A DIVINE CONSISTENCY PROOF FOR MATHEMATICS

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Abstract. We present familiar principles involving objects and classes (of objects), pairing (on objects), choice (selecting elements from classes), positive classes (elements of an ultrafilter), and definable classes (definable using the preceding notions). We also postulate the existence of a divine object in the formalized sense of lying in every definable positive class. ZFC (even extended with certain hypotheses just shy of the existence of a measurable cardinal) is interpretable in the resulting system. This establishes the consistency of mathematics relative to the consistency of these systems. Measurable cardinals are used to interpret and prove the consistency of the system. Positive classes and various kinds of divine objects have played significant roles in theology.

INTRODUCTION

This work came about from our interactions at and reflection on two conferences hosted by the John Templeton

Foundation. The first of these meetings was held in honor of the 100th birthday of Kurt Gödel, in Vienna, in 2006. The second was held in honor of the 100th birthday of Sir John Templeton, in Heidelberg, in 2012.

Interactions at the second of these meetings have previously led to [Fr12b], where a basic mathematical property of a mathematical object in a mathematical structure, immediately suggested by familiar ideas about God, is shown to hold within some artificially created mathematical structure. Specifically, there is a relational structure of finite relational type with a unique undefinable element, in the usual sense of first order predicate calculus with equality.

At the Vienna meeting, there was discussion of Gödel's formalization of earlier ontological arguments for the existence of God. This relied heavily on modal logic, but what we found particularly striking was the use of "positive properties", which has a substantial history and goes back at least to Leibniz.

From a theological standpoint, it is perhaps natural to view the attribute of positive as a facility that God has created and given the world access to, perhaps in order to help direct us into "positive behaviors".

It is clear that Gödel was using "positive properties" as, mathematically speaking, an ultrafilter on properties. In fact, at least implicitly, he was using "positive properties" as an ultrafilter on extensions of properties. I.e., whether a property is positive depends only on what objects it holds of. For discussion of ultrafilters and positivity, see section 3 below.

It occurred to us that perhaps this highly intriguing ultrafilter, viewed as an ultrafilter on classes of objects, can be used to prove the consistency of mathematics.

At the time, we just didn't see how to get such ambitious mathematical mileage out of this "positivity ultrafilter", at least in any simple basic conceptual way. This goal seemed particularly remote since the positivity ultrafilter, as discussed by Gödel and implicitly by others, is what is called a "trivial ultrafilter", i.e., an ultrafilter consisting merely of the classes containing some given special point.

In particular, following Gödel, and going back at least to Leibniz (see [Go95], and also [An90], [Le23], [Le56], [L369], [Le96], [Lo06], [Op95], [Op96], [Op06], [Op12], [So87], [So04]), a class of objects is positive if and only if it contains God. And trivial ultrafilters are, as the name suggests, mathematically trivial. So it would appear that one cannot expect to do anything substantial, mathematically, with the positivity ultrafilter.

Interactions with scholars from theology at the second meeting rekindled our interest in trying again to do something powerful with this mathematically trivial positivity ultrafilter.

It occurred to us that if we take God out of the class of all objects, treating God as exceptional, but keeping the positivity ultrafilter, pruned to be over the class of objects excluding God, then the positivity ultrafilter is no longer trivial. In fact, it is a nontrivial ultrafilter over the class of all objects without God.

It is well known that in the context of set theory, certain kinds of ultrafilters are enough to prove the consistency of mathematics, as formalized by the usual ZFC axioms. In particular, a nontrivial countably complete ultrafilter serves this purpose.

However, to straightforwardly state that the positivity ultrafilter is a nontrivial countably complete ultrafilter, together with the mathematical infrastructure needed to state and make use of this, is far too technically brutal to be of fundamental philosophical or theological meaning.

So we encountered the following challenge. To find fundamental properties of the positivity ultrafilter which, in the context of a conceptually very minimal mathematical infrastructure, has sufficient power to construct a model of ZFC, and perhaps more.

This paper offers our response to this challenge. Here is a succinct six part description of our framework.

1. Objects and classes of objects, linked by membership. This is arguably viewed as within our capacity to clearly imagine, as it is an accepted useful working component of modern mathematics. It should be noted that in modern mathematics, objects are usually restricted to objects with

a definite mathematical purpose, rather than a completely general notion of object. See section 1.

2. Pairing of objects - a combining of any two objects into one. This is generally viewed as within our capacity to clearly imagine. E.g., the pair of two objects can be taken to be the idea of having the first followed by the second. In various guises, this is also an accepted useful working component of modern mathematics. See section 1.

3. A choice operator CHO that picks an element out of each nonempty class of objects. Formally, $x \in A \rightarrow \text{CHO}(A) \in A$, where CHO(A) is read "chosen element from A". This appears to be beyond our capacity to clearly imagine. I.e., we don't have an understanding of how to go about making the choices. However, this continues to be an accepted useful working component of modern mathematics. See section 2.

4. The positivity attribute on classes of objects (the positivity ultrafilter). This seems definitely beyond our capacity to clearly imagine. I.e., we don't have an understanding of definite criteria for positivity. The notion of ultrafilter is also an accepted useful working component of modern mathematics. But a specific preferred ultrafilter, particularly on the entire universe of objects, has not appeared as a generally accepted component of modern mathematics. See section 3.

5. The definability attribute on classes of objects, reflecting the standard notion of "class of objects definable from the preceding concepts 1-4 without parameters". We have a clear understanding of this notion, relative to the primitives to which it is being applied (here, in 1-4 above). This is an essential feature of modern mathematical logic, and also has useful interactions with several branches of mathematics - especially real and complex algebraic geometry. It is exemplified by Tarski's formal treatment of truth in formalized languages. See section 4.

Thus the positivity attribute is the one feature above that has such a deep philosophical and theological meaning. The additional axiom that creates the vast logical power, is the Divine Object axiom.

Recall that there does not exist an object which lies in all positive classes, as we treat God as exceptional and outside the class of objects. This is reflected by the positivity ultrafilter being nontrivial.

The crucial innovation is this: an object is **divine** if and only if it lies in all **definable** positive classes.

6. The Divine Object axiom. There is a divine object. See section 5.

We can give the following natural theological interpretation of this development. God created a certain structure, with several components, which he gave the world intellectual access to. This intellectual access is reflected by the system T_5. Using T_5, man creates a mathematical universe, or model of the mathematical universe, through set theory, as presented in sections 6 and 7 in this paper.

The axioms of T_5 are so simple that a promising research project is suggested: to analyze all such simple systematizations, and determine what their mathematical ramifications are. As we prove in section 7, T_5 corresponds, logically, to something close to ZFM = ZFC + "there exists a measurable cardinal". See section 5.

In modern set theory, there are developments that have led far beyond the scope of normal modern mathematics. One of these is the detailed study of ultrafilters on arbitrary sets (or the closely related abstract measure theory). This has lead to a study of certain set theoretic hypotheses postulating the existence of ultrafilters with special properties. One of these set theoretic hypotheses is "there exists a measurable cardinal". The usual ZFC axioms augmented with this hypothesis is what we use to interpret our system with the Divine Object axiom. See section 5.

We then show that we can interpret the usual ZFC axioms for mathematics in our system (section 6). In fact, we interpret ZFC augmented with a set theoretic hypothesis close to, but a little weaker, than "there exists a measurable cardinal" (section 7).

In sections 1-5, we gently build up to T_5 with a series of theories T_1,T_2,T_3,T_4. The impatient reader can safely begin with T_5 in section 5.

After writing this paper, we have seen how to incorporate God as an object, as part of a wider supernatural world. In this second approach, we have both the "Real World" and the wider "Supernatural World". Thus we have a more complex framework in that we have two sorts of objects. But a number of axioms are simplified. In particular, in this second approach, we do not need the positivity ultrafilter.

So a tradeoff is emerging, where we use one sort of object and the positivity ultrafilter (this is what we do here), or two sorts of objects without the positivity ultrafilter (this is what we will do in [Fr13]).

We view [Fr11], [Fr12a], [Fr12b], [Fr13], as part of a wider program which we call Concept Calculus. In Concept Calculus, we aim to identify groups of informal concepts throughout the informal and semiformal intellectual landscape, and formulate fundamental transparent principles which are of sufficient power to interpret the usual formalizations for mathematics and beyond. These provide formal interpretations of mathematics, consistency proofs of mathematics, and relative consistency proofs of mathematics, as in Theorems 6.27 and 7.34.

1. T_1: Objects, classes, pairing.

In sections 1-5, we present, in cumulative stages, the axioms of our theory T_5 that we use to interpret ZFC in section 6. The reader can dispense with this background material and go directly to section 5. T_1 - T_4 admit familiar standard models constructed well within ZFC, but T_5 does not.

In this section, we focus on a particularly simple system T_1 that provides the basic apparatus needed to support the substantial and flexible use of objects and classes of objects required for the interpretation of ZFC.

The two sorted T_1 is very similar to the two sorted system Z_2 of second order arithmetic. The main difference is that in Z_2, the objects are nonnegative integers, with arithmetic operations taken as primitives. In T_1, the objects are not constrained in any way, and are meant to include all objects whatsoever, as reflected in type 0 of the usual Russell theory of simple types (in modern formulation). The classes of objects are reflected in type 1 of the usual Russell theory of simple types (in modern formulation).

For presentations of Z_2, see [Si99], p. 4, and http://en.wikipedia.org/wiki/Second-order_arithmetic. We follow the treatment in [Si99] using the two sorted language for nonnegative integers and sets of nonnegative integers, with $0,1,+,\bullet,<,=$ on nonnegative integers, $\in$ between nonnegative integers and sets of nonnegative integers, set induction, and full comprehension.

The crucial issue is to determine just what should take the place, in T_1, of the arithmetic operations on the objects that are used in Z_2.

Our solution is to use a pairing function P. The Pairing axiom below is all that is required. We establish a detailed connection between T_1 and Z_2.

We first introduce the language L_1 of T_1, which has two sorts: objects and classes of objects (classes). L_1 has object variables v_i, $i \geq 1$, and class variables A_i, $i \geq 1$. L_1 uses a binary relation symbol $\in$ between objects and classes, a binary function symbol P on objects, and = between objects. (P is not used in the usual Russell theory of simple types, or in Z_2).

Here $v_i \in A_j$ means that the object v_i is a member of the class A_j. $P(v_i,v_j)$ is the object which is the ordered pair of v_i and v_j.

The object terms of L_1 are inductively defined by

i. For all $i \geq 1$, v_i is an object terms of L_1.
ii. For all object terms s,t, $P(s,t)$ is an object term of L_1.

The atomic formulas of L_1 are given as follows.

i. For object terms s,t of L_1, $s = t$ is an atomic formula of L_1.
ii. For all $i \geq 1$ and object terms t of L_1, $t \in A_i$ is an atomic formula of L_1.

where $i \geq 1$ and s,t are terms of L_1.

The formulas of L_1 are inductively defined as follows.

i. Every atomic formula of L_1 is a formula of L_1.
ii. For all formulas φ,ψ of L_1, $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ are formulas of L_1.

iii. For all $i \geq 1$ and formulas φ of L_1, $(\forall v_i)(\varphi)$, $(\exists v_i)(\varphi)$, $(\forall A_i)(\varphi)$, $(\exists A_i)(\varphi)$ are formulas of L_1.

We use $\leftrightarrow$ as an abbreviation. I.e., $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

There are standard axioms and rules of inference for L_1, with a corresponding completeness theorem. The system T_1, in language L_1, has the following nonlogical axioms.

NONLOGICAL AXIOMS FOR T_1

PAIRING

$(\exists v_1,v_2)(v_1 \neq v_2)$.
$P(v_1,v_2) = P(v_3,v_4) \to v_1 = v_3 \wedge v_2 = v_4$.

L_1 COMPREHENSION

$(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, where φ is a formula of L_1 in which A_1 is not free.

We make some remarks concerning the axioms of T_1.

1. As we shall see, T_1 is mutually interpretable with Z_2. In fact, we establish that fragments of T_1 are mutually interpretable with fragments of Z_2.

T_1 does not use Extensionality. Thus we can also think of the variables A_i as ranging over properties of objects, where $v_i \in A_j$ means that A_j holds at v_i. The matter of Extensionality is taken up in sections 2 and 8. Specifically, we will be incorporating Extensionality in T_2. In sections 8, we discuss the avoidance of Extensionality.

2. Pairing. Given any two objects x,y, we can form the ordered pair P(x,y). We can raise the issue of just what P(x,y) really is. There is the conceptual idea of x followed by y. So we can define P(x,y) as the conceptual idea of x followed by y. Furthermore, P is definable, as we have just defined it. Also, the Pairing axiom, line 2, is immediate.

There are of course many ways to justify the existence of at least two distinct objects. One way is to use the approach in the previous paragraph, where we take any object x, and consider the conceptual idea of x followed by x, and also the conceptual idea of x followed by x followed by x. Obviously we can argue that x cannot be the same as x followed by x, but perhaps more convincing is that x followed by x, and x followed by x followed by x are of different lengths, and therefore are distinct. (The existence of at least one object x is normally considered to be a logical truth).

3. L_1 Comprehension. This is the standard class existence principle. The objects and classes of objects, with L_1 Comprehension, constitutes the first two stages of a common version of the Russell theory of simple types. L_1 Comprehension alone has the trivial model with exactly 1 object and exactly 2 classes. The usual way of achieving strength is to add an axiom of infinity, involving the existence of an ordering on objects with a certain property. Or alternatively, to use the inclusion relation among classes as the ordering. Under these approaches, we need to use classes of classes of objects, or even classes of classes of classes of objects, or alternatively, binary relations on classes.

In contrast, here we use only objects and classes of objects, with the Pairing axiom doing not only the work of the axiom of Infinity, but also enough work to avoid having to use higher types or binary relations. Also, Pairing has the explicit interpretation as discussed above.

In sections 6 and 7 we will need only L5 Comprehension where φ has only a few class quantifiers. In this paper, we will not go into just how many quantifiers are needed, but preliminary indications are that two suffice. There is also the matter of how many quantifiers over objects are needed, which is also beyond the scope of this paper.

We will consider fragments of L_1 Comprehension, and compare them with corresponding fragments of Z_2. These fragments of Comprehension are finitely axiomatizable.

The following result goes back to Tarski, but we include a proof here. It is of independent interest, and allows us to avoid a number of details when establishing finite and non finite axiomatizability. However, in section 3 we cannot easily take this route, and use another method to establish Theorem 3.6.

LEMMA 1.1. Let S_1,S_2 be two theories with commuting interpretations π_1,π_2. I.e., for all sentences φ of S_1, $\varphi \leftrightarrow$

$\pi_2(\pi_1(\varphi))$ is provable in S_1. Then S_1 is finitely axiomatizable if S_2 is finitely axiomatizable. If K is a finite axiomatization of S_1 then $\pi_1(K)$ is a finite axiomatization of S_2 (S_1).

Proof: Let S_1,S_2,π_1,π_2 be as given. Let ψ be a sentence of S_2. We want $\psi \leftrightarrow \pi_1(\pi_2(\psi))$ provable in S_2. It suffices to show that $\pi_2(\psi \leftrightarrow \pi_1(\pi_2(\psi))) = \pi_2(\psi) \leftrightarrow \pi_2(\pi_1(\pi_2(\psi)))$ is provable in S_1. This follows from the hypothesis applied to $\pi_2(\psi)$.

Let K finitely axiomatized S_1. Let S_2 prove ψ. Then S_1 proves $\pi_2(\psi)$. Hence K proves $\pi_2(\psi)$. Therefore $K \rightarrow \pi_2(\psi)$ is provable from nothing. Hence $\pi_1(K) \rightarrow \pi_1(\pi_2(\psi))$ is provable from nothing. Therefore $\pi_1(K)$ proves ψ. QED

LEMMA 1.2. T_1 is interpretable in Z_2.

Proof: We take the objects for T_1 to be Gödel numbers of terms in 0 and the binary function symbol P. We take the classes for T_1 to be the sets in Z_2. QED

THEOREM 1.3. T_1 and Z_2 have commuting interpretations.

Proof: Let π_1 be the interpretation of T_1 in Z_2 given by the proof of Lemma 1.2. We now give the interpretation π_2 of Z_2 in T_1.

We have to develop arithmetic in T_1. By the first line of Pairing, fix $a \neq b$. Set $0 = P(a,b)$. By the second part of Pairing, 0 is not of the form $P(v,v)$.

We say that A is good if and only if $0 \in A \wedge (\forall v \in A)(P(v,v) \in A)$. Let ω be any class consisting of the objects lying in every good A. Let 1 be $P(0,0)$. Obviously $v,w \in \omega \rightarrow P(v,v) \neq 0 \wedge (P(v,v) = P(w,w) \rightarrow v = w)$. Also if $A \subseteq \omega$, $0 \in A$, and $(\forall v \in A)(P(v,v) \in A)$, then ω,A have the same elements (because A is good). So $P(v,v)$, $v \in \omega$, serves as an inductive successor function on ω.

Using P, we have full access to ternary relations on ω, and hence also binary functions on ω, including comprehension involving quantification over them. So we can develop $+,\bullet,<$ and their standard properties in the usual way.

For Z_2, the sets will be the subsets of ω in T_1. Set induction in Z_2 gets interpreted using the inductive successor function P(v,v) on ω in T_1. Set comprehension in Z_2 gets interpreted using L_1 Comprehension in T_1.

To check that π_1,π_2 are commuting interpretations, we can view them as model constructions, where it suffices to show that if we compose the two model constructions, first from models M of Z_2 to models M' of T_1 via π_1, and second from models M' of T_1 to models M'' of Z_2 via π_2, then we obtain a model M'' of Z_2 that is isomorphic with M.

So begin with a model M of Z_2. We build a model M' of T_1 within M, according to π_1. N hasn't really changed, but has merely been reassembled in the form of closed terms. Now build a model M'' of Z_2 within M', according to π_2. The resulting $(\omega,0,S)$ can be determined from the point of view of M, and seen to be a second order successor system, according to M. Therefore, the resulting model M'' is isomorphic to M via an isomorphism that can be clearly defined in M. QED

DEFINITION 1.1. Let $n \geq 0$. $T_1[n]$ is the subsystem of T_2, where in L_1 comprehension, the formula φ is required to start with at most n class quantifiers, followed by a formula without class quantifiers. $Z_2[n]$ is the subsystem of Z_2, where in set comprehension, φ is required to start with at most n set quantifiers, followed by an arithmetic formula (i.e., a formula without set quantifiers). As in [Si99], p. 16, ACA_0 is the fragment of Z_2 where comprehension is for arithmetic formulas (i.e., without set quantifiers), and Π^1_n-CA_0 is the fragment of Z_2 where comprehension is for Π^1_n formulas. These start with a series of n alternating set quantifiers, starting with $\forall$, followed by an arithmetic formula.

LEMMA 1.4. For $n \geq 0$, $Z_2[n]$ is equivalent to Π^1_n-CA_0.

Proof: Obviously Π^1_n-CA_0 is a fragment of $Z_2[n]$. We show that $Z_2[n]$ is a logical consequence of Π^1_n-CA_0.

Let φ be a formula for comprehension in $Z_2[n]$. In the series of at most n set quantifiers in front, we can collapse blocks of like set quantifiers to one, using manipulations already supported in ACA_0. We therefore obtain $\leq n$ alternating set quantifiers. We can obviously pad this with more alternating set quantifiers, if need be, to obtain exactly n alternating set quantifiers. If they begin with $\forall$, then we are done. If they begin with $\exists$, then we can take the negation, and obtain the complement of the set we need. But the existence of complements is obviously supported by ACA_0. QED

LEMMA 1.5. For $n \geq 0$, $T_1[n]$ is interpretable in $Z_2[n]$.

Proof: Let $n \geq 0$. We use the construction in Lemma 1.1. Each class quantifier in T_1 is interpreted by a like set quantifier in Z_2. QED

THEOREM 1.6. For $n \geq 1$, $T_1[n]$, $Z_2[n]$, $\Pi^1_n\text{-}CA_0$ have commuting interpretations.

Proof: Let $n \geq 1$. We interpret $Z_2[n]$ in $T_1[n]$. We use the same interpretation of Z_2 in T_1 presented in the proof of Theorem 1.3. We can prove the existence of an ω in $T_1[1]$, and use ω as a set parameter.

We need to show that $T_1[n]$ proves the interpretation of $\{m: (\forall x_1)(\exists x_2)\ldots(Qx_n)(\varphi(m))\}$ exists. But this gets interpreted as $\{m \in \omega: (\forall A_1 \subseteq \omega)(\exists A_2 \subseteq \omega)\ldots(QA_n \subseteq \omega)(\varphi'(m))\}$ exists, where in φ', all quantifiers are relativized to ω. This is provable in $T_1[n]$ by obvious quantifier manipulations.

Also note that set induction gets interpreted as $(\forall A \subseteq \omega)(0 \in A \wedge (\forall n \in \omega)(n \in A \rightarrow n+1 \in A) \rightarrow \omega \subseteq A)$, which is provable in $T_1[1]$.

We can now give the model operation isomorphism argument that we gave in the proof of Theorem 1.3. QED

THEOREM 1.7. T_1, Z_2 are not finitely axiomatizable. For all $n \geq 1$, $T_1[n]$, $Z_2[n]$ are finitely axiomatizable. $Z_2[0] = ACA_0$ is finitely axiomatizable.

Proof: Z_2 is not finitely axiomatizable, since Z_2 proves the consistency of each of its finite fragments (see [Si99], section VII.7). For $n \geq 1$, $Z_2[n]$ is finitely axiomatizable, by standard normal form theorems. Also $Z_2[0] = ACA_0$ is finitely axiomatizable (see [Si99], Lemma VIII.1.5).

Now observe that in the proof of Theorems 1.2 and 1.5, we have actually given commuting interpretations. Now apply Lemma 1.8. QED

THEOREM 1.8. $T_1[0]$ is interpretable in RCA_0.

Proof: Let P be one of the standard pairing functions (N,P) treated in [Te72], [Te74]. The first order theory of these (N,P) is shown to be just beyond elementary recursive (see [FR79], p. 163). We interpret the objects of $T_1[0]$ to be elements of N, and the classes of $T_1]0]$ to be the subsets of N first order definable over (N,P). L_1 comprehension is immediate in RCA_0. QED

QUESTION 1.1. What is the interpretation power of $T_1[0]$? Is $T_1[0]$ finitely axiomatizable?

By [FR79], Chapter 8, the first order theory of any pairing function (N,P) is at least just beyond elementary recursive, and by [Te72], [Te74], for various standard pairing functions P, the theory of (N,P) is at most just beyond elementary recursive. These considerations should be at least close to enough to answer Question 1.1.

DEFINITION 1.2. The standard models of T_1 are of the form $(D,\wp(D),\in,P)$, where $|D| \geq 2$, and $P:D^2 \to D$ is one-one.

2. T_2: Extensionality, choice operator.

In this section, we extend L_1 and T_1 to L_2 and T_2. T_2 will also be mutually interpretable with Z_2.

The language L_2 extends L_1 with the unary function symbol CHO from classes to objects, and = between classes. Here CHO refers to a choice operator.

The object terms, atomic formulas and formulas of L_1 are extended in the obvious way to L_2, incorporating the new function symbol CHO, and = on classes.

The system T_2, in language L_2, has the following nonlogical axioms.

NONLOGICAL AXIOMS FOR T_2

PAIRING

$(\exists v_1,v_2)(v_1 \neq v_2)$.

$P(v_1,v_2) = P(v_3,v_4) \rightarrow v_1 = v_3 \wedge v_2 = v_4$.

L_2 COMPREHENSION

$(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, where φ is a formula of L_2 in which A_1 is not free.

EXTENSIONAITY

$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow A_1 = A_2$.

CHOICE OPERATOR

$v_1 \in A_1 \rightarrow CHO(A_1) \in A_1$.

We make some remarks concerning the axioms of T_2.

1. L_2 Comprehension. We again consider fragments of L_2 Comprehension here, and compare them with the corresponding fragments of Z_2. These fragments of Comprehension are shown to be finitely axiomatizable.

2. Extensionality. Classes (of objects) are normally considered identical if they have the same elements, whereas properties (of objects) are not considered identical just because they hold of the same objects.

We could use properties (of objects) rather than classes (of objects), avoid the extensionality axiom, and define extensional equality as: holding of the same objects. We take this approach in section 8.

Extensional equality arises because the power of the Choice Operator for section 6 rests on CHO(A) depending only on the objects in the class A, or only on the objects for which the property A holds.

4. Choice Operator. A Choice Operator operating locally - i.e., at a single class - can be given immediately. If A has an element then CHO(A) is any element of A; otherwise CHO(A) is any object. We can view the (global) Choice Operator as an obvious extrapolation of the trivial local version achieved by an "infinite mind".

In fact, we can invoke the omnipotence of God - that God can do anything "logically possible". The logical possibility here should be manifested by the triviality of what a Choice Operator does locally.

In this connection, it is interesting to determine precisely what kind of simply described global tasks of this kind are "logically possible" as this can be viewed as achieving more understanding of the nature of God's omnipotence.

LEMMA 2.1. T_2 is interpretable in Z_2.

Proof: We take the objects for T_2 to be the Gödel numbers of terms in 0 and the binary function symbol P. We take the classes for T_1 to be the sets in Z_2. We interpret = between classes for T_1 as extensional equality for sets in Z_2. We interpret CHO in T_2 by CHO(A) = min(A) if A $\neq \emptyset$; 0 otherwise. QED

THEOREM 2.2. T_2,Z_2 have commuting interpretations.

Proof: We use the interpretation from the proof of Lemma 2.1 as π_1. We reuse π_2. We follow the argument given for Theorem 1.3. QED

DEFINITION 2.1. For $n \geq 0$, $T_2[n]$ is the subsystem of T_2, where in L_2 comprehension, the formula φ is required to start with at most n class quantifiers, followed by a formula without class quantifiers.

LEMMA 2.3. For $n \geq 0$, $T_2[n]$ is interpretable in $Z_2[n]$.

Proof: Let $n \geq 0$. We use the construction in Lemma 1.1. Each class quantifier in T_1 is interpreted by a like set quantifier in Z_2. QED

THEOREM 2.4. For $n \geq 1$, $T_2[n]$, $Z_2[n]$, Π^1_n-CA_0 have commuting interpretations. $T_2[0]$ is interpretable in RCA_0.

Proof: See Theorem 1.6. The interpretations given for Lemma 1.5 easily extend to accommodate Extensionality and Choice Operator. QED

THEOREM 2.5. T_2 is not finitely axiomatizable. For all $n \geq 1$, $T_2[n]$ is finitely axiomatizable.

Proof: By Lemma 1.1 and Theorem 2.4. QED

QUESTION 2.1. What is the interpretation power of $T_2[0]$? Is $T_2[0]$ finitely axiomatizable?

DEFINITION 2.2. The standard models of T_2 are of the form $(D,\wp(D),\in,P,CHO)$, where $|D| \geq 2$, $P:D^2 \rightarrow D$ is one-one, and $CHO:\wp(D) \rightarrow D$, where $A \neq \emptyset \rightarrow CHO(A) \in A$.

3. T_3: Positive classes.

The notion of "positive property (of objects)" is explicitly used in [Go95], p. 403-404. Gödel had an initial version in 1941. The informal theological idea of "positive property" goes back at least to Leibniz, and plays a significant role in Theology. See [Go95], p. 388-402, and also [An90], [Le23], [Le56], [L369], [Le96], [Lo06], [Op95], [Op96], [Op06], [Op12], [So87], [So04].

In our framework, we can view the attribute of Positive as a facility that God has created and given the world access to, perhaps in order to direct us into "positive behaviors". In our context, we are not incorporating any direct access to God, but we do have access to various of God's creations that he has chosen to give us access to.

The attribute of Positive can be viewed as such a fundamental facility.

Gödel uses positive properties of objects, whereas we use positive classes of objects. Gödel rightly does not assume extensionality for properties of objects. This raises the possibility that our use of positive classes is a greater commitment than Gödel's use of positive properties. However, this is not the case, as we now show. In addition, this is reflected in the development in section 8.

Gödel assumes that

i. The conjunction of any two positive properties is a positive property.
ii. Every property or its negation is positive, with exclusive or.
iii. If God exists, then God possesses all positive properties.

and. But Gödel does not assume extensionality for properties. Thus the conjunction of any two properties is a property (not necessarily the only property) that holds of exactly the objects that the two properties hold of. The negation of any property is a property (not necessarily the property) that holds of exactly the objects that the given property does not hold of.

It is clear that "every positive property holds of some object" is at least implicit in [Go95], p. 403-404. It follows from i-iii and "God exists", which Gödel is claiming to prove (prove is necessarily true).

It now follows that if properties P,Q hold of the same objects, then P is positive if and only if Q is positive. For suppose P,Q hold of the same objects, and P is positive. Suppose Q is not positive. Then the negation of Q is positive, and its conjunction with P is therefore positive. But this conjunction does not hold of any object. Contradiction.

This argument surfaces in the treatments that avoid extensionality in section 8.

We now return to our context of classes of objects with extensionality, and the theory T_3. Since positivity is so very closely related to the purely mathematical notion of ultrafilter, we make a brief mathematical digression.

DEFINITION 3.1. An ultrafilter over a set X is a $K \subseteq \wp(X)$ such that

i. $A,B \in K \rightarrow A \cap B \in K$.
ii. $A \in K \leftrightarrow X\backslash A \in K$.
iii. $\emptyset \notin K$.

THEOREM 3.1. Let K be an ultrafilter over X. Then $X \in K$. If $A \subseteq B$ and $B \in K$, then $A \in K$. There is an ultrafilter over X if and only if X is nonempty.

Proof: Well known and straightforward. QED

DEFINITION 3.2. A trivial ultrafilter over X is an ultrafilter of the form $\{A \subseteq X: x \in A\}$, for some $x \in X$. It is easy to see that there can be at most one such x. A nontrivial ultrafilter is an ultrafilter which is not trivial.

THEOREM 3.2. Let K be an ultrafilter over X. The following are equivalent.

i. K is nontrivial.
ii. Every element of K has at least two elements.
iii. Every element of K is infinite.
There is a nontrivial ultrafilter over X if and only if X is infinite.

Proof: The equivalences are well known and straightforward. The last claim is a well known application of Zorn's Lemma, and cannot be proved in ZF, even in the case X = N. QED

We can simplify the definitions of ultrafilter and nontrivial ultrafilter in the following uniform way.

THEROEM 3.3. K is an ultrafilter over X if and only if
i. $X = A \cup B \rightarrow A \in K \vee B \in K$.
ii. $A,B \in K \rightarrow |A \cap B| \geq 1$.
K is a nontrivial ultrafilter over X if and only if
iii. $X = A \cup B \rightarrow A \in K \vee B \in K$.
iv. $A,B \in K \rightarrow |A \cap B| \geq 2$.

Proof: Let $K \subseteq \wp(X)$. If K is an ultrafilter then i,ii follow immediately. If K is a nontrivial ultrafilter then iii,iv follow immediately by Theorem 3.2.

Suppose i,ii hold. Suppose $A,B \in K$. Now $X = (A \cap B) \cup (X\backslash A \cup X\backslash B)$. Hence $A \cap B \in K \vee X\backslash A \cup X\backslash B \in K$. Hence $A \cap B \in K \vee X\backslash A \in K \vee X\backslash B \in K$. By ii, $A \cap B \in K$.

Suppose $A \in K$. Then $X\backslash A \notin K$ by ii. Suppose $A \notin K$. Now $X = A \cup X\backslash A$. Hence $A \in K \vee X\backslash A \in K$. Therefore $X\backslash A \in K$.

Clearly $\emptyset \notin K$, by ii.

Now suppose iii,iv. By the preceding argument, K is an ultrafilter over X. By Theorem 3.2, we have only to show that $A \in K \rightarrow |A| \geq 2$. This follows from iv by setting B = A. QED

In this section, we extend L_2 and T_2 to L_3 and T_3. T_3 will also be mutually interpretable with Z_2.

L_3 extends L_2 by the unary relation symbol POS on classes. POS(A) is read "the class A is positive".

The object terms, atomic formulas and formulas of L_2 are extended in the obvious way to L_3, incorporating the new atomic formulas $POS(A_i)$, $i \geq 1$.

The system T_3, in language L_3, has the following nonlogical axioms.

NONLOGICAL AXIOMS FOR T_3

PAIRING

$P(v_1,v_2) = P(v_3,v_4) \rightarrow v_1 = v_3 \wedge v_2 = v_4$.

L_3 COMPREHENSION

$(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, where φ is a formula of L_3 in which A_1 is not free.

EXTENSIONAITY

$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow A_1 = A_2$.

CHOICE OPERATOR

$v_1 \in A_1 \rightarrow CHO(A_1) \in A_1$.

POSITIVE CLASSES

$(\forall v_1)(v_1 \in A_1 \vee v_1 \in A_2) \rightarrow POS(A_1) \vee POS(A_2)$.
$POS(A_1) \wedge POS(A_2) \rightarrow (\exists v_1 \neq v_2)(v_1,v_2 \in A_1 \wedge v_1,v_2 \in A_2)$.

This completes the presentation of the nonlogical axioms of T_3.

Here is an alternative axiom group for Positive Classes.

POSITIVE CLASSES ALTERNATIVE

$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \notin A_2) \rightarrow (POS(A_1) \leftrightarrow \neg POS(A_2))$.
$(\forall v_1)(v_1 \in A_1 \leftrightarrow (v_1 \in A_2 \wedge v_1 \in A_3)) \rightarrow (POS(A_2) \wedge POS(A_3) \rightarrow POS(A_1))$.
$POS(A_1) \rightarrow (\exists v_1 \neq v_2)(v_1,v_2 \in A_1)$.

Note that we have dropped the first line of Pairing. It can be easily recovered from Positive Classes.

THEOREM 3.4. T_3 proves the existence of at least two objects. T_3 proves T_2. T_3 is logically equivalent to T_3 with Positive Classes replaced by Positive Classes Alternative. Let $(D,K,P,\in,CHO,POS)$ satisfy T_3, where $K \subseteq \wp(D)$. Then POS is a nontrivial ultrafilter over D.

Proof: For the first claim, from Positive Classes we have that the class of all objects, V, is positive or the empty class is positive. The latter is impossible, since it has no elements. Hence there are at least two objects.

The second claim is immediate from the first claim.

For the third claim, we first derive Positive Classes Alternative. For the first line, it suffices to show that $\neg$(A is positive $\wedge$ V\A is positive). This is clear from Positive Classes. For the second line of Positive Classes Alternative, we assume A,B are positive, and show $A \cap B$ is positive. Suppose $A \cap B$ is not positive. By line 1 of Positive Classes Alternative, $V\backslash A \cup V\backslash B$ is positive. By Positive Classes, V\A is positive or V\B is positive. These violate line 1 of Positive Classes Alternative. The third line of Positive Classes Alternative follows from the second line of Positive Classes by setting $A_2 = A_1$.

We now derive Positive Classes from Positive Classes Alternative. Let $V = A \cup B$. Suppose A,B are not positive. By Positive Classes Alternative, V\A and V\B are positive, $V\backslash A \cap V\backslash B$ is positive, $V\backslash(V\backslash A \cap V\backslash B)$ is not positive, $A \cup B$ is not positive, V is not positive, $\emptyset$ is positive, violating line 3 of Positive Classes Alternative. Finally, Let A,B be positive. By Positive Classes Alternative, $A \cap B$ is positive, and so $A \cap B$ has at least two distinct elements.

Let $(D,K,P,\in,CHO,POS)$ satisfy T_3, where $K \subseteq \wp(D)$. POS is an ultrafilter over D since we have Positive Classes Alternative (this needs only $POS(A1) \rightarrow (\exists v_1)(v_1 \in A_1)$). Line 3 of Positive Classes Alternative and Theorem 3.2 guarantees that POS is a nontrivial ultrafilter. We can also use Theorem 3.3 and Positive Classes. QED

DEFINITION 3.3. For $n \geq 0$, $T_3[n]$ is the subsystem of T_3, where in L_3 Comprehension, the formula φ is required to have at most n class quantifiers.

THEROEM 3.5. T_3,Z_2 are mutually interpretable. For all $n \geq 1$, $T_3[n],Z_2[n]$ are mutually interpretable. $T_3[0]$ is interpretable in ACA_0.

Proof: Recall the interpretation of T_1 in Z_2 given for Theorem 1.5. We now have to interpret POS using a defined nontrivial ultrafilter over N. This cannot be done in Z_2. However, we can use the well known construction of the constructible hierarchy in Z_2. For a treatment within ATR_0 and its applications to Z_2 and strong fragments of Z_2 with and without choice principles, see sections VII.4 - VII.6 in [Si99].

Thus we interpret the objects in T_3 as nonnegative integers, and the classes in T_3 as constructible subsets of N. For CHO, we can of course continue to use the least element construction. It is well known that we obtain the full comprehension of Z_2.

Now it is easily proved in ACA_0 that for any nontrivial ultrafilter U on a countable Boolean algebra B of subsets of N, and $A \subseteq N$, $A \notin B$, there is a unique nontrivial ultrafilter on the Boolean algebra generated by $B \cup \{A\}$, with $B \in U$, extending U.

This allows us to make an obvious transfinite construction along the constructible hierarchy of subsets of N that results in an explicitly definable ultrafilter. We use this ultrafilter to interpret POS. This establishes the first claim.

For $n \geq 2$, this argument is robust enough to interpret $T_3[n]$ in $Z_2[n]$, as $\Pi^1_2\text{-}CA_0$ is rather substantial and well supports the constructible hierarchy for subsets of N.

To interpret $T_3[1]$ in $Z_2[1]$, it is better to think in terms of the first ω hyperjumps. We build a nontrivial ultrafilter arithmetically in the first hyperjump for sets recursive in the first hyperjump. We then extend this to a nontrivial ultrafilter arithmetically in the second hyperjump for sets recursive in the second hyperjump, etcetera. The classes in $T_3[1]$ are interpreted as the sets recursive in some iterated hyperjump. We have to be careful not to use more than set induction, so we don't have that for all n, the n-th hyperjump exists. Only for a cut of n. Then we build the nontrivial ultrafilter as indicated.

Any single class quantifier statement in $T_3[1]$ becomes an arithmetic statement in the hyperjump of its class parameters under this interpretation. Thus the interpretation of single class quantifier comprehension in $T_3[1]$ is provable in $Z_2[1]$.

To interpret $T_3[0]$ in $Z_2[0] = ACA_0$, we follow the above approach, thinking in terms of the first ω Turing jumps. Again we have to be careful not to use more than set induction, so we don't have that for all n, the n-th Turing jump exists. Again, only for a cut of n. We build the nontrivial ultrafilter as above. QED

THEOREM 3.6. T3 is not finitely axiomatizable. For $n \geq 1$, $T_3[n]$ is finitely axiomatizable.

Proof: Since we are not claiming that T_3,Z_2 have commuting interpretations, use a different kind of argument to show that T_3 is not finitely axiomatizable. Let K be a finite axiomatization of T_3. The interpretation of T_3 in Z_2 given in the proof of Theorem 3.5 shows that Con(K) is provable in Z_2. Let n be such that Con(K) is provable in $Z_2[n]$. Now $Z_2[n]$ is finitely axiomatizable, and interpretable in T_3. Let K' be a finite fragment of T_3 such that $Z_2[n]$ is interpretable in K'. Using the proof of K' in K, we see that $Z_2[n]$ proves Con(K'). Hence $Z_2[n]$ proves Con($Z_2[n]$), and therefore Z_2 is inconsistent. This is a contradiction.

Let $n \geq 1$. To see that $T_3[n]$ is finitely axiomatizable, we argue directly. We obtain the expected normal form theorem for formulas in L3 without class quantifiers, via a formula beginning with 1 class quantifier followed by a formula with no class quantifiers. The class quantifier can be taken to be either $\forall$ or $\exists$. This works even though POS is present. QED

QUESTION 3.1. What is the interpretation power of $T_3[0]$? Is $T_3[0]$ finitely axiomatizable?

There are many ways to justify both parts of Positive Classes, or, alternatively, all three parts of Positive Classes Alternative.

A particularly straightforward way to motivate the first line of Positive Classes is to assume the existence of a perfect object v. Let $V = A_1 \cup A_2$. Then $v \in A_1 \vee v \in A_2$.

We claim that if $v \in A_1$ then A_1 is positive. To argue this, suppose A_1 is not positive. Then v would not belong to A_1, since if v is perfect, v would choose not to belong to A_1.

So we have argued that $v \in A_1 \rightarrow A_1$ is positive, and $v \in A_2 \rightarrow A_2$ is positive. Therefore A_1 is positive or A_2 is positive.

Unfortunately, if we admit a perfect object v, then the second part of Positive Classes fails. This is because the class consisting of exactly v must be positive; i.e., $v \in \{v\} \rightarrow \{v\}$ is positive. But $\{v\}$ has exactly one element.

In order to reconcile the discussion with Positive Classes, we must deliberately exclude any perfect object from the totality of objects. E.g., redefine the objects to consist of the imperfect objects.

A very natural viewpoint is this: no object is perfect, but objects can "approximate perfection" without limit. I.e., any level of "approximate perfection" can be achieved by imperfect objects.

In particular, all objects have flaws. This is an interesting statement in its own right, formalized as follows.

UNIVERSAL FALLIBILITY

$(\forall v_1)(\exists A_1)(v_1 \in A_1 \wedge \neg POS(A_1))$.

Note that Universal Fallibility is provable in T_3 as follows. Let v be an object. Since positive classes have at least two distinct elements, $\{v\}$ is not positive. Hence the complement of $\{v\}$ is positive (e.g., use the second line of Positive Classes), yet does not contain v.

We continue the argument for Positive Classes. Now suppose all objects lie in A_1 or A_2. If A_1,A_2 are not positive, then objects are certainly not approximating perfection without limit, in that every object is flawed even with respect to just A_1 and A_2. So this justifies the first part of Positive Classes.

For the second line of Positive Classes, let A_1,A_2 be positive classes. There must exist an object v in both A_1 and A_2 (since we can approximate perfection without limit).

Suppose v is the unique object in both A_1 and A_2. By Universal Fallibility, let $v \notin A_3$, where A_3 is positive. Since we can approximate perfection without limit, let w lie in A_1,A_2,A_3. Then $v \neq w$, and we have the desired contradiction.

DEFINITION 3.4. The standard models of T_3 are of the form $(D,\wp(D),\in,P,CHO,POS)$, where $P:D^2 \to D$ is one-one, $CHO:\wp(D) \to D$, $A \neq \emptyset \to CHO(A) \in A$, and POS is a nontrivial ultrafilter over D.

4. T_4: Definable classes.

We follow the usual convention in mathematical logic that "definable" means "definable with parameters", and "0-definable" means definable without parameters.

In this section, we extend L_3 to L_4 by the unary relation symbol DEF on classes. The intended meaning of DEF(A) is "the class A is L_3 0-definable". This is the same as "the class A is L_4 0-definable without DEF".

The object terms, atomic formulas and formulas of L_3 are extended in the obvious way to L_4, incorporating the new unary relation symbol DEF on classes.

The system T_4, in language L_4, has the following nonlogical axioms.

NONLOGICAL AXIOMS FOR T_4

PAIRING

$P(v_1,v_2) = P(v_3,v_4) \to v_1 = v_3 \wedge v_2 = v_4$.

L_4 COMPREHENSION

$(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, where φ is a formula of L_4 in which A_1 is not free.

EXTENSIONAITY

$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \to A_1 = A_2$.

CHOICE OPERATOR

$v_1 \in A_1 \to CHO(A_1) \in A_1$.

POSITIVE CLASSES

$(\forall v_1)(v_1 \in A_1 \vee v_1 \in A_2) \rightarrow POS(A_1) \vee POS(A_2)$.
$POS(A_1) \wedge POS(A_2) \rightarrow (\exists v_1 \neq v_2)(v_1,v_2 \in A_1 \wedge v_1,v_2 \in A_2)$.

0-DEFINABLE CLASSES

$(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi) \wedge DEF(A_2) \wedge \ldots \wedge DEF(A_n) \rightarrow DEF(A_1)$, where φ is a formula of L_4 without DEF, with at most the free variables $v_1,A_2,\ldots,A_n$, $n \geq 1$.

DEFINITION 4.1. For $n \geq 0$, $T_4[n]$ is the subsystem of T_4, where in L_3 Comprehension and 0-Definable Classes, the formula φ is required to have at most n class quantifiers.

THEOREM 4.1. T_4,Z_2 are mutually interpretable. For all $n \geq 1$, $T_4[n],Z_2[n]$ are mutually interpretable. $T_4[0]$ is interpretable in ACA_0.

Proof: This is immediate from Theorem 3.5, since we can interpret DEF(A) as A = A. We won't be able to interpret DEF(A) in this trivial way for T_5, since T_5 proves $\neg(\forall A)(DEF(A))$. QED

THEOREM 4.2. T_4 is not finitely axiomatizable. For $n \geq 1$, $T_4[n]$ is finitely axiomatizable.

Proof: Suppose T_4 is equivalent to K, where K is finite. Now replace each $DEF(A_i)$ with $A_i = A_i$ in all axioms of T_4 and K, obtaining T_4' and K'. Then T_4',K' are logically equivalent. But T_4' is logically equivalent to T_3. This contradicts that T_3 is not finitely axiomatizable. The argument given for the second claim of Theorem 3.6 adapts to this context. QED

QUESTION 4.1. What is the interpretation power of $T_4[0]$? Is $T_4[0]$ finitely axiomatizable?

DEFINITION 4.2. The standard models of T_4 are of the form $(D,\wp(D),\in,P,CHO,POS,DEF)$, where $P:D^2 \rightarrow D$ is one-one, $CHO:\wp(D) \rightarrow D$, $A \neq \emptyset \rightarrow CHO(A) \in A$, POS is a nontrivial ultrafilter over D, and DEF is the set of all subsets of D that are 0-definable over $(D,\wp(D),\in,P,CHO,POS)$.

We can formulate 0-Definable Classes in the following reduced form.

0-DEFINABLE CLASSES (binary)

$(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi) \wedge DEF(A_2) \wedge ... \wedge DEF(A_n) \rightarrow DEF(A_1)$, where φ is a formula of L_4 without DEF, with at most the free variables $v_1,A_2,...,A_n$, $1 \leq n \leq 3$.

THEOREM 4.2. T_3 is logically equivalent to T_3 formulated with 0-Definable Classes (binary). This is also true for all $T_3[n]$, $n \geq 0$.

Proof: We work with 0-Definable Classes (binary), and the rest of T_3. We establish $(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi) \wedge DEF(A_2) \wedge ... \wedge DEF(A_n) \rightarrow DEF(A_1)$, where φ has no DEF, and at most the free variables $v_1,A_2,...,A_n$, holds. We can assume that $n \geq 3$.

Assume $(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, $DEF(A_2),...,DEF(A_n)$. With n-2 uses of the binary form, we obtain $DEF(A_{n+1})$, with $A_{n+1} = \{P(x_2,...,x_n): x_2 \in A_2 \wedge ... \wedge x_n \in A_n\}$, where here we have used P associated to the left. We replace the free occurrences of A_i in φ, $2 \leq i \leq n$, by $\{x_i: (\exists x_2,...,x_{i-1},x_{i+1},...,x_n)(P(x_2,...,x_n) \in A_{n+1}\}$, and then simplify by removing these set abstraction terms in the usual way. Thus we have defined A_1 with the single parameter A_{n+1}, where $DEF(A_{n+1})$. Hence $DEF(A_1)$.

Since this argument does not introduce any class quantifiers, the second claim is also established. QED

5. T_5: Divine objects.

Here we finally introduce the theory T_5 that we use in section 6 to interpret ZFC.

The language of T_5 is the same as the language of T_4, and T_5 merely extends T_4 by the Divine Object axiom.

Since in section 1, we suggested that the reader can safely start reading this section, we now give a fully self contained presentation of T_5.

The language L_5 of T_5 has two sorts: objects and classes of objects (classes). L_5 has object variables v_i, $i \geq 1$, and class variables A_i, $i \geq 1$. L_4 uses a binary relation symbol $\in$ between objects and classes, a binary function symbol P on objects (pairing), = between objects, a unary function

CHO from classes to objects (choice), a unary relation symbol POS on classes (positive classes), and a unary relation symbol DEF on classes (definability without DEF).

Note that L_5 is the same as the language L_4 presented in section 4.

The object terms of L_5 are inductively defined by

i. For all $i \geq 1$, v_i is an object term of L_5.
ii. For all object terms s,t, P(s,t) is an object term of L_5.
iii. For all $i \geq 1$, CHO(A_i) is an object term of L_5.

The atomic formulas of L_5 are given as follows.

i. For all $i,j \geq 1$, $A_i = A_j$ is an atomic formula of L_5.
ii. For object terms s,t of L_5, $s = t$ is an atomic formula of L_5.
iii. For all $i \geq 1$ and object terms t of L_5, $t \in A_i$ is an atomic formula of L_5.
iv. For all $i \geq 1$, POS(A_i), DEF(A_i) are atomic formulas of L_5.

The formulas of L_5 are inductively defined as follows.

i. Every atomic formula of L_5 is a formula of L_5.
ii. For all formulas φ,ψ of L_5, $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ are formulas of L5.
iii. For all $i \geq 1$ and formulas φ of L_5, $(\forall v_i)(\varphi)$, $(\exists v_i)(\varphi)$, $(\forall A_i)(\varphi)$, $(\exists A_i)(\varphi)$ are formulas of L_5.

We use $\leftrightarrow$ as an abbreviation. I.e., $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

There are standard axioms and rules of inference for L_5, with a corresponding completeness theorem. The system T_5, in language L_5, has the following nonlogical axioms.

NONLOGICAL AXIOMS FOR T_5

PAIRING

$P(v_1,v_2) = P(v_3,v_4) \rightarrow v_1 = v_3 \wedge v_2 = v_4$.

L_5 COMPREHENSION

$(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, where φ is a formula of L_5 in which A_1 is not free.

EXTENSIONAITY

$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow A_1 = A_2$.

CHOICE OPERATOR

$v_1 \in A_1 \rightarrow CHO(A_1) \in A_1$.

POSITIVE CLASSES

$(\forall v_1)(v_1 \in A_1 \vee v_1 \in A_2) \rightarrow POS(A_1) \vee POS(A_2)$.
$POS(A_1) \wedge POS(A_2) \rightarrow (\exists v_1 \neq v_2)(v_1,v_2 \in A_1 \wedge v_1,v_2 \in A_2)$.

0-DEFINABLE CLASSES

$(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi) \wedge DEF(A_2) \wedge \ldots \wedge DEF(A_n) \rightarrow DEF(A_1)$, where φ is a formula of L_5 without DEF, with at most the free variables $v_1,A_2,\ldots,A_n$, $n \geq 1$.

DIVINE OBJECT

$(\exists v_1)(\forall A_1)(DEF(A_1) \wedge POS(A_1) \rightarrow v_1 \in A_1)$.

As discussed in section 3, T_5 proves that there is no object that lies in every positive class. I.e., there is no perfect object. In section 3, we discussed the informal principle that we can approximate perfection without limitation.

With this background, we say that an object is **divine** if and only if it is an element of all 0-definable positive classes. The Divine Object axiom asserts that there exists a divine object.

Divine objects turn out to be surprisingly powerful approximations to perfection. In section 6, we use the existence of a divine object to give an interpretation of ZFC in T_5. This is extended in section 7 by giving an interpretation of ZFC together with a rather substantial large cardinal hypothesis - the existence of arbitrarily large strong Ramsey cardinals. This is just short of the well known large cardinal hypothesis that asserts the existence of a measurable cardinal. See Definition 7.18. These interpretations also provide consistency and relative consistency proofs in the usual way.

But how do we know that T_5 is consistent - i.e., free of contradiction?

We now present the standard models of T_5 using measurable cardinals from set theory.

DEFINITION 5.1. In set theory, a von Neumann cardinal κ is measurable if and only if κ is uncountable, and there is nontrivial κ complete ultrafilter U over κ. I.e., a nontrivial ultrafilter U over κ such that the intersection of fewer than κ many elements of U is an element of U. This is equivalent to: the intersection of fewer than κ many elements of U is nonempty.

DEFINITION 5.2. ZFM is the theory ZFC + "there exists a measurable cardinal".

ZFM is a very well studied system which is very much stronger than ZFC alone. See, e.g., [Ka94], [Je06].

DEFINITION 5.3. The standard models of T_5 are $(\kappa,\wp(\kappa),\in,P,CHO,POS,DEF)$, where $P:\kappa^2 \to \kappa$ is one-one, $CHO:\wp(\kappa) \to \kappa$, $A \neq \emptyset \to CHO(A) \in A$, POS is a nontrivial κ complete ultrafilter over κ, and DEF is the set of all subsets of κ that are 0-definable over $(\kappa,\wp(\kappa),\in,P,CHO,POS)$.

DEFINITION 5.4. EFA = exponential function arithmetic, is our notation for a system that later became known as $I\Sigma_0(\exp)$. See [HP93], p. 37.

The first claim of Theorem 5.1 is an interpretability result. The second claim is a consistency proof. The third claim is a relative consistency proof. All three of these results establish that T_5 is "all right" assuming that ZFM is "all right".

The main point of this paper goes the other direction. That ZFC is "all right" assuming T5 is "all right" - see Theorem 6.27. Moreover, that something just shy of ZFM is "all right" assuming T_5 is "all right" - see Theorem 7.34.

THEOREM 5.1. ZFM proves that the structures in Definition 5.3 are models of T_5. T_5 is interpretable in ZFM. ZFM proves the consistency of T_5. EFA proves $Con(ZFM) \rightarrow Con(T_5)$.

Proof: All of the axioms of T_4 obviously hold in the $(\kappa,\wp(\kappa),\in,P,CHO,POS,DEF)$ of Definition 5.3. Since there are fewer than κ many (in fact, only countably many) subsets of κ that are 0-definable over $(\kappa,\wp(\kappa),\in,P,CHO,POS)$, the intersection of those that have POS (i.e., lie in the nontrivial κ complete ultrafilter) is nonempty (in fact, has POS). Any element of this intersection is therefore divine.

This provides an interpretation of T_5 in ZFM. Also ZFM obviously proves the consistency of T_5 since ZFM proves the existence of a model of T_5. The last claim follows from the interpretability of T5 in ZFM.

The third claim follows immediately from the fact that the first claim is provable in EFA. QED

Note that we have only used that the intersection of countably many elements of the ultrafilter over κ is nonempty. Thus we only need a weaker condition on the ultrafilter, called countable completeness.

DEFINITION 5.5. A countably complete ultrafilter over a set D is an ultrafilter U where the intersection of any countable number of elements of U is an element of U. This is equivalent to: the intersection of any countable number of elements of U is nonempty.

It is well known that the existence of a nontrivial countably complete ultrafilter is equivalent to the existence of a measurable cardinal, over ZFC. See, e.g., [Ka94], p. 23.

It is natural to expand our set theoretic models of T_5 in the following way.

DEFINITION 5.6. The preferred models of T_5 are $(D,\wp(D),\in,P,CHO,POS,DEF)$, where $P:D^2 \rightarrow D$ is one-one, $CHO:\wp(D) \rightarrow D$, $A \neq \emptyset \rightarrow CHO(A) \in A$, POS is a nontrivial countably complete ultrafilter over D, and DEF is the set of all subsets of D that are 0-definable over $(D,\wp(D),\in,P,CHO,POS)$.

6. Interpreting ZFC in T_5.

In this section, we interpret ZFC in T_5. We work in T_5, developing the interpretation. We carry out the development in T_4 for as long as we can, before using the powerful Divine Object axiom of T_5.

DEFINITION 6.1. V is the class of all objects. Relations on V are treated as classes using the pairing function P. Functions are treated as relations.

DEFINITION 6.2. A well ordering is a pair $(A,<)$, where A and $< \subseteq A^2$ are classes, such that < is irreflexive, transitive, connected, and where every nonempty $B \subseteq A$ has an R least element. Define $x \leq y \leftrightarrow x < y \vee x = y$. Define $<[x] = \{y: y < x\}$, $\leq[x] = \{y: y \leq x\}$, $>[x] = \{y: x < y\}$, $\geq[x] = \{y: x \leq y\}$. x is a < limit point if and only if x is not < least and x has no immediate predecessor in <. If x is not < greatest, define x+1 to be the immediate successor of x in <. If x is not a limit point and not < least, define x-1 to be the immediate predecessor of x in (A,<).

DEFINITION 6.3. $A \subseteq V$ is finite if and only if A is empty, or there is a well ordering (A,<) with a < greatest object and without a < limit point. A is infinite if and only if A is not finite.

LEMMA 6.1. (T_1). Let (A,<) be a well ordering with a < greatest element and without a < limit point. Then < induction holds. I.e., suppose $0 \in B \wedge (\forall x \in A)(x \in B \rightarrow x+1 \in B)$, where 0 is < least. Then $A \subseteq B$.

Proof: Let A,<,0,B be as given. Let x be < least such that $x \notin B$. Now $x = 0 \vee$ x is a < successor $\vee$ x is a < limit point. None of these three disjuncts are possible. QED

LEMMA 6.2. (T_1). Let (A,<) be a nonempty linear ordering (irreflexive, transitive, connected), with no < greatest element. Then A is infinite.

Proof: Let (A,<) be as given, and assume A is finite. Let (A,<') be a well ordering, with a <' greatest element, and with no <' limit points. We prove by <' induction (Lemma 6.1) that $(\forall x \in A)(\exists y \in A)(\forall z <' x)(z < y)$. This induction is clear using that there is no < greatest element. Now apply this quantified statement to the <' greatest element, resulting in a contradiction. QED

LEMMA 6.3. (T_1). V is infinite.

Proof: Suppose V is finite. Let < be a well ordering of V with a < greatest object, and with no < limit points.

We first prove by < induction on x that every one-one f:≤[x] → ≤[x] is onto. This is obvious if x is < least. Suppose this is true for x. Let f:≤[x+1] → ≤[x+1] be one-one. Let g be f restricted to the subdomain ≤[x].

case 1. f(x+1) = x+1. By the induction hypothesis, rng(g) = ≤[x]. Therefore f is onto.

case 2. f(x+1) ≠ x+1. Now f(x+1) ∉ rng(g). If x+1 ∉ rng(g) then by the induction hypothesis, rng(g) = ≤[x], contradicting f(x+1) ∉ rng(g).

If x+1 ∈ rng(g), then let g(b) = x+1. Let h be the same as g except h(b) = f(x+1). By the induction hypothesis, rng(h) = ≤[x]. Hence rng(g) = ≤[x+1]\{f(x+1)}. Hence rng(f) = ≤[x+1]. This completes the induction argument.

By setting x to be the < greatest object, we see that every one-one f:V → V is onto. But let a ≠ b (since there are at least two objects). Then the function f(x) = P(a,x) is one-one, but does not achieve P(b,b). This is a contradiction. QED

DEFINITION 6.4. We say that (A,R) is critical if and only if (A,R) is a well ordering, and for all x ∈ A, x = CHO({y: ¬R(y,x)}).

LEMMA 6.4. (T_2). Let (A,R),(B,S) be critical. There is a longest common initial segment. (A,R) is an initial segment of (B,S) or (B,S) is an initial segment of (A,R).

Proof: Let A,B,R,S be as given. By well foundedness, there is a longest initial segment (C,T) of (A,R) that is an initial segment of (B,S). If C = A then (A,R) is an initial segment of (B,S). If C = B then (B,T) is an initial segment of (B,S), and so S = T and (B,S) is an initial segment of (A,R).

The remaining case is where $C \neq A \wedge C \neq B$. Since $C \subseteq A \cap B$, let x be R least outside C, and y be S least outside C. Now the R predecessors of x and the S predecessors of x are the same (the elements of C). By the definition of critical, x = y. This contradicts that (C,T) is longest. QED

We refer to the second claim of Lemma 6.4 as critical comparability.

LEMMA 6.5. (T_2). Let A be the union of the domains of the critical well orderings, and R be the union of the relations of the critical well orderings. Then (A,R) is critical.

Proof: Let A,R be as given. Obviously $R \subseteq A_2$ and (A,R) is irreflexive. To see that (A,R) is transitive, let x R y R z. Let (B,S),(C,T) be critical, where $S(x,y) \wedge T(y,z)$. By critical comparability, $T(x,z) \vee S(x,z)$. Hence R(x,z).

To see that (A,R) is connected, let $x \neq y$, $x,y \in A$. Let $x \in B$, $y \in C$, (B,S),(C,T) critical. By critical comparability, $T(x,y) \vee T(y,x) \vee S(x,y) \vee S(y,x)$. Hence $R(x,y) \vee R(y,x)$.

To see that (A,R) is well ordered, let $E \subseteq A$, $E \neq \emptyset$. Let $x \in E$, $x \in B$, (B,S) critical. Let y be the S least element of $E \cap B$. We claim that y is the R least element of E. To see this, let z R y, $z \in E$. Let T(z,y), (C,T) critical. By critical comparability, S(z,y). This contradicts the choice of y.

To see that (A,R) is critical, let $x \in A$, $x \in B$, (B,S) critical. Then $x = CHO(V\backslash\{a: a\ S\ x\})$. It suffices to prove that $(\forall y)(R(y,x) \leftrightarrow S(y,x))$. Clearly $S(y,x) \rightarrow R(y,x)$. Suppose R(y,x). Let T(y,x), (C,T) critical. Since $x,y \in B \cap C$, we have S(y,x). QED

LEMMA 6.6. (T_2). There is a unique critical well ordering (V,<). (T_4). DEF(<) holds.

Proof: Let (A,R) be as given by Lemma 6.3. Suppose $A \neq V$. Let $x = CHO(V\backslash A)$. Then we can extend (A,R) by putting $x \notin A$ on top. The result is a critical $(A \cup \{x\},R')$, contradicting the definition of (A,R). Note that (A,R) has been explicitly defined without parameters using only

∈,P,CHO, and equality between objects and between classes. Hence DEF(<) by 0-Definable Classes. QED

DEFINITION 6.5. A is negative if and only if A is not positive. $DEF(A_1,...,A_n) \leftrightarrow DEF(A_1) \wedge ... \wedge DEF(A_n)$.

LEMMA 6.7. (T_4). There is a well ordering (V,<*) with DEF(<*) such that each ≤*[x] is negative. Every finite set is negative.

Proof: Let < be as given by Lemma 6.6. Suppose some ≤[x] is positive. Let x be < least such that ≤[x] is positive. Then <[x] is positive and DEF({x}).

Let <* be the result of removing ≥[x] and putting it at the bottom. Then (V,<*) is a well ordering, and DEF(<*). Let y ∈ V. If y <* x then ≤[y] is negative, and so ≤*[y] is negative, since ≥[x] is negative. If y ≥ x then ≤*[y] ⊆ ≥[x], and so ≤*[y] is also negative.

For the second claim, let A be finite. We can assume A ≠ ∅. Let (A,<) be a well ordering with a greatest element and no limit point. It suffices to show that for all x ∈ A, ≤[x] is negative. Let x be the < least counterexample. Then x is not the < least element of A. Now ≤[x-1] is negative. Hence ≤[x] is negative, and we have a contradiction. QED

DEFINITION 6.6. <* is as given by Lemma 6.6. 0 is the <* least element. 1 is the immediate successor of 0 in <*. x+1 is the immediate successor of x in <*, if it exists.

LEMMA 6.8. (T_4). There is no <* greatest object. <* induction holds if <*[x] is finite. I.e., suppose 0 ∈ A ∧ (∀x ∈ A)(<*[x] finite → x+1 ∈ A). Then (∀x)(<*[x] finite → x ∈ A).

Proof: Let x be <* greatest. Then ≤*[x] is negative by Lemma 6.7. But ≤*[x] = V, which is positive. There is no <* greatest object, and hence every x+1 exists.

For the second claim, let A be as given, and let x be <* least such that x ∉ A and <*[x] is finite. Then x ≠ 0 and x is not a <* successor. Hence x is a <* limit point, which contradicts that <*[x] is finite, by Lemma 6.2. QED

DEFINITION 6.7. sup(A) is the <* least object x with $(\forall y \in A)(y <^* x)$, provided such an x exists. Otherwise, we take sup(A) = ∞.

DEFINITION 6.8. (x,f) is a reduction if and only if
i. dom(f) = V ∧ sup(rng(f)) = x.
ii. For all y <* x, $f^{-1}(y)$ is negative.
iii. DEF(f).
(x,f) is a semi reduction if and only if the above holds, with sup(rng(f)) ≤* x.

DEFINITION 6.9. We use γ for the <* least object such that there is a reduction (γ,f), provided there is some reduction. Otherwise, we take γ = ∞, which is not considered to be an object. If γ ≠ ∞ then we use J for a fixed function such that (γ,J) is a reduction. If γ = ∞ then we use J for the identity function on V. We extend <* to V ∪ {∞}. We continue to use letters x,y,z,w,u,v,... for objects (elements of V) only.

LEMMA 6.9. (T_4). If (x,f) is a reduction then DEF({x}). If (x,f) is a semi reduction then (sup(rng(f)),f) is the unique reduction with second term f. If (x,f) is a semi reduction then γ ≤* x. Suppose γ ≠ ∞. Then DEF({γ}), and γ is a <* limit point.

Proof: Let (x,f) be a reduction. Since x = sup(rng(f)), we have DEF({x}). The second claim is immediate. For the third claim, let (x,f) be a semi reduction. Then (sup(rng(f)),f) is a reduction, and so γ ≠ ∞ and γ ≤* sup(rng(f)) ≤* x.

Now suppose γ ≠ ∞. By the first claim, DEF({γ}). Obviously γ ≠ 0.

Let γ = x+1. Let f(y) = H(y) if J(y) <* x; 0 if J(y) = x. Note that $f^{-1}(0) = J^{-1}(0) \cup J^{-1}(x)$, which is negative. Also DEF(f), since DEF({γ},{x},f,{0}). Hence (x,f) is a semi reduction, x <* γ. Hence there is a reduction (y,f), y ≤* x <* γ, by the previous claim. This is a contradiction. QED

Note that if γ = ∞, then we can still use γ as if DEF({γ}).

LEMMA 6.10. (T_4). If $x <^* \gamma$ then $J^{-1}[<^*[x]]$ is negative. Let $f:<^*[\gamma] \to <^*[x]$ be partially defined, dom(f) is positive, $x <^* \gamma$, and DEF(f). Then some $f^{-1}(y)$ is positive.

Proof: If $\gamma = \infty$ then $J^{-1}[<^*[x]] = <^*[x]$, which is negative by Lemma 6.7. Assume $x <^* \gamma \neq \infty$. Suppose $J^{-1}[<^*[x]]$ is positive. Define $f:V \to <^*[x]$ by $f(y) = J(y)$ if $J(y) <^* x$; 0 otherwise. Since the default set is negative, we see that (x,f) is a semi reduction, $x <^* \gamma$. By Lemma 6.9, $\gamma \leq^* x$, which is a contradiction.

For the second claim, let f be as given. We can assume that x is $<^*$ least such that $\text{rng}(f) \subseteq <^*[x]$. Extend f to $g:<^*[\gamma] \to <^*[x]$, where g is 0 off of dom(f). Then DEF(g). If all $g^{-1}(y)$ are negative, then (x,g) is a reduction. But this contradicts the definition of γ. QED

LEMMA 6.11. (T_4). Let $x <^* \gamma$, $f:<^*[x] \to <^*[\gamma]$, and DEF(f). Then $\sup(\text{rng}(f)) <^* \gamma$.

Proof: let x,f be as given. Suppose $\sup(\text{rng}(f)) = \gamma$. Note that DEF({x}). Define $g:V \to <^*[x]$ by $g(z) = \min\{w <^* x: J(z) <^* f(w)\}$. Note that g is well defined since $\sup(\text{rng}(f)) = \gamma$. Also DEF(g).

Let $y <^* x$. We claim that $g^{-1}(y) \subseteq J^{-1}[<^*[f(y)]]$. To see this, suppose $g(z) = y = \min\{w <^* x: J(z) <^* f(w)\}$. In particular, $y \in \{w <^* x: J(z) <^* f(w)\}$. I.e., $J(z) <^* f(y)$, as required.

By Lemma 6.10, $J^{-1}[<^*[f(y)]]$ is negative, since $f(y) < \gamma$. This establishes that (x,g) is a semi reduction, $x <^* \gamma$. This is a contradiction. QED

DEFINITION 6.10. We treat $f:A \to \wp(B)$ as the corresponding class $\{P(x,y): y \in f(x)\}$, which is also a relation on $A \cup B$.

LEMMA 6.12. (T_4). Let $x <^* \gamma$. There is no one-one $f:<^*[\gamma] \to \wp(<^*[x])$ with DEF(f). Let $g:<^*[\gamma] \to \wp(<^*[x])$ be partially defined and one-one, with DEF(g). Then dom(g) is a $<^*$ bounded subclass of $<^*[\gamma]$.

Proof: For the first claim, let $x <^* \gamma$. Let $f:<^*[\gamma] \to \wp(<^*[x])$ be one-one, with DEF(f).

We first define a special $E \subseteq <^*[x]$. For each $w <^* x$, we look at $\{z: w \in f(J(z))\}$ and $\{z: w \notin f(J(z))\}$. One of these is positive and the other is negative. E consists of the w for which the first of these two sets is positive. Obviously DEF(E).

We now define $g:V \to <^*[x]$. Set $g(z) = \min(E \Delta f(J(z))\}$ if this exists; 0 otherwise. Clearly DEF(g). We show that (x,g) is a semi reduction. This is a contradiction since $x < \gamma$.

In this definition of g, the default applies for g(z) if and only if $f(J(z)) = E$ (in which case, we are taking the min of $\emptyset$, which does not exist). I.e., when $J(z) = f^{-1}[(E)$. Here we are inverting f at the point E, as f is one-one. (If $f^{-1}(E)$ does not exist, or if it lies outside rng(J), then there is no default in the definition of g). Thus the default applies only for z in a single $J^{-1}(u)$, $u <^* \gamma$, or not at all. In any case, the default applies only on a negative class of z's.

It now suffices to show that for all $w <^* x$, $\{z: \min(E \Delta f(J(z)) = w\}$ is negative. This is because $\{z: \min(E \Delta f(J(z)) = w\}$ is the same as $g^{-1}(w)$ off of the negative default class of z's.

case 1. $w \in E$. Then $\{z: \min(E \Delta f(J(z)) = w\} \subseteq \{z: w \notin f(J(z))$. If $g(z) = w$ then $w \notin f(J(z))$. Hence z lies in the negative set $\{z: w \notin f(J(z))\}$. This set is negative because $w \in E$.

case 2. $w \notin E$. Then $\{z: \min(E \Delta f(J(z)) = w\} \subseteq \{z: w \in f(J(z))$. This set is negative because $w \notin E$.

This completes the proof of the first claim. For the second claim, let $g:<^*[\gamma] \to \wp(<^*[x])$ be partially defined, with DEF(g). Suppose dom(g) is $<^*$ unbounded in γ. Let h be the unique order preserving comparison between $(<^*[\gamma],<^*)$ and $(\mathrm{dom}(g),<^*)$. Then DEF(h). By Lemma 6.11, it is clear that h is an isomorphism. Now let $f:<^*[\gamma] \to \wp(<^*[x])$, where $f(z) = g(h(z))$. Then DEF(f), and apply the first claim to f,x. QED

LEMMA 6.13. (T_4). Suppose $<^*[x]$ is finite. Then DEF(x). Every $A \subseteq\ <^*[x]$ has DEF(A).

Proof: The first claim is by an obvious induction on the $x <^* \gamma$ with $<^*[x]$ finite (see Lemma 6.8). For the second claim, we prove by induction on the $x <^* \gamma$ with finite $<^*[x]$, that $(\forall A)(A \subseteq\ <^*[x] \wedge A \text{ finite} \rightarrow \text{DEF}(A))$. This is clearly true for $x = 0$. Suppose this is true for fixed $x <^* \gamma$ with finite $<^*[x]$. Let $A \subseteq\ <^*[x+1]$ be finite. Then $A\backslash\{x\}$ is finite, and so by the induction hypothesis, $\text{DEF}(A\backslash\{x\})$. Since $\text{DEF}(\{x\})$, we have DEF(A). QED

We now use the Divine Object axiom for the first time.

DEFINITION 6.11. x is divine if and only if $(\forall A)(\text{DEF}(A) \wedge \text{POS}(A) \rightarrow x \in A)$. Note that by the Divine Object axiom in T_5, there is a divine object.

LEMMA 6.14. (T_5). Let x be divine. $(\forall y)(\text{DEF}(\{y\}) \rightarrow y <^* x)$. $\neg\text{DEF}(\{x\}) \wedge\ <^*[x]$ is infinite. If $\gamma \neq \infty$ then $x >^* \gamma$. $(\exists z)(<^*[z] \text{ is infinite})$. The class K of all divine objects has $\neg$DEF(K).

Proof: Let x is divine. Let $\text{DEF}(\{y\})$. By Lemma 6.7, $>^*[y]$ is positive and $\text{DEF}(>^*[y])$. Hence $x \in\ >^*[y]$, $y <^* x$.

From the first claim, $\neg\text{DEF}(\{x\})$. Suppose $<^*[x]$ is finite. By the induction in Lemma 6.8, we see that $\text{DEF}(<^*[x])$, which is a contradiction.

Suppose $\gamma \neq \infty$. Then $\text{DEF}(\{\gamma\})$, $\text{DEF}(<^*[\gamma])$, and by Lemma 6.7, $\leq^*[\gamma]$ is negative. Hence $x \geq^* \gamma$.

By the second claim, $<^*[x]$ is infinite. Hence $(\exists z)(<^*[z] \text{ is infinite})$.

If the class K of all divine objects has DEF(K), then let x be the $<^*$ least element of K. Clearly $\text{DEF}(\{x\})$ and x is divine. This contradicts the second claim. QED

DEFINITION 6.12. ω is the $<^*$ least object such that $<^*[\omega]$ is infinite.

LEMMA 6.15. (T5). $\text{DEF}(\omega)$. $\omega <^* \gamma$.

Proof: DEF(ω) by Definition 6.12. By Lemma 6.9, γ is a $<*$ limit point. Hence $<*[\gamma]$ is infinite. Therefore $\omega \leq* \gamma$.

Now suppose $\gamma = \omega$. Then (ω,J) is a reduction. Let x be divine, and $J(x) = n <* \omega$. Now J-1(n) is negative. By Lemma 6.13, DEF({n}), DEF($J^{-1}(n)$). Therefore $x \notin J^{-1}(n)$. But $x \in J^s(n)$, and we have a contradiction. Hence $\omega <* \gamma$. QED

We now want to code the constructible hierarchy up through γ. For this purpose, we will need the "0-definable regularity" of γ afforded by Lemma 6.11.

Throughout this section, and even section 7, we will operate under ignorance as to whether $\gamma = \infty$ or $\gamma \neq \infty$. We are able to argue uniformly. We will also freely use ω, and so we will from now on just consider our claims to be proved in T_5.

We now sharpen Lemma 6.11.

DEFINITION 6.13. A is a $<*$ bounded subclass of $<*[\gamma]$ if and only if $\sup(A) <* \gamma$. B is a $<*$ bounded subclass of $<*[\gamma]^2$ if and only if there exists $x <* \gamma$ such that for all $(y,z) \in B$, $y,z <* x$.

LEMMA 6.16. Let $f:V \rightarrow <*[\gamma]$ be partially defined, where DEF(f). Then f maps $<*$ bounded subclasses of $<*[\gamma]$ onto $<*$ bounded subclasses of $<*[\gamma]$. Let $g:V^2 \rightarrow <*[\gamma]$ be partially defined, where DEF(g). Then g maps $<*$ bounded subclasses of $<*[\gamma]^2$ onto $<*$ bounded subclasses of $<*[\gamma]$.

Proof: The first claim follows from the second easily by using g(x,x).

For the second claim, let g be as given. Suppose this is false, and let x be $<*$ least such that $x <* \gamma$ and $g[<*[x]^2]$ is $<*$ unbounded in $<*[\gamma]$. Then DEF({x}).

We claim that for all $y <* \gamma$, $\sup(\{g(y,z): z <* x\}) <* \gamma$. Suppose this is false, and let y be $<*$ least such that $y <* x$ and $\sup(\{g(y,z): z <* x\}) = \gamma$. Then DEF({y}), and we can apply Lemma 6.11 to the function g(y,z), with y fixed. Hence $\sup(\{g(y,z): z <* x\}) <* \gamma$. This contradicts the choice of y.

We now define $h:<^*[\gamma] \rightarrow <^*[\gamma]$ by $h(y) = \sup(\{g(y,z): z <^* x\}) <^* \gamma$, using the claim in the previous paragraph. Then by Lemma 6.11, $\sup(\text{rng}(h)) <^* \gamma$. But clearly $\sup(\text{rng}(h)) = \sup(\{g(y,z): z <^* x\}) <^* \gamma$. This contradicts the choice of x. QED

The pairing function P is rather uncontrolled. E.g., P(0,0) might even be larger than γ (although of course this can't happen if $\gamma = \infty$). We need a pairing function that maps $<^*[\gamma]^2$ into $<^*[\gamma]$.

DEFINITION 6.14. $(x,y) <' (z,w) \leftrightarrow \max(x,y) <^* \max(z,w) \vee (\max(x,y) = \max(z,w) \wedge (x,y) <_{lex} (z,w))$. Here max and $<_{lex}$ are with respect to $<^*$.

LEMMA 6.17. $(V^2,<')$ is a well ordering.

Proof: $<'$ is obviously irreflexive. Let $(x,y) <' (z,w) \wedge (z,w) <' (u,v)$. Clearly $\max(x,y) \leq^* \max(z,w) \leq^* \max(u,v)$. If not all three of these max's are equal, then $(x,y) <' (u,v)$. So assume that all three max's are equal. Then $(x,y) <_{lex} (z,w) <_{lex} (u,v)$. Therefore $(x,y) <' (u,v)$.

Let $(x,y) \neq (z,w)$. If $\max(x,y) \neq \max(z,w)$ then $(x,y) <' (z,w) \vee (z,w) <' (x,y)$. So assume that $\max(x,y) = \max(z,w)$. If $(x,y) <_{lex} (z,w)$ then $(x,y) <' (z,w)$. If $(z,w) <_{lex} (x,y)$ then $(z,w) <' (x,y)$.

Let $A \subseteq V^2$ be nonempty. Cut down to the elements of A with $<^*$ least possible max. Then cut down to those with $<^*$ least possible first coordinate. Then pick the one with $<^*$ least second coordinate. This is the $<'$ least element of A. QED

DEFINITION 6.15. We say that f is a $(<',x,y)$ function if and only if
i. $f:<'[(x,y)] \rightarrow V$.
ii. For all $(z,w) <' (x,y)$, $f(z,w) = \sup(\{f(u,v): (u,v) <' (z,w)\})$.

LEMMA 6.18. If f is a $(<',x,y)$ function and g is a $(<',z,w)$ function, then $f \subseteq g \vee g \subseteq f$. For all $x,y <^* \gamma$, there is a unique $(<',x,y)$ function. The range of this function is a $<^*$ bounded subset of $<^*[\gamma]$.

Proof: The first claim is easily proved by using the <' least (u,v) such that $f(u,v) \neq g(u,v)$.

We now prove the remaining claims. Let (x.y) be <' least with x,y <* γ, such that there is no (<',x,y) function f with sup(rng(f)) < γ. Then x is unique such that there exists y such that (x,y) is <' least with this property, and y is unique such that (x,y) is <' least with this property. Hence DEF({x}), and therefore DEF({y}).

Clearly (x,y) is not <' least. I.e., $(x,y) \neq (0,0)$.

case 1. (x,y) is a <' immediate successor. Let (x,y) be the <' immediate successor of (z,w). Then z,w <* γ. Also DEF({z},{w}). Let f be the (<',z,w) function. By the choice of (x,y), sup(rng(f)) <* γ. Also DEF(f). Extend f at (z,w) by f'(z,w) = sup(rng(f)). Then DEF(f'), and sup(rng(f')) <* γ. This contradicts the choice of (x,y).

case 2. (x,y) is a <' limit point. Let g be the union of the various (<',z,w) functions, where (z,w) <' (x,y). This union is a function by the first claim. Clearly DEF(g), and g is a (<',x,y) function. Note that g is a partial function from $\leq^*[\max(x,y)]^2$ into <*[γ]. By Lemma 6.16, sup(rng(g)) <* γ. This contradicts the choice of (x,y).

Thus we have a contradiction. Hence For all (x,y) <* γ, there is a (<',x,y) function f with sup(rng(f)) <* γ. By the first claim, f is unique. QED

DEFINITION 6.16. We define P*(x,y) for x,y <* γ as follows. Let (x',y') be the <' immediate successor of (x,y). Let f be the unique <'(x',y') function. Define P*(x,y) = f(x',y'). For other x,y, P*(x,y) is undefined.

LEMMA 6.19. $P^*:<^*[\gamma]^2 \rightarrow <^*[\gamma]$ is one-one with DEF(P*). x,y $\leq^*$ P*(x,y).

Proof: DEF(P*) by Definition 6.16. Now let P*(x,y) = P*(z,w). We want $x = z \wedge y = w$. By symmetry, assume (x,y) <' (z,w). Let (x',y'),(z',w') be the <' immediate successors of (x,y),(z,w), respectively. Clearly x',y',z',w' <* γ, (x',y') <' (z',w').

By Lemma 6.18, let f,g be the (<',x',y'),(<',z',w') functions, respectively. By Lemma 6.18, $f \subseteq g$. Also P*(z,w)

= g(z,w) = sup({g(u,v): (u,v) <' (z,w)}). Hence f(x,y) = g(x,y) = P*(x,y) <* f(z,w) = P*(z,w), which is a contradiction.

For the second claim, suppose this is false. Let (x,y) be <' least such that x,y ≤* P*(x,y) is false. Then (x,y) ≠ (0,0). Suppose (x,y) is the <' immediate successor of (z,w). Clearly max(x,y) ≤* max(z,w)+1 ≤* P*(z,w)+1 = P*(x,y). Let (x,y) be a <' limit point. For all (z,w) <' (x,y), z,w ≤* P*(z,w). Now P*(x,y) = sup({P*(z,w): (z,w) <' (x,y)} ≥* sup({max(z,w): (z,w) <' (x,y)}) ≥* max(x,y). Thus x,y ≤* P*(x,y), which contradicts the choice of (x,y). This is a contradiction. QED

DEFINITION 6.17. We extend P* to any standard arity ≥ 2, left associatively, by P*(x_1,...,x_n) = P*(...(P*(x_1,x_2),...,x_n). We say that f is a finite sequence if and only if f:<*[x] → V, for some x <* ω. The length of f, lth(f), is x.

DEFINITION 6.18. Let f be a finite sequence of length x. A P* sequence for f is a finite sequence g of length x such that

i. 0 <* x → g(0) = P*(x,0,f(0)).

ii. (∀y+1 <* x)(g(y+1) = P*(x,g(y),f(y+1))).

LEMMA 6.20. Every finite sequence into <*[γ] has a unique P* sequence into <*[γ].

Proof: By obvious induction arguments along lengths x <* ω. Use x,y <* γ → P*(x,y) <* γ. QED

DEFINITION 6.19. Let f be a finite sequence of length x into <*[γ]. y is a P* code for f if and only if y = g(x-1) if x ≠ 0; P(0,0,0) otherwise, where g is the P* sequence for f. A P* code is a P* code for some finite sequence into <*[γ]. B has a P* code if and only if B is a finite sequence into <*[γ].

LEMMA 6.21. Two finite sequences into <*[γ] have the same P* code if and only if they are equal. (∀x <* γ)(∃y <* γ)(∀z,w <* x)(P*(z,w) <* y).

Proof: Suppose x,y are finite sequences into <*[γ] with different lengths. Their P* codes are of the form

P*(lth(x),z,w) and P*(lth(y),u,v), respectively. Hence their P* codes are not equal.

We prove by induction on x <* ω that any two finite sequences f,g, with lth(f) = x, and the same P* code, are equal. The basis case x = 0 is immediate, as there is exactly one finite sequence of length 0. Suppose this is true for fixed x ≠ 0.

Let f',g' be the P* sequences for f,g, respectively. We prove by backwards induction i = x-1,...,0, that f(i) = g(i). To see this, clearly f'(x-1) = g'(x-1), as these are the P* codes for f,g, respectively. Suppose f'(z) = g'(z), z ≠ 0. Then P*(x,f'(z-1),f(z)) = P*(x,g'(z-1),g(z)), and so f'(z-1) = g'(z-1).

Since f' = g', we see that for all 0 ≠ i <* x, f(i) = g(i), and also f(0) = g(0). Hence f = g.

The second claim follows immediately from Lemma 6.16. QED

LEMMA 6.22. Let x <* γ. The P* codes of finite sequences into <*[x] form a <* bounded subclass of <*[γ].

Proof: Suppose this is false, and let x <* γ be the <* least counterexample. Then DEF({x}). We first show that for all y <* ω, the P* codes of length y sequences into <*[x] form a <* bounded subclass of <*[γ].

Let y <* ω. We show by induction that for all i <* y, {g(i): g is a P* sequence for some finite sequence of length y into <*[x]} is <* bounded in <*[γ]. The basis case is obvious, and the induction step is clear using the second claim of Lemma 6.21.

By setting i = y-1, we see that the P* codes of finite sequences of length y into <*[x] are bounded in <*[γ]. (The case y = 0 is trivial).

Define g:<*[ω] → <*[γ] by g(y) = the sup of the P* codes of length y sequences into <*[x]. Then DEF(g). By Lemma 6.16, sup(rng(g)) <* γ. QED

DEFINITION 6.20. Let x be a P* code. lh(x) is lth(f), where the P* code of f is x. x<y> is f(y), where the P* code of f is x. Note that x<y> is defined if and only if y <* lh(x).

We are now in a position to code the constructible hierarchy below γ. We freely assume standard Gödel numberings of formulas in the language LST of set theory, which is based on $\in$,=, and using variables $v_1,v_2,\ldots$.

DEFINITION 6.21. Let $R \subseteq A^2$. SAT(A,R,x,y) if and only if
i. x <* ω is a Gödel number of a formula φ of LST.
ii. y is the P* code of a finite sequence f into A, where all subscripts of variables appearing in φ are <* lth(f).
iii. (A,R) satisfies φ at the assignment f, where R interprets $\in$.

Definition 6.19 is formalized in T_5 using Tarski's truth definition (satisfaction relation definition). This makes reference to a provably unique class that forms the relevant satisfaction relation.

DEFINITION 6.22. A wfes (well founded extensional structure) is an (A,R), $R \subseteq A^2$, where $(\forall x,y \in A)((\forall z)(R(z,x) \leftrightarrow R(z,y)) \rightarrow x = y)$, every nonempty subset of A has an element with no R predecessor from A, and A is a <* bounded subclass of <*[γ].

DEFINITION 6.23. Let (A,R) be given, $R \subseteq A^2$. x is a basic code over (A,R) if and only if
i. x is the P* code of a nonempty finite sequence f into A.
ii. $\{u \in A$: SAT(A,R,f(0),u,f(1),...,f(lth(f)-1))$\}$ is not of the form $\{a \in A$: R(a,b)$\}$, $b \in A$.
iii. $\{u \in A$: SAT(A,R,f(0),u,f(1),...,f(lth(f)-1))$\}$ is not $\{u \in A$: SAT(A,R,g(0),u,g(1),...,g(lth(g)-1))$\}$, where the P* code of nonempty g is <* x.
x is a code over (A,R) if and only if x = P*(y,sup(A)), where y is a basic code over (A,R).

The last line of Definition 6.23 is to guarantee that the codes are higher than A in <*.

DEFINITION 6.24. Let (A,R) be given, $R \subseteq A^2$. FODO(A,R) = (B,S), where B is A union the class of all codes over (A,R), and S(u,x) if and only if R(u,x) $\vee$ ($u \in A \wedge$ SAT(A,R,(f(0),u,f(1),...,f(lth(f)-1)))), where x = P*(y,sup(A)) and y is the P* code of f.

DEFINITION 6.25. We say that (A,R) is an initial segment of (B,S) if and only if $R \subseteq A^2 \wedge S \subseteq B^2 \wedge A = B \cap$ <*[sup(A)] $\wedge$ $R = S \cap A^2$. The union of a family of (A,R)'s is (B,S), where B is the union of the A's and S is the union of the R's. sup(A,R) = sup(A).

LEMMA 6.23. Every wfes (A,R) is a proper initial segment of the wfes FODO(A,R).

Proof: Let A,R be as given. The first component of FODO(A,R) is a <* bounded subclass of <*[γ] by Lemma 6.21. Extensionality in FODO(A,R) is clear by the construction in Definition 6.25. That (A,R) is an initial segment of FODO(A,R) is also clear by the construction in Definition 6.25, as the new elements are ≥ sup(A) using Lemma 6.19. For the well foundedness of FODO(A,R), let X be a nonempty subset of dom(FODO(A,R)). If X ∩ A ≠ ∅, then use an R least element of X. Otherwise, use any new element.

It remains to show that FODO(A,R) is not (A,R). We have only to show that some (A,R) definable B ⊆ A is not the set of R predecessors of any x ∈ A. Take B = {x ∈ A: ¬R(x,x)}. QED

Recall the convention in Definition 6.10.

DEFINITION 6.26. f is an L function if and only if f:<*[x] → ℘(<*[γ]), where
i. x <* γ.
ii. 0 <* x → f(0) = (∅,∅).
iii. y+1 <* x → f(y+1) = FODO(f(y)).
iv. Let y <* x, and y be a <* limit point. $f(y) = \cup_{z<^*y} f(z)$.

In T_5, f:V → ℘(V) are interpreted as corresponding relations on V.

LEMMA 6.24. If f,g are L functions then f ⊆ g ∨ g ⊆ f. For all x <* γ there exists a unique L function f:<*[x] → ℘(<*[γ]). This unique f has sup(∪rng(f)) <* γ. Furthermore, for all y ≤* z <* x, f(y) is an initial segment of f(z), and f(y) is a wfes.

Proof: Let f,g be L functions, and suppose (∃x)(f(x) ≠ g(x)). Let x be <* least such that f(x) ≠ g(x). Then f(x) = g(x), which is a contradiction. Hence f,g are comparable.

We now prove that for all x <* γ, there exists an L function f:<*[x] → <*[γ] with sup(∪rng(f)) <* γ, and for all y ≤* z <* x, the wfes f(y) is a proper initial segment of the wfes f(z). Suppose this is false, and let x be the <* least counterexample. Obviously x ≠ 0, and DEF(x).

case 1. x is a <* limit point. Let f be the union of the L functions whose domain is some <*[y], y <* x. This union is a function by the first claim. Clearly f:<*[x] → ℘(<*[γ]). By Lemma 6.16, sup(rng(f)) <* γ. Also, the initial segment property holds for f, since the initial segment property holds for the L functions whose domain is some <*[y], y <* x.

case 2. x = y+1. Let f be the L function with f:<*[y] → <*[γ]. If y = 0 then extend f at 0 by f'(0) = (∅,∅). If y = z+1 then extend f at y by f'(y) = FODO(f(z)). Since sup(∪rng(f)) <* γ, we see that sup(∪rng(f')) <* γ, by Lemma 6.22. Also the initial segment property holds by Lemma 6.23. This contradicts the choice of x.

Assume y is a <* limit point. Let f be the L function with f:<*[y] → <*[γ]. Let (A,R) = $\cup_s$f(z). Since y <* x, sup(∪rng(f)) <* γ. From the initial segment property for f, we see that (A,R) is a wfes. Hence FODO(A,R) is a wfes. Extend f at y by f'(y) = (A,R). Then f' is an L function with f':<*[x] → <*[γ], sup(∪rng(f')) <* γ, and we have the initial segment property. This contradicts the choice of x.

Thus we have arrived at a contradiction, and so for all x <* γ, there exists an L function f:<*[x] → <*[γ] with sup(∪rng(f)) <* γ, where for all y ≤* z <* x, f(y) is an initial segment of f(z), and f(y) is a wfes. QED

DEFINITION 6.27. Let x <* γ. (L[x],E[x]) = f(x), where f is the L function with domain <*[x+1].

LEMMA 6.25. (L[0],E[0]) = (∅,∅). For all x <* γ, (L[x+1],E[x+1]) = FODO(L[x],E[x]). For all <* limit points x <* γ, L[x] = $\cup_{y<^*x}$L[y], E[x] = $\cup_{y<^*x}$E[y]. For all x <* y <* γ, (L[x],E[x]) is an initial segment of (L[y],E[y]). For all x,y,z <* γ, (E[x](y,z) → y <* z) ∧ (z ∈ E[y]\E[x] → sup(E[x] <* z).

Proof: The first four claims follow easily from Lemma 6.24 using Definition 6.29. For the last claim, let x be the <* least counterexample. Clearly $x \neq 0$. Let $x = w+1$. Since w has the property, x has the property by the FODO construction, and $x,y \leq P^*(x,y)$ from Lemma 6.19. Let x be a <* limit point. Then x has the property, using that each y <* x has the property, and the initial segment claims in Lemma 6.24. We have contradicted the choice of x, and hence have a contradiction. QED

DEFINITION 6.28. $L = \cup_{x<^*\gamma} L[x]$, $E = \cup_{x<^*\gamma} E[x]$.

LEMMA 6.26. DEF(L,E). $E(x,y) \rightarrow x <^* y$. $L \cap <^*[y] \subseteq L[y]$.

Proof: DEF(L,E) is clear from the definitions of L,E. The second claim is by Lemma 6.24 and the last claim of Lemma 6.25. For the third claim, let $z \in L \cap <^*[y]$. Let $z \in L[w]$, $z <^* y$. If $w \leq^* y$ then $z \in L[y]$. If $w >^* y$ then by Lemma 6.24, (L[y],E[y]) is a proper initial segment of (L[w],E[y]). Hence $z \in L[y]$. QED

THEOREM 6.27. T_5 proves (L,E) is a model of ZF. T_5 proves the consistency of ZFC, when formulated with standard formalized syntax. ZFC is interpretable in T_5. EFA proves $Con(T5) \rightarrow Con(ZFC)$.

Proof: In T_5, we can define the satisfaction relation over (L,E), no matter whether $\gamma = \infty$ or $\gamma < \infty$. The axioms of extensionality, pairing, union, infinity, foundation are easily verified. This leaves separation, power set, and replacement.

Separation is verified by adapting the standard argument using Skolemization, in order to pull down a formula φ with unbounded quantifiers, and parameters from some L[x], to a formula ψ with quantifiers bounded to some L[y], where φ,ψ are equivalent for all parameters from some L[x]. Here ψ depends only on φ, and y depends only on x,φ. In the sense internal to (L,E), the required set lies in L[y+1]. This pulling uses the regularity of γ given by Lemma 6.16.

Replacement is established by first using the regularity of γ given by Lemma 6.16, and then applying separation.

For power set, we show that $(\forall x)(\exists y)(\forall z)((L,E)$ satisfies $z \subseteq x \rightarrow z \in L[y])$. This suffices, since we have already established Separation in (L,E).

Suppose this is false, and let x be the <* least counterexample. Then DEF(x).

We define partial $f:<^*[\gamma] \rightarrow \wp(<^*[\gamma])$ by $f(z) = \{w: E(w,z)\}$ if (L,E) satisfies $z \subseteq x$; undefined otherwise. Then DEF(f). By extensionality in (L,E), f is one-one. Also for all $z \in \text{dom}(f)$, $f(z) \in \wp(<^*[x])$ by Lemma 6.26. By Lemma 6.12, let $y = \sup(\text{dom}(f))$. By Lemma 6.26, $L \cap <^*[y] \subseteq L[y]$. Hence $(\forall z)((L,E)$ satisfies $z \subseteq x \rightarrow z \in L[y])$. This contradicts the choice of x.

Thus we have proved the first two claims. For the third claim, note that ZFC is interpretable in ZF. The fourth claim follows easily from the second claim. QED

7. Interpreting a strong extension of ZFC in T_5.

In this section, we extend Theorem 6.26 to ZFC + "there are arbitrarily large strong Ramsey cardinals". See Definition 7.18.

We use the development and notation from section 6. The ideas are an adaptation of the well known inner model construction L[U], where U is a nontrivial κ complete ultrafilter on a measurable cardinal κ, to the present context. See, e.g., [Ka94], p. 261.

One difference in the present construction is that we are not sitting inside a model of ZFC above γ. Recall that we have only $\gamma \in V \vee \gamma = \infty$, and we are arguing uniformly under both cases. Thus we will not be getting a model of ZFC + "there exists a measurable cardinal". We know that we cannot get such a model from T_5 because of the last claim in Theorem 5.1.

Another difference is that we cannot normalize the ultrafilter given by POS before starting the construction. Usually, in set theory, with L[U] type constructions, the filter is normalized first before constructing from it. We will, however, still get an internal γ complete ultrafilter, which, internally, can be normalized - even though we are not sitting within a model of ZFC. Then the normal γ

complete ultrafilter is used in familiar ways to prove the existence of arbitrarily large strong Ramsey cardinals below γ. This will be enough to get a model of ZFC with arbitrarily large strong Ramsey cardinals.

We argue entirely in T5.

DEFINITION 7.1. Let (A,T) be a well ordering. An end extension of (A,T) is a well ordering (A',T') such that
i. $A \subseteq A' \wedge T \subseteq T'$.
ii. $y \in A \wedge T'(x,y) \rightarrow x \in A$.
A proper end extension of (A,T) is an end extension which is not (A,T).

In this section, in a typical well ordering (A,T), A is a $<^*$ unbounded subclass of $<^*[\gamma]$, and (A,T) is much longer than $(<^*[\gamma],<^*)$. The end extensions (B,W) of (A,T) will typically have many $x \in A$ and $y \in B\backslash A$ with $x >^* y$. This is in contrast with the initial segment notion in Definition 6.27, which is tied to $<^*$.

DEFINITION 7.2. We often consider $(A,T \cap A^2)$. It is convenient to define $(A,T/) = (A,T \cap A^2)$, as the expression for A may be long.

The following definition was made in section 6 only for $<^*$ (Definition 6.14). Here we need the more general notion.

DEFINITION 7.3. Let (A,T) be a wo. We define $<_{A,T}$ on A^2 as follows. $(x,y) <_{A,T} (z,w) \leftrightarrow T(\max(x,y),\max(z,w)) \vee (\max(x,y) = \max(z,w) \wedge (x,y) <_{lex} (z,w))$. Here max and $<_{lex}$ are with respect to (A,T).

LEMMA 7.1. If (A,T) is a wo then $(A^2,<_{A,T})$ is a well ordering.

Proof: See Lemma 6.17. QED

LEMMA 7.2. Let $A \subseteq <^*[\gamma]$ be $<^*$ unbounded in $<^*[\gamma]$, DEF(A). There is a unique isomorphism from $(A,<^*/)$ onto $(<^*[\gamma],<^*/)$. This unique isomorphism f has DEF(f).

Proof: Let A be as given. First show that the isomorphisms from $(A \cap <^*[x],<^*/)$ onto some $(<^*[y],<^*/)$ cohere. Then show that for all $x <^* \gamma$, there exists a unique isomorphism from $(A \cap <^*[x],<^*/)$ onto some $(<^*[y],<^*/)$, using the regularity given by Lemma 6.11. Finally, take the union f of these coherent isomorphisms. By Lemma 6.11, f is onto. QED

LEMMA 7.3. Let $A \subseteq <^*[\gamma]$ be $<^*$ unbounded in $<^*[\gamma]$, $0 \neq x \leq^* \gamma$, and DEF(A,{x}). There exists $f:A \to <^*[x]$ such that for all $y <^* x$, f-1(y) is $<^*$ unbounded in $<^*[\gamma]$, and DEF(f).

Proof: Let A,x be as given. In light of Lemma 7.2, it suffices to find $f:<^*[\gamma] \to <^*[x]$ such that for all $y <^* x$, $f^{-1}(y)$ is $<^*$ unbounded in $<^*[\gamma]$, and DEF(f). Take $f:<^*[\gamma] \to <^*[x]$ given by $f(y) = z$ if $(\exists w <^* \gamma)(z <^* x \wedge y = P^*(z,w))$; 0 otherwise. Then for all $z <^* x$, $f^{-1}(z) = \{P^*(z,w): w <^* \gamma\}$, which is unbounded in $<^*[\gamma]$ by Lemma 6.11 and that P^* is one-one. QED

DEFINITION 7.4. The f constructed in the proof of Lemma 7.2 is called the x splitting of A.

LEMMA 7.4. There exists $A \subseteq <^*[\gamma]$, where
i. $<^*[\omega] \subseteq A$.
ii. $<^*[\gamma]\backslash A$ is $<^*$ unbounded in $<^*[\gamma]$.
iii. POS(A)
iv. DEF(A).

Proof: Let f be the 2 splitting of $<^*[\gamma]$. Then $POS(f^{-1}(0)) \vee POS(f^{-1}(1))$. Take $A = f^{-1}(0) \cup <^*[\omega]$ if $POS(f^{-1}(0))$; $f^{-1}(1) \cup <^*[\omega]$ otherwise. We are using Lemma 6.7. Here $0 <^* 1 <^* 2$ are the first three elements of $<^*[\omega]$. QED

DEFINITION 7.5. Let X be the A constructed in the proof of Lemma 7.4. Let θ be the $<^*$ least element of $<^*[\gamma]\backslash A$. Clearly $\theta >^* \omega$.

All of the well orderings that we will be now using will end extend $X \cup \{\theta\}$, with θ moved to just above X.

θ is very important to the rest of this section. Ultimately, θ will become an internal strongly inaccessible cardinal with large cardinal properties, and $V(\theta)$ will be a model of ZFC with arbitrarily large strong Ramsey cardinals.

DEFINITION 7.6. A good wo is a well ordering (A,T) such that
i. $A \subseteq <^*[\gamma]$.

ii. $<^*[\gamma]\backslash A$ is $<^*$ unbounded in $<^*[\gamma]$.
iii. (A,T) is an end extension of $(X \cup \{\theta\},\{(x,y): (x <^* y \wedge x,y \in X) \vee (x \in X \wedge y = \theta)\})$.

We impose condition ii because we will want to be end extending good wo's, and we need some room.

DEFINITION 7.7. (A,T) is an adequate wo if and only if
i. (A,T) is a good wo.
ii. $(A^2,<_{A,T})$ is order isomorphic to (A,T).
$P_{A,T}$ is the unique order isomorphism from $(A^2,<_{A,T})$ onto (A,T).

LEMMA 7.5. $(X \cup \{\theta\},\{(x,y): (x <^* y \wedge x,y \in X) \vee (x \in X \wedge y = \theta)\})$ is a good wo whose two components have DEF. Let (A,T) be a good wo, DEF(A,T). There is an adequate wo (B,W) which is a proper end extension of (A,T), with DEF(B,W).

Proof: The first claim is immediate. For the second claim, we put ω successive copies of $(A^2,<_{A,T})$ on top of (A,T). We have to replace ordered pairs with points, and we also have to disjointify. For the n-th copy, $n <^* \omega$, use points from $f^{-1}(n+1)$, where f is the ω splitting of $<^*[\gamma]\backslash A$. Specifically, for each $n <^* \omega$, let $g_n:<^*[\gamma] \to f^{-1}(n+1)$ be the unique $<^*$ preserving bijection, and $h_n:A^2 \to f^{-1}(n+1)$ be given by $h_n(y,z) = g_n(P_{A,T}(y,z))$. The n-th successive copy of $(A_2,<_{A,T})$ will be the push of h_n, so that its domain is a $<^*$ unbounded subclass of $f^{-1}(n+1)$. Note that $f^{-1}(0)$ remains unused. QED

An adequate wo (A,T) is a pretty good place to build hierarchies over its $<^*$ unbounded domain, because of its closure under the natural pairing function $P_{A,T}$. However, it is more convenient to have the stronger closure property in Definition 7.8.

DEFINITION 7.8. f is special for (A,T) if and only if
i. (A,T) is an adequate wo.
ii. dom(f) = A.
iii. f(0) = (A,T).
iv. If T(x,y) then f(x),f(y) are adequate wo's, and f(y) is a proper end extension of f(x).
v. $\cup$rng(f) is an adequate wo.

LEMMA 7.6. Let (A,T) be an adequate well ordering, DEF(A,T). There is a special f for (A,T), DEF(f).

Proof: Let (A,T) be as given. We consider specific approximations to the desired f. These are based on the γ splitting F of <*[γ]\A. We require that f(0) = (A,T), and that f at limits in (A,T) be the union of the earlier values. At successors x' in (A,T), we require that f(x') be the end extension of f(x) constructed in the proof of Lemma 7.5, then trivially modified so that the new points lie in F^{-1}(x').

There is a longest such f. If dom(f) = A then we are done. Assume dom(f) ≠ A. Then DEF(f), and so we can extend f. Hence dom(f) = A. In the construction, we never use F^{-1}(0), and so condition v holds. QED

DEFINITION 7.9. (A,T) is a special wo if and only if (A,T) is an adequate wo which is order isomorphic to (B,T ∩ B^2), where B = {x ∈ A: {y: T(y,x)} is closed under $P_{A,T}$}}. We write $P\#_{A,T}$ for this order isomorphism.

LEMMA 7.7. Let (A,T) be an adequate wo, DEF(A,T). There is a special wo (B,W) which is a proper end extension of (A,T), with DEF(B,W).

Proof: Let (A,T) be as given. We construct (A,T) = $(A_0,T_0),(A_1,T_1)$, ..., and functions $f_0,f_1,\ldots$, such that each f_i is special for (A_i,T_i), and each (A_{i+1},T_{i+1}) is the union of the range of f_i. Specifically, each f_i is given by the explicit construction used in the proof of Lemma 7.6.

The union (A',T') of the (A_i,T_i) meets all of the requirements of a special wo, except that <*[γ]\A' might be <* bounded in <*[γ]. This can be easily fixed using the 2 splitting of A'\(X ∪ {θ}), resulting in the desired (B,W), DEF(B,W). QED

We now develop the constructible hierarchy relative to POS, along any special wo (A,T). We can build this without assuming DEF(A,T).

DEFINITION 7.10. Let (A,T) be a special wo, and (D,T ∩ D^2,R,U) be given, where D is a proper initial segment of A under T, X ∪ {θ} ⊆ D, R ⊆ D^2, U ⊆ D. Furthermore, we assume that (D,R) satisfies extensionality, and R(x,y) → T(x,y). We define the continuation of (D,T ∩ D^2,R,U) in (A,T) as the following quadruple (D',T ∩ D'^2,R',U'). We have D ⊆ D', and

D' is a proper initial segment of A under T. The sets {y: R'(y,x)}, x ∈ D', comprise exactly the subsets of D that are first order definable over $(D,T \cap D^2,R,U)$, without repetition (extensionality). We accomplish this by assigning, to each x ∈ D'\D, one finite sequence $(\varphi,y_1,...,y_n)$, φ a formula in the language of $(D,T \cap D^2,R,U)$, with free variables assigned the elements $y_1,...,y_n \in D$, with the free variable v of φ reserved to define the associated subset of D definable over $(D,T \cap D^2,R,U)$. The constraint is that extensionality is to be maintained. Suppose that we have used, in this way, all y above D but below a given x, under T. For x, we use the "least" finite sequence that will maintain extensionality when used in this manner. (If we cannot maintain extensionality, then the construction has already been completed). Here "least" is in terms of the ordering T of finite sequence codes arising out of the pairing function $P_{A,T}$. We take U' to consist of all x ∈ D' such that {y: R'(y,x)} ⊆ X and POS({y: R'(y,x)}).

Definition 7.10 is a more careful form of Definitions 6.23, 6.24. Here we are working in an environment (A,T), of which there are many - rather than previously in one specific environment (<*[γ],<*). Here we retain (A,T) as part of a richer structure, that also incorporates a piece of POS. So we make sure that we use initial segments of (A,T) as sets of points. In Definition 6.29, we didn't care about L[x] skipping over points in <*[γ].

LEMMA 7.8. Definition 7.10 is well defined. $(D,T \cap D^2,R,U)$ is a proper initial segment of $(D',T' \cap D'^2.R',U')$ in the sense that $(D',T' \cap D'^2)$ is a proper end extension of $(D,T \cap D^2)$, $R = R' \cap D^2$, and $U = U' \cap D$.

Proof: For the first claim, the issue is whether there is enough room to go through the finite sequences from D in the manner stipulated. This follows from the adequacy of (A,T): specifically the closure of A under $P_{A,T}$.

For the second claim, it suffices to show that some E ⊆ D that is $(D,T \cap D^2,R,U)$ definable if not the set of R predecessors of any x ∈ D. Take E = {x ∈ D: ¬R(x,x)}. QED

DEFINITION 7.11. Let (A,T) be a special well ordering. For each x ∈ A, we define L[x,A,T,POS] to be a proper initial segment of A, E[x,A,T,POS] ⊆ T ∩ L[x,A,T,POS]2, POS*[x,A,T] ⊆ L[x,A,T,POS], as follows. We take L[0,A,T,POS] = X ∪ {θ}, E[0,A,T,POS] = T ∩ $(X \cup \{\theta\})^2$, POS*[0,A,T] = {θ}. If x is a limit point in (A,T), then L[x,A,T,POS] = ∪{L[y,A,T,POS]: T(y,x)}, E[x,A,T,POS] = ∪{E[y,A,T,POS]: T(y,x)}, POS*[x,A,T] = ∪{POS*(y,A,T): T(y,x)}. Let x' be the immediate successor of x in (A,T). We take (L[x',A,T,POS], T/,E[x,A,T,POS],POS*[x,A,T)) to be the continuation of (L[x,A,T,POS],T/,E[x,A,T,POS],POS*[x,A,T]) in (A,T), assuming L[x,A,T,POS] is a proper initial segment of A.

Definitions 7.10, 7.11 are formalized in T5 using approximating functions, as in section 6.

LEMMA 7.9. Definition 7.11 is well defined for all x ∈ A.

Proof: The issue is whether we run out of points in A during the construction. We must check that if x is a limit point in (A,T), and we have defined L[x,A,T,POS], E[x,A,T,POS], POS*[x,A,T), that we have not used up all of A. We use $P\#_{A,T}$ for this purpose. It is easy to show that for each x ∈ A, the construction can be made uniquely for all y with T(y,x), so that L[y,A,T,POS] ⊆ $P\#_{A,T}(2+y)$. Since (A,T) is a special wo, A is closed under $P\#_{A,T}$. QED

DEFINITION 7.12. Let (A,T) be a special wo. E[A,T,POS] = $\cup_{x \in A}$E[x,A,T,POS]. POS*[A,T] = $\cup_{x \in A}$POS*[x,A,T]. Let M[A,T] = (A,T,E[A,T,POS],POS*[A,T]). For purely set theoretic statements in M[A,T], quantifiers range over A, and ∈ is interpreted as E[A,T,POS].

LEMMA 7.10. Let (A,T) be a special well ordering. M[A,T] is well founded, and satisfies extensionality, pairing, union, infinity, foundation, Δ_0 separation, and "every set is in one-one correspondence with an ordinal". E[A,T,POS](x,y) → T(x,y). X forms a proper initial segment of the set theoretic ordinals of M[A,T], with θ as the next set theoretic ordinal of M[A,T]. The ordinals of M[A,T] under E[A,T,POS] are isomorphic to (A,T).

Proof: Left to the reader. T_5 is sufficient to formalize the statement that M[A,T] satisfies Δ_0 separation. QED

We want to work with an M[A,T] which satisfies "V(θ) exists". Some work is needed in order to find such M[A,T] within T_5. We need to develop canonical codes for some M[A,T].

LEMMA 7.11. There is a sentence φ in ∈,=,POS such that the following holds. Let (A,T) be a special well ordering. M[A,T] is the unique expansion of (A,T) satisfying φ which interprets POS correctly, in the following sense. For any x ∈ A, POS(x) holds if and only if the ∈ predecessors of x forms a subset of X which actually has POS.

Proof: Left to the reader. QED

DEFINITION 7.13. Let (A,T) be a special wo. For x ∈ A, define $M[A,T]_x$ = {y: E[A,T,POS](y,x)}. B is M[A,T] internal if and only if B = $M[A,T]_x$ for some necessarily unique x ∈ A. B $<_{A,T,POS}$ C if and only if (∃x,y)(B = M[A,T]x ∧ C = $M[A,T]_y$ ∧ T(x,y)).

DEFINITION 7.14. (A,T) is a great wo if and only if
i. (A,T) is a special wo.
ii. Some subset of X that is definable over M[A,T] with parameters from X, is not M[A,T] internal.
iii. ii holds for any special wo (B,W) such that (A,T) is an end extension of (B,W).

LEMMA 7.12. Let (A,T) be a great wo. Some subset of X definable over M[A,T] without parameters, is not M[A,T] internal.

Proof: This amounts to being able to kill the parameters used for the subset of X. First note that θ is defined as the least ordinal with POS*[A,T]. Hence X is also defined without parameters. Now use T least counterexamples to successively kill the parameters. QED

LEMMA 7.13. Let (A,T) be a great wo. Every element of A is definable over M[A,T] with parameters from X.

Proof: Let (A,T) be a great wo. By Lemma 7.12, let {x ∈ A: φ(x) holds in M[A,T]} be not M[A,T] internal, where φ has no parameters.

Let B be the set of all elements of A which are definable over M[A,T] with parameters from X. Using T least examples, we see that the restriction of M[A,T] to B, M[A,T]|B, is an elementary substructure of M[A,T].

It is easy to see that $(B,T \cap B^2)$ is a special wo. Since it is a restriction of M[A,T] that contains X, it interprets POS correctly. By Lemma 7.11, M[A,T] satisfies the sentence φ used there. Hence M[A,T]|B also satisfies this φ. Hence by Lemma 7.11, $M[A,T]|B = M[B,T \cap B^2]$. Clearly $\{x \in A: \varphi(x)$ holds in $M[A,T]\} = \{x \in A: \varphi(x)$ holds in $M[B,T \cap B^2]\}$ is not M[A,T] internal and not $M[B,T \cap B^2]$ internal. Also $(B,T \cap B^2)$ is obviously not longer than (A,T).

Suppose $(B,T \cap B^2)$ is shorter than (A,T). Then $(B,T \cap B^2)$ is isomorphic to a proper initial segment of (A,T). Therefore $\{x \in A: \varphi(x)$ holds in $M[B,T \cap B^2]\}$ is M[A,T] internal. This is a contradiction.

Thus $(B,T \cap B^2)$ and (A,T) are isomorphic, and therefore M[A,T] and $M[B,T \cap B^2]$ are isomorphic. The isomorphism must be the identity on X. Since every element of B is definable over $M[A,T]|B = M[B,T \cap B^2]$ with parameters from X, it is also the case that every element of A is definable over M[A,T] with parameters from X. We can also conclude that B = A (although this is not needed). QED

DEFINITION 7.15. Let (A,T) be a great wo. CODE(A,T) is the set of all sentences with parameters from X that hold in M[A,T]. Using $P_{A,T}$, CODE(A,T) is viewed as a subset of X.

LEMMA 7.14. Let (A,T),(B,W) be great wo's. We can explicitly recover an isomorphic copy of M'[A,T] from CODE(A,T). The following are equivalent.
i. CODE(A,T) = CODE(B,W).
ii. (A,T),(B,W) are isomorphic.

Proof: Let A,T,B,W be as given. Using CODE(A,T), we construct an isomorphic copy of M[A,T], and therefore of (A,T), as follows. First identify definitions of prospective points, with parameters from X, under the equivalence relation of equality. Then define the components of the prospective copy of M(A,T), acting on the equivalence classes. Then explicitly pick a unique representative from each equivalence class, using the explicit well ordering of X. By Lemma 7.13, this results in an isomorphic copy of M[A,T].

The second claim follows immediately from the first claim. QED

LEMMA 7.15. Let (A,T),(B,W) be great wo's. If (A,T) is shorter than (B,W), then CODE(A,T) is M[B,W] internal. CODE(A,T) is not definable over M[A,T]. In particular, CODE(A,T) is not M[A,T] internal.

Proof: Any great wo (B,W) has easily enough power to internally construct CODE(A,T) for each proper initial segment (A,T) that is a great wo.

Suppose CODE(A,T) is definable over M[A,T]. Then E = {(x,y): x codes the Gödel number of a formula φ and a finite sequence from X, and $y \in X$, and $\varphi(x,y)$} enumerates all subsets of X definable over M[A,T]. Since E can be read off of CODE(A,T), E is definable over M[A,T]. Therefore $\{x \in X: (x,x) \notin E\}$ is definable over M[A,T]. Fix $x \in X$ such that $\{y: (x,y) \in E\} = \{x \in X: (x,x) \notin E\}$. Then $(x,x) \in E \leftrightarrow (x,x) \notin E$. QED

LEMMA 7.16. There is a great wo, (A,T), with DEF(A,T).

Proof: We start with the good wo of Lemma 7.5, $(X \cup \{\theta\},\{(x,y): (x <^* y \wedge x,y \in X) \vee (x \in X \wedge y = \theta)\})$. The proof of Lemma 7.5, second claim, produces a specific adequate wo (A,T). The proof of Lemma 7.7 produces a specific proper end extension (B,W) of (A,T) which is a special wo. We claim that (B,W) is a great wo. To see this, we repeat these specific constructions internally in M[B,W]. We obtain $R \subseteq X^2$, definable over M[B,W], such that the R_x, $x \in X$, comprise exactly the M[B,W] internal subsets of X. $\{x: \neg R(x,x)\}$ is therefore definable over M[B,W], yet not M[B,W] internal. Clearly DEF(B,W). QED

LEMMA 7.17. Let (A,T) be a great wo, DEF(A,T). There is a great wo (B,W) which is a proper end extension of (A,T), with DEF(B,W).

Proof: Let A,T be as given. Apply the construction in the proof of Lemma 7.7 to (A,T), obtaining the special wo (B,W), DEF(B,W). As in the proof of Lemma 7.16, repeat this specific construction applied to (A,T), internally in M[B,W]. Argue as in the proof of Lemma 7.16 that there is a subset of X, definable over M[B,W], which is not M[B,W] internal. QED

LEMMA 7.18. Let $x \in X$. There is a great wo (A,T) such that the following holds.
i. M[A,T] satisfies "V(x) exists and is in one-one correspondence with some $u \in \theta$".
ii. For any great wo (B,W) longer than (A,T), the V(x) of M[B,W] is the same as the V(x) of M[A,T].

Proof: Assume the hypothesis. Suppose this is false, and let $x \in X$ be a $<^*$ least counterexample. Clearly DEF({x}), and $x \neq 0$.

case 1. x is the immediate successor of y in X. Clearly DEF({y}). Let (A,T) be a great wo and $u \in X$ be $<^*$ least such that

i. M[A,T] satisfies "V(x) exists and is in one-one correspondence with u".
ii. For any great wo (B,W) longer than (A,T), the V(x) of M[B,W] is the same as the V(x) of M[A,T].

Then DEF({u}). We now show that there is a great wo (B,W) and $v \in X$ such that

iii. M[B,W] satisfies "$\wp$(u) exists and is one-one correspondence with v".
iv. For any great wo (C,Y) longer than (B,W), the $\wp$(u) of M[C,Y] is the same as the $\wp$(u) of M[B,W].

To establish iii,iv, look at all of the M[B,W] internal subsets of u for the various great wo's (B,W), under the various $<_{B,W,POS}$. Suppose that the resulting length is at least that of $(<^*[\gamma],<^*)$. Then we can find a one-one $f:<^*[\gamma] \to \wp(<^*[u])$ with DEF(f), using that DEF({x},{y},{u}). This contradicts Lemma 6.12.

Thus the resulting length is that of some $(<^*[v],<^*/)$, $v <^* \gamma$, DEF({v}). We now show that we can find a single special wo (B,W) long enough so that all of the internal subsets of u living in the various great wo's are M[B,W] internal.

Clearly $v \neq 0$. Suppose v is a successor in $<^*$. We use any great wo which has all of these subsets of u, internally. Since v is a successor, $\wp$(u) exists internally in this great wo, and also the internal one-one correspondence with the analog of v in X.

Thus we now assume that v is a <* limit point.

For each w <* v, let h(w) be CODE(C,Y), for the shortest great wo (C,Y) whose internal subsets of u are of length at least (<*[w],<*/) under $<_{C,Y,POS}$. Note how the use of these codes, with Lemma 7.14, avoids use of any form of choice which is unavailable in T_5.

From each h(w), we can recover a copy of the corresponding great wo with code h(w), using Lemma 7.14. We can then put these great wo's together to form a single long special wo, (B,W).

It only remains to verify that (B,W) is a great wo. Obviously, if there is a longest h(w), w <* v, then there is nothing to prove, since h(w) is already great. So we assume that there is no longest h(w), w <* v.

To verify this, we shift the work along X, using v' ∈ X with the same position in X that v has in <*[γ]. Internally in M[B,W], define h'(w), w ∈ X, w <* v', as CODE(C,Y), for the shortest great wo (C,Y) whose internal subsets of u are of length at least the ordinal w.

Since each code is a subset of X, and these codes are indexed by {w ∈ X: w <* v'}, we can join them together to form a subset of X that is definable over M[B,W]. But this join cannot be M[B,W] internal. If it is M[B,W] internal, then it is M[C,Y] internal, for some great (C,Y) that is a proper initial segment of (B,W), and CODE(C,Y) is M[C,Y] internal. This contradicts Lemma 7.15.

So we have shown that M[B,W] is a great wo. But in order to get the desired contradiction, we need to properly end extend (B,W) to another great wo. Note that DEF(B,W). Hence we can apply Lemma 7.17.

case 2. x is a <* limit point in X. For each y <* x, y ∈ X, let f(y) be CODE(A,T), where (A,T) is a great wo of least length such that

i. M[A,T] satisfies "V(y) exists and is in one-one correspondence with some u ∈ θ".
ii. For any great wo (B,W) longer than (A,T), the V(y) of M[B,W] is the same as the V(y) of M[A,T].

We also let g(y) be the least u ∈ θ in clause i. Clearly DEF({x},f,g). By the regularity in Lemma 6.16, fix w to be the sup in X of the values of g. Note that DEF({w}).

By Lemma 7.14, for each y <* x, y ∈ X, recover a copy of the corresponding great wo with code f(y), and then piece these together to form a single long special wo, (B,W), DEF(B,W). By redoing the construction of (B,W) within M[B,W] as in case 1, we see that (B,W) is a great wo.

By Lemma 7.17, let (C,Y) be a great wo which is a proper end extension of (B,W). Then M[C,Y] satisfies that V(x) exists. Also, by Lemma 7.10, in M[C,Y], V(x) is in one-one correspondence with an ordinal. Using w and the regularity in Lemma 6.16, we see that in M[C.Y], V(x) is in one-one correspondence with an ordinal < θ. Hence x is not a counterexample, and we have a contradiction.

Since both cases lead to contradictions, x is not a counterexample, and the Lemma is proved. QED

LEMMA 7.19. There is a great wo (A,T) such that the following holds.
i. M[A,T] satisfies "V(θ) exists and is in one-one correspondence with θ".
ii. For any great wo (B,W) longer than (A,T), the V(θ) of M[B,W] is the same as the V(θ) of M[A,T].
iii. DEF(A,T).
iv. M[A,T] satisfies "θ is a strongly inaccessible cardinal".

Proof: First construct a great wo (A,T) with i,ii above, in a manner that is entirely analogous to how we argued case 2 in the proof of Lemma 7.18. I.e., for each x ∈ θ there exists a least code of a great wo that stabilizes V(x) in the sense of Lemma 7.18, with internal one-one correspondence with some u ∈ θ. Then piece these codes together to form a special wo (B,W) which stabilizes V(θ), and in light of Lemma 7.10, has V(θ) is one-one correspondence with θ. By redoing the construction inside (B,W), we see as before that (B,W) is a great wo. We will also have DEF(B,W) because of Lemma 7.14.

For iv, use this M[B,W] with i-iii above, and Lemma 6.12 and 6.16. We can apply Lemmas 6.12, 6.16, since DEF[B,W]. QED

We now normalize POS.

DEFINITION 7.16. We fix the great wo (A,T), built from CODE(A,T), given by Lemma 7.19.

For the remainder of this section, the only great wo that we use is (A,T). By "internal" we will always mean internal to M[A,T].

We will be making a number of arguments internally in M[A,T]. We do not have to worry about using choice within M[A,T], as all internal objects are given by points in A, and we have access to the well ordering T of A.

However, we have to be careful about the separation in M[A,T] that we use. We have V(θ) as internal in M[A,T]. However, we have only Δ_0 separation in M[A,T], and we have to be careful not to use separation in M[A,T] with unbounded quantifiers. Of course, we have access to POS internally.

We now normalize the ultrafilter POS in M[A,T].

DEFINITION 7.17. f is internally adequate if and only if $f:\theta \to \theta$, f is internal, and for all $x \in \theta$, $\neg$POS($f^{-1}(x)$). f <# g if and only if f,g are adequate and POS({x: f(x) <* g(x)}). We say that f is <# minimal if and only if f is internally adequate and there is no g <# f.

LEMMA 7.20. There exists f such that f is <# minimal and DEF(f).

Proof: Suppose there is no <# minimal f. Externally, we build internally adequate $f_0,f_1,\ldots$ as follows. f_0 is the identity on θ. f_{n+1} is the M[A,T] least f_{n+1} <# f_n. Let F(n,x) = $f_n(x)$, $x \in \theta$. We are not claiming that F is internal, but clearly DEF(F).

For all $x \in X$, let h(x) be the least n such that $f_n(x) \in \ldots \in f_0(x)$ is false. h:X $\to$ <*[ω] by well foundedness. Clearly DEF(h). Since ω <* γ, by Lemma 6.10, some $h^{-1}(n)$ is positive. I.e., fix n such that {$y \in \theta$: $f_n(x) \in \ldots \in f_0(x)$ is false} is positive.

We claim that $\{y \in \theta: f_n(y) \in ... \in f_0(y)\}$ is positive. This is proved by straightforward induction up through n. This contradicts the choice of n. We have a contradiction.

So there is a <# minimal f. Choose f to be the M[A,T] least <* minimal f. Then DEF(f). QED

DEFINITION 7.18. Fix H to be the # minimal f constructed in Lemma 7.20. Define POS#(E) if and only if $E \subseteq \theta$ is internal, and POS(H^{-1}[E]).

LEMMA 7.21. POS# is a nontrivial θ complete ultrafilter in the following sense.

i. If $x \in \theta$ then ¬POS#({x}).

ii. If $E \subseteq \theta$ is internal, then POS#(E) $\leftrightarrow$ ¬POS#($\theta \backslash E$).

iii. Let $f:\theta \to x$, $x \in \theta$, be partially defined and internal, where POS#(dom(f)). There exists $y \in x$ such that POS#($f^{-1}(y)$).

Proof: i follows from the internal adequacy of H. ii follows from the same statement for POS.

Suppose iii is false, and let f be the M[A,T] least counterexample. Let x be least such that $f:\theta \to x$. Then DEF(f,{x}).

Define $G:\theta \to x$ by G(y) = f(H(y)), $y \in \theta$. Then $H^{-1}[f^{-1}(y)] = G^{-1}(y)$. Clearly DEF(G). By Lemma 6.10, there exists $y \in x$ such that POS($G^{-1}(y)$), POS($H^{-1}[f^{-1}(y)]$), POS#($f^{-1}(y)$). QED

LEMMA 7.22. POS# is normal in the following sense. Let $f:\theta \to \theta$ be internal and partially defined, where POS#({x: $f(x) \in x$}). There exists $c < \theta$ such that POS#($f^{-1}(c)$).

Proof: Suppose this is false, and let f be the M[A,T] least counterexample. Note that POS#(dom(f)). Extend f to $f':\theta \to \theta$ by setting f' to be 0 on $\theta \backslash$dom(f).

There is no $c \in \theta$ such that POS#($f^{-1}(c)$). Hence there is no $c \in \theta$ such that POS#($f'^{-1}(c)$), since $f'^{-1}(c) = f^{-1}(c) \cup (\theta \backslash \text{dom}(f))$.

Let G(x) = f'(H(x)), $x <^* \gamma$. Then DEF(f',{x},G), and G is internal.

case 1. G is not internally adequate. Let POS($G^{-1}(c)$). Now $G^{-1}(c) = \{y: f'(H(y)) = c\} = \{y: H(y) \in f'^{-1}(c)\} = H^{-1}(f'^{-1}(c))$. Hence POS($H^{-1}(f'^{-1}(c))$), and so POS#($f^{-1}(c)$).

case 2. G is internally adequate. Since POS#($\{x: f(x) <^* x\}$), we have POS($\{x: f(H(x)) <^* H(x)\}$). Hence G <# H. But this contradicts that H is <# minimal.

Since both cases lead to contradictions, the Lemma has been proved. QED

LEMMA 7.23. POS# diagonal intersection. Let $R \subseteq \theta^2$ be internal, where for all $x \in \theta$, POS#(R_x). Then POS#($\{x \in \theta: (\forall y \in x)(x \in R_y)\}$).

Proof: Let R be as given. Suppose $\neg$POS#($\{x \in \theta: (\forall y \in x)(x \in R_y)\}$). Then POS#($\{x \in \theta: (\exists y \in x)(x \notin R_y)\}$). Let $f:\theta \to \theta$ be the partial function given by $f(x) = (\mu y \in x)(x \notin R_y)$. Then f is internal, POS#($\{x: f(x) \in x\}$). Let $c \in \theta$ be such that POS#($f^{-1}(c)$). Now $f^{-1}(c) \subseteq \theta\backslash R_c$. Hence POS#($\theta\backslash R_c$)), contradicting that POS#(R_c). QED

DEFINITION 7.19. We make the following definitions in set theory. $[E]^k$ is the set of all k element subsets of E. $[E]^{<\omega}$ is the set of all nonempty finite subsets of E. κ is a Ramsey cardinal if and only if κ is a cardinal such that the following holds. Let $f:[\kappa]^{<\omega} \to \alpha$, $\alpha < \kappa$. There exists unbounded $E \subseteq \kappa$ such that for all $n \geq 1$, f is constant on $[E]^n$. The strong Ramsey cardinals are defined the same way with "unbounded" replaced by "stationary". I.e., "meets every closed unbounded subset of κ".

LEMMA 7.24. Let $f:[E]^{k+1} \to x$ be internal, POS#(E), $x \in \theta$. There exists internal $E' \subseteq E$ with POS#(E') such that the following holds. For each $x \in [E]^k$, f is constant on the $x \cup \{y\}$, $\max(x) \in y \in E'$.

Proof: Let f be as given. We define internal sets $C_x \subseteq (x,\theta)$, for $x \in \theta$. Fix $x \in \theta$.

For each $y \in [E]^k$, there exists unique $u \in x$ such that POS#($\{z: f(y \cup \{z\}) = u\}$), by Lemma 7.21. Let $B_y = \{z: \max(y) \in z \wedge f(y \cup \{z\}) = u\}$. We have POS#($B_y$).

By Lemma 7.23, POS#(C_x), where C_x = {z ∈ θ: (∀y ∈ z)(z ∈ B_y)}.

By Lemma 7.23, POS#(E'), where E' = {y ∈ E: (∀x ∈ y)(y ∈ C_x)}). Let z ∈ $[E]^k$, max(z) = x. Then f is constant on the z ∪ {y}, where x ∈ y ∈ E'. QED

We need the following refinement of Lemma 7.24.

LEMMA 7.25. In Lemma 7.24, if the M[A,T] level of f is at most the limit point u ∈ A, θ ∈ u, then E' can be taken to be of M[A,T] level < u.

Proof: This is evident, as the constructions made in the proof of Lemma 7.24 are very explicit. QED

LEMMA 7.26. Let k be such that the following holds. For all limit points u ∈ A, x ∈ θ ∈ u, every f:$[E\}^k$ → x of M[A,T] level < u, POS#(E), is constant on some E' ⊆ E, POS#(E'), of M[A,T] level < u. Then this holds with k replaced by k+1.

Proof: Let k be as given. Let u ∈ A be a limit point, x ∈ θ < u, f:$[E]^{k+1}$ → x of M[A,T] level < u, POS#(E). Let E' be as given by Lemma 7.24, E' of M[A,T] level < u. Let g:$[E']^k$ → x be given by g(y) = f(y ∪ {z}), for max(y) ∈ z ∈ E'. Then g is of M[A,T] level < u. By hypothesis, let E'' ⊆ E', POS#(E''), g constant on $[E'']^k$, where E'' is of M[A,T] level < u. Note that f is constant on $[E'']^{k+1}$. QED

LEMMA 7.27. Let k be a positive integer. Let u ∈ A be a limit point, x ∈ θ ∈ u, f:$[E]^k$ → x be internal of level < u, POS#(E). There exists E' ⊆ E, POS#(E'), where f is constant on $[E']^k$, and of M[A,T] level < u.

Proof: By external induction on k. The basis case k = 1 is obvious using Lemma 7.21. The induction step is from Lemma 7.26. We use external induction since the induction hypothesis has unbounded quantification over u ∈ A. QED

LEMMA 7.28. Let f:$[E]^{<\omega}$ → x be internal, x ∈ θ, E ⊆ θ, POS#(E). There exists internal E' ⊆ E, POS#(E'), where f is constant on every $[E']^k$.

Proof: Let f,E,x be as given, where E is of M[A,T] level < u, $\theta \in$ u, u a limit point. For all k $\geq$ 1, let E_n be the M[A,T] least subset of E such that f is constant on $[E]^k$ and POS#(E_n). By Lemma 7.27, the En are of M[A,T] level < u, and so the sequence $E_1,E_2,\ldots$ is M[A,T] internal. By Lemma 7.21, we see that $\cap_k E_k$ is internal, and POS#($\cap_k E_k$). QED

LEMMA 7.29. Let E $\subseteq \theta$ be internal and internally closed and unbounded in θ. Then POS#(E). If E $\subseteq \theta \wedge$ POS#(E), then E is internally stationary in θ.

Proof: Let E be as given. Let f:$\theta\backslash$E $\rightarrow \theta$ be given by f(x) = the greatest y $\in$ x that lies in D if 0 $\in$ x $\in \theta$; 0 otherwise. By Lemma 7.21, if POS#($\theta\backslash$E), some $f^{-1}(c)$ is unbounded in θ. Hence POS#(E). The second claim follows immediately. QED

LEMMA 7.30. Let f:$\theta \rightarrow$ V(θ) be internal, where for all x $\in$ θ, f(x) $\subseteq$ V(x). There exists internal E $\subseteq \theta$, POS#(E), such that for all x $\in$ y $\in$ E, f(x) = f(y) $\cap$ V(x).

Proof: Let f be as given. For each x $\in$ V(θ), let g(x) = 1 if POS#({y $\in \theta$: x $\in$ f(y)}); 0 otherwise, and let h(x) = {y $\in \theta$: x $\in$ f(y)} if g(x) = 1; {y $\in \theta$: x $\notin$ f(y)} otherwise. For each y $\in \theta$, let $B_y = \cap_{x\in V(y)}$h(x). By Lemma 7.21, for all y $\in \theta$, POS#(By). By Lemma 7.23, POS#({y $\in \theta$: y $\in B_y$}). If y $\in B_y \wedge$ z $\in B_z \wedge$ y $\in$ z, then f(y),f(z) have the same elements from V(y). QED

LEMMA 7.31. Let E $\subseteq \theta$ be internal, POS#(E). POS#({x $\in \theta$: E $\cap$ x is internally stationary in x}).

Proof: Note that the set in question involves only quantifiers ranging over the V(θ) of M[A,T]. Let E be as given. Suppose POS#({x $\in \theta$: E $\cap$ x is not internally stationary in x}). Then POS#(B), where B = {x $\in \theta$: x is an internal limit ordinal $\wedge$ E $\cap$ x is not internally stationary in x}. (The set of limit ordinals $\in \theta$, both in the internal and external sense, must have POS# by Lemma 7.22). For each internal limit ordinal x $\in$ B, let $C_x \subseteq$ x be the M[A,T] least internally closed and unbounded set in x that is disjoint from E. By Lemma 7.30, let B' $\subseteq$ B be internal, POS#(B'), where for all x $\in$ y from B', $C_x = C_y \cap$ V(x). Then the union of the C_x, $x \in E'$, is internally closed unbounded in θ, and disjoint from E. This contradicts that POS#(E), by Lemma 7.29. QED

LEMMA 7.32. POS#($\{x \in \theta$: x is an internal strong Ramsey cardinal$\}$).

Proof: Suppose B = $\{x \in \theta$: x is not an internal strong Ramsey cardinal$\}$ has POS#(E). Let C = $\{x \in \theta$: x is an internal cardinal $\wedge$ x is not an internal strong Ramsey cardinal$\}$. Then POS#(C). For each $x \in C$, let f_x be M[A,T] least such that $f:[x]^{<\omega} \to x$, $\sup(\text{rng}(f)) \in x$, and for no $E \subseteq x$, E internally stationary in x, is it the case that for all k, f is constant on $[E]^k$. Then each $f_x \subseteq V(x)$, and so we can apply Lemma 7.30. We obtain internal $E' \subseteq \theta$, POS#(E'), such that the f_x, $x \in E'$, are comparable under $\subseteq$. Let $F:[\theta]^{<\omega} \to \theta$ be the union of the f_s, $x \in E'$. Then $\sup(\text{rng}(F)) = y \in \theta$.

By Lemma 7.28, let $D \subseteq \theta$, POS#(D), where F is constant on every $[D]^k$. By Lemma 7.31, POS#($\{x \in \theta$: $D \cap x$ is internally stationary in x$\}$). Let $x \in \theta$, x an internal cardinal, $D \cap x$ is internally stationary in x. Clearly f_x is constant on every $[D \cap x]^k$. This contradicts the choice of f_x. QED

LEMMA 7.33. The $V(\theta)$ of M[A,T] under E(A,T,POS) satisfies ZFC + "there are arbitrarily large strong Ramsey cardinals".

Proof: By Lemma 7.19, the $V(\theta)$ of M[A,T] under E(A,T,POS) satisfies ZFC. Now apply Lemma 7.32. QED

THEOREM 7.34. ZFC + "there are arbitrarily large strong Ramsey cardinals" is interpretable in T_5. T_5 proves the consistency of ZFC + "there are arbitrarily large strong Ramsey cardinals", when formulated with standard formalized syntax. EFA proves $\text{Con}(T_5) \to$ Con(ZFC "there are arbitrarily large strong Ramsey cardinals"). T_5 is interpretable in ZFM. ZFM proves the consistency of T_5. EFA proves Con(ZFM) $\to$ $\text{Con}(T_5)$.

Proof: By Theorem 5.1 and Lemma 7.33. Since the theories involved are not finitely axiomatizable, the EFA results do not immediately come from the existence of the interpretations. The simplest route is to verify that EFA is sufficient to prove that the interpretations work. QED

8. Without Extensionality.

In [Go95], positivity is used as an attribute on properties of objects, not on classes of objects. As discussed in section 3, Gödel assumed that positivity behaves as an ultrafilter, and this leads to an easy derivation that positivity depends only on the extension of properties. I.e., if two properties hold of the same objects, then one is positive if and only if the other is positive. Adams in [Go95] also discusses the Leibniz view of positivity, in some detail, citing parts of [Le23], [Le56] [Le69].

This would suggest that we might be able to simply drop Extensionality from T_5, and be able to retain Theorem 7.34, or at least Theorem 5.1. However, we doubt this, as a difficulty arises in connection with Choice Operator.

We now present the system T_5(prop), where "prop" abbreviates "properties". For ease of comparison with T_5, we take the language L_5(prop) of T_5[prop] to be the same as the language L_5 of T_5, except we remove equality between class variables. Here $x \in A$ read "the property A holds of the object x". Extensionality is dropped, and the axioms that are modified are asterisked. The only substantial modification is to Choice Operator, where a second line is added.

We show that T_s(prop) proves the strongest form of extensionality that can be formulated without equality between classes. From this result, it is easy to conclude that Theorem 7.34 holds for T_5(prop).

NONLOGICAL AXIOMS FOR T_5(prop)

PAIRING

$P(v_1,v_2) = P(v_3,v_4) \rightarrow v_1 = v_3 \wedge v_2 = v_4$.

L_5(prop) COMPREHENSION

$(\exists A_1)(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi)$, where φ is a formula of L_5(prop) in which A_1 is not free.

AUGMENTED CHOICE OPERATOR

$v_1 \in A_1 \rightarrow CHO(A_1) \in A_1$.
$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow CHO(A_1) = CHO(A_2)$.

POSITIVE CLASSES

$(\forall v_1)(v_1 \in A_1 \vee v_1 \in A_2) \rightarrow POS(A_1) \vee POS(A_2)$.
$POS(A_1) \wedge POS(A_2) \rightarrow (\exists v_1 \neq v_2)(v_1,v_2 \in A_1 \wedge v_1,v_2 \in A_2)$.

0-DEFINABLE CLASSES (prop)

$(\forall v_1)(v_1 \in A_1 \leftrightarrow \varphi) \wedge DEF(A_2) \wedge \ldots \wedge DEF(A_n) \rightarrow DEF(A_1)$, where φ is a formula of L_5(prop) without DEF, with at most the free variables $v_1,A_2,\ldots,A_n$, $n \geq 1$.

DIVINE OBJECT

$(\exists v_1)(\forall A_1)(DEF(A_1) \wedge POS(A_1) \rightarrow v_1 \in A_1)$.

Here is the strongest form of extensionality in the language of L_5(prop), where we have dropped equality between classes.

EXTENSIONLTIY WITHOUT CLASS EQUALITY

$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow (\varphi \leftrightarrow \psi)$, where φ,ψ are formulas of L_5(prop), and ψ is the result of replacing zero or more free occurrences of A_1 by A_2.

THEOREM 8.1. T_5(prop) proves Extensionality Without Class Equality.

Proof: By standard predicate calculus manipulations, it suffices to prove this for atomic formulas φ,ψ. Working in T_5(prop), assume $(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2)$.

case 1. $s = t$, where s,t are object terms. The relevant substitutions are of the form $s' = t'$, where s',t' are obtained from s,t by replacing some A_1's by A_2, respectively. Now the A_1's appear only as part of subterms $CHO(A_1)$, and Augmented Choice Operator proves $CHO(A_1) = CHO(A_2)$. Therefore we obtain $s = s'$, $t = t'$ by the axioms of equality (for the equality that we do have, namely between objects). Hence we have $s = t \leftrightarrow s' = t'$.

case 2. $t \in A_i$, where t is an object term. The relevant substitutions are of the form $t' \in A_j$, where the reasoning in case 1 yields $t = t'$. Now if $i = j$ then we are done. Otherwise, $i = 1$ and $j = 2$. But $t \in A_1 \leftrightarrow t' \in A_2$ is provable, using $t = t'$.

case 3. $DEF(A_i)$. We have only to prove $DEF(A_1) \leftrightarrow DEF(A_2)$. By 0-Definable Classes (prop), we have the universal closure of

$$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \wedge DEF(A_2) \rightarrow DEF(A_1)$$

and therefore

$$(\forall v_1)(v_1 \in A_2 \leftrightarrow v_1 \in A_1) \wedge DEF(A_1) \rightarrow DEF(A_2).$$

Hence we have $DEF(A_1) \leftrightarrow DEF(A_2)$.

case 4. $POS(A_i)$. We have only to prove $POS(A_1) \leftrightarrow POS(A_2)$.

Assume $POS(A_1)$. By L_5(prop) Comprehension, let A_3 be such that $(\forall v_1)(v_1 \in A_3 \leftrightarrow v_1 \notin A_1)$. By $(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2)$, we have $(\forall v_1)(v_1 \in A_2 \vee v_1 \in A_3)$. By Positivity, $POS(A_2) \vee POS(A_3)$.

We claim that $\neg POS(A_3)$. Assume $POS(A_3)$. By Positivity, $(\exists v_1 \neq v_2)(v_1,v_2 \in A_1 \wedge v_1,v_2 \in A_3)$. Let v_1 be such that $v_1 \in A_1$, $v_1 \in A_3$. Then $v_1 \notin A_1$. Contradiction.

By $POS(A_2) \vee POS(A_3)$, $\neg POS(A_3)$, we have $POS(A_2)$.

We have established the universal closure of

$$(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2) \rightarrow (POS(A_1) \rightarrow POS(A_2))$$

and therefore

$$(\forall v_1)(v_1 \in A_2 \leftrightarrow v_1 \in A_1) \rightarrow (POS(A_2) \rightarrow POS(A_1)).$$

Hence we have established $POS(A_1) \leftrightarrow POS(A_2)$.

QED

THEOREM 8.2. T_5(prop) and T_5 are mutually interpretable. Theorem 7.34 holds for T_5(prop).

Proof: As in the proof of Theorem 7.34, we want to use that EFA proves that a particular interpretation of T_5 in

T_5(prop) works. This will be clear below. Note that Theorem 8.1 is provable in EFA. This will be enough to establish Theorem 7.34 for T_5(prop) based on Theorem 7.34 for T_5.

In an interpretation that does not treat equality as equality, we have to be particularly careful. Here we do interpret equality between objects as equality between objects, but we obviously don't interpret equality between classes as equality between classes, since we don't have equality between classes in the language.

The simplest approach to interpretations is to use the associated theories with only relation symbols and no equality. Here all equality relations on sorts are viewed simply as binary relation symbols, all constant symbols are replaced by unary predicates, and all function symbols are replaced by relation symbols of one higher arity. In the associated theory, we include not only the modified form of the nonlogical axioms, but also the axioms for equality, now formulated in the modified language, and now viewed as nonlogical axioms.

Thus an interpretation of T in T' is an interpretation of modified T in modified T'. We take this approach now in our interpretation of T_5 in T_5(prop).

The axioms for equality can always be formulated in terms of equivalences between atomic formulas. With these caveats in mind, we now give the interpretation of T_5 in T_5(prop).

We interpret the objects of T_5 as the objects of T_5(prop), the classes of T_5 as the classes of T_5(prop), = between objects of T_5 as = between objects of T_5(prop), the ternary relation P of T_5 as the ternary relation P of T_5(prop), $\in$ of T_5 as $\in$ of T_5(prop), DEF of T_5 as DEF of T_5(prop), POS of T_5 as POS of T_5(prop).

We interpret $A_1 = A_2$ in T_5 as $(\forall v_1)(v_1 \in A_1 \leftrightarrow v_1 \in A_2)$ in T_5(prop).

We interpret the binary relation CHO of T_5 as the binary relation CHO of T_5(prop).

We now have to show that the interpretation of all of the axioms of modified T_5, including all of the equality axioms in its language, are provable in modified T_5(prop).

1. Equality axioms. The proof of Theorem 8.1 establishes this.

2. Nonlogical axioms. We have only to consider the nonlogical axioms of modified T_5 where = between classes occurs. The only such axiom is modified L_5 Comprehension. But its interpretation is clearly follows from modified L_5(prop) Comprehension.

QED

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