2$). The result is a two element emulator E′ of E. It is clear that E′ is a maximal emulator of E. It is also clear that E′ is negatively stable. QED

So for the Little Student Theorem we need only consider E of cardinality 2, where its elements are order equivalent elements of $Q[-1,1]^2$.

<u>*Lexicographic order on X^k, where X is any linearly ordered set*</u>. Well known linear ordering on X^k. Look for the first term where they differ, and that rules.

LEMMA 5.2.1. Suppose the following E all have a negatively stable maximal emulator: $E = \{(p,q),(r,s)\}$ where $p < q \wedge (p,q) <_{lex} (r,s) \wedge (p,q),(r,s)$ are order equivalent. Then all $|E| = 2$ have a negatively stable maximal emulator (i.e., the little student theorem).

Proof: Let $D = \{(p,q),(r,s)\}$. We want to get a negatively stable maximal emulator.

case 1. $p \leq q$. Then $r \leq s$. Set $E = \{(p,q),(r,s)\}$ or $\{(r,s),(p,q)\}$ depending on how these pairs compare lexicographically. Obviously $D = E$.

case 2. $p \geq q$. Then $r \geq s$. Set $E = \{(q,p),(s,r)\}$. By coordinate switching from Lecture 4, D has a negatively stable maximal emulator if and only if E does. Thus we are thrown back to case 1.

case 3. $p = q$. Then $r = s$. We need to prove $\{(p,p),(q,q)\}$ has a negatively stable maximal emulator. $\{(p,p): p \in Q[-1,1]\}$ is obviously the unique maximal emulator of $\{(p,p),(q,q)\}$, $p \neq q$. It obviously is negatively stable. QED

For future reference, note that in case 3 above, the negatively stable maximal emulator is in fact order invariant.

So from Lemma 5.2.1 we need only work with the E's with that stated property. Now we get into the nuts and bolts of what these E's look like so we can prove that they all have a negatively stable maximal emulator.

LEMMA 5.2.2. Let $E = \{(p,q),(r,s)\} \subseteq Q[-1,1]^2$, $|E| = 2$, $p < q \wedge (p,q) <lex (r,s) \wedge (p,q),(r,s)$ are order equivalent. Exactly one of the following holds.

i. $E = \{(p,q),(p,s)\} \wedge p < q < s$.

ii. $E = \{(p,q),(r,q)\} \wedge p < r < q$.

iii. $E = \{(p,q),(q,s)\} \wedge p < q < s$.

iv. $E = \{(p,q),(r,s)\} \wedge p < r < q < s$

v. $E = \{(p,q),(r,s)\} \wedge p < r < s < q$

vi. $E = \{(p,q),(r,s)\} \wedge p < q < r < s$.

Proof: Let E,p,q,r,s be as given. Obviously $|\{p,q,r,s\}| \in \{2,3,4\}$.

case 1. $|\{p,q,r,s\}| = 2$. So r,s are both among p,q. Hence $(p,q) = (r,s)$, which is impossible.

case 2. $|\{p,q,r,s\}| = 3$. We split into cases according to the sole equation.

case 2.1. $p = q$. Impossible.

case 2.2. $p = r$. Then $q < s$ by $(p,q) <lex (r,s)$. This is i.

case 2.3. $p = s$. Impossible.

case 2.4. $q = r$. Then $p < q = r < s$. This is iii.

case 2.5. $q = s$. This is ii.

case 2.6. $r = s$. Impossible.

case 3. $|\{p,q,r,s\}| = 4$. By $(p,q) <lex (r,s)$, obviously $p \leq r$. Hence $p < r$. Obviously $p < q < r$ or $p < r < q$. In the first case $p < q < r < s$, and in the second case $p < r < q < s$ or $p < r < s < q$. These are iv,v,vi.

These cases are obviously mutually exclusive. QED

LEMMA 5.2.3. Let $E = \{(p,q),(p,r)\}$ where $p < q < r$. E has an order theoretic negatively stable maximal emulator. E has no finite and no order invariant maximal emulator.

Proof: Prove that $\{(-1,a): -1 < a \leq 1\}$ is an order theoretic negatively stable maximal emulator of E. Prove that E has no finite maximal emulator. We can replace -1 by any number other than 0 or 1. QED

LEMMA 5.2.4. Let E = {(p,r),(q,r)} where p < q < r. E has an order theoretic negatively stable maximal emulator. E has no finite and no order invariant maximal emulator.

Proof: Prove that {(a,-1/2): -1 ≤ a < -1/2} is an order theoretic negatively stable maximal emulator of E. Prove that E has no finite maximal emulator. We can replace -1/2 by any number other than -1,0,1. QED

LEMMA 5.2.5. Let E = {(p,q),(q,r)} where p < q < r. E has a two element negatively stable maximal emulator. Every maximal emulator of E is of cardinality 2.

Proof: Prove that S = {(-1,-1/2),(-1/2,-1/3)} is a two element negatively stable maximal emulator. Prove that every maximal emulator of E is of cardinality 2. QED

LEMMA 5.2.6. Let E = {(p,q),(r,s)} where p < r < q < s. E has a piecewise linear negatively stable maximal emulator. No maximal emulator of E is order theoretic.

Proof: Prove that {(a,a+.4): -1 ≤ a < -.6} is a piecewise linear negatively stable maximal emulator of E. Prove that no maximal emulator of E is order theoretic using lecture 4. QED

LEMMA 5.2.7. Let E = {(p,q),(r,s)} where p < r < s < q. E has a piecewise linear negatively stable maximal emulator. No maximal emulator of E is order theoretic.

Proof: Prove that {(-1+a,-.5-a): 0 ≤ a < .25} is a piecewise linear negatively stable maximal emulator of E. Prove that no maximal emulator of E is order theoretic using Theorem 4.7. QED

LEMMA 5.2.8. Let E = {(p,q),(r,s)}, p < q < r < s. E has an algorithmic negatively stable maximal emulator. No negatively stable maximal emulator of E is semi algebraic. No maximal emulator of E is semi algebraic with the sole exception of the stable maximal emulator {(-1,1)}.

Proof: From lecture 4, let S be an algorithmic maximal emulator of E containing (0,1). Since 0,1 do not appear in S\{(0,1)}, S is negatively stable. Now let S be a negatively stable maximal emulator of E and semi algebraic. If S is finite then S = {(-1,1)}, which is not negatively stable. So S is infinite. Now use

that the set of first terms of elements of S to complete the argument (lecture 4), and also use this to verify the last claim. QED

EXPLORATORY QUESTION. What kind of simple descriptions can we give of a maximal emulator of the above E other than {(-1,1)}? What can we say about their computational complexity?

LITTLE STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 2$, has an algorithmic negatively stable maximal emulator.

Proof: Let E be as given. If $|E| \leq 1$ then use the Baby Student Theorem. Suppose $|E| = 2$. We now apply Lemma 5.2.1. By Lemma 5.2.2, this reduces to those cases i-vi. These were handled in that order, by Lemmas 5.2.3 - 5.2.8. QED

We have seen in Lemma 5.2.8 that we cannot replace "algorithmic" by "semi algebraic" in the Little Student Theorem. However, what if we are just looking for stability rather than negative stability? See Theorem 4.7.1 from the last lecture.

THEOREM 5.2.9. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 2$, has a semi linear stable maximal emulator. {(-1,0),(1/2,1)} does not have a semi algebraic negatively stable maximal emulator.

Proof: We have semi linear (negative) stability in Lemmas 5.2.3 - 5.2.7. QED

5.3. COUNTING PROBLEMS

We now give a framework for stating more precise information including giving some quantitative information (counting). This depends on having a suitable equivalence relation(s) on the $E \subseteq Q[-1,1]^2$, $|E| \leq 2$.

DEFINITION 5.3.1. $E,E' \subseteq Q[-1,1]^2$ are order isomorphic/switching if and only if E is order isomorphic to E' or csw(E').

THEOREM 5.3.1. Order isomorphic/switching is an equivalence relation on the $E \subseteq Q[-1,1]^2$.

We write iso/switch for this equivalence relation.

QUESTION. How many equivalence classes are there under iso/switch on the $E \subseteq Q[-1,1]^2$, $|E| = 2$?

QUESTION. How does this relate to the number of equivalence classes of the $E \subseteq Q[-1,1]^2$, $|E| = 2$, up to "S is an emulator of E and E is an emulator of S". Also "S is a maximal emulator of E and E is a maximal emulator of S".

QUESTION. Raise 2 to 3 throughout.

The ingredients needed to calculate this for $|E| = 2$ are probably already present above.

For each $E \subseteq Q[-1,1]^2$ of cardinality 2, up to iso/switch, we associate a list of 18 numbers. Each number is a cardinality $\leq c$. We associate the number of

(order invariant, order theoretic, piecewise linear, semi algebraic, algorithmic), (purely stable, negatively stable) maximal emulators up to iso/switch.

PROBLEM. What can we say about the distribution of these 18 tuples of numbers? Individually or jointly?

In the last lecture, we will work with the $E \subseteq Q[-1,1]^2$, $|E| = 3$. And we ask this same question.

5.4. BIG PICTURE SUMMARY

THEOREM 5.4.1. (Friedman machinery). Every subset of $Q[-1,1]^2$ has a negatively stable maximal emulator.

This proof gives no information about how "good" the negatively stable maximal emulator is, and in particular it doesn't address the following open question:

OPEN QUESTION. Does every subset of $Q[-1,1]^2$ have an algorithmic negatively stable maximal emulator? Also replace "algorithmic" by various computational complexity classes.

THEOREM 4.7.1. Every subset of $Q[-1,1]^2$ has a algorithmic purely stable maximal emulator.

We have proved today that the open question is yes if the given subset of $Q[-1,1]^2$ has at most 2 elements. Next lecture we prove that if the given subset of $Q[-1,1]^2$ has at most 3 elements.

OPEN QUESTION. Does every subset of $Q[-1,1]^2$ with 4 elements have an algorithmic negatively stable maximal emulator? Also replace "algorithmic" by various computational complexity classes.