ADVENTURES IN GÖDEL INCOMPLETENESS

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Abstract. We begin with a discussion of various forms of G1 put into the following general form: If a first theory satisfies one or more adequacy conditions then it has one or more wildness properties. We give a list of familiar adequacy conditions and wildness properties. We propose an investigation into the myriad forms of G1 in this framework. Some such forms of G1 will be well known, some well known to be false, and some yet to be investigated. We expect many will suggest further investigations. We then discuss various new "no interpretation" forms of G2. These are fundamental model theoretic formulations of G2 in the following sense. The proofs of them from G2 are entirely straightforward applications of G2 and Gödel Completeness. The derivation of G2 is also straightforward and does not rely on any of the ingredients in the known proofs of G2. We also give corresponding fundamental model theoretic characterizations of the consistency statement Con(T) for finitely axiomatized T. We then discuss G2/1-con which is G2 with the strengthened hypothesis of 1-consistency and the weakened conclusion of the unprovability of 1-consistency. We give the long since known, if not well known, proof of G2/1-Con which is much simpler than the proof of G2. It is best proved by what we call "transparent diagonalization" which is the kind of informative diagonalization used by Cantor in his proof that there are uncountably many infinite sequences of 0’s and 1’s. A by product of this proof is the association of a crucially important set of objects to T that gets properly expanded by T + 1-Con(T) - namely the provably recursive functions. Since so much of the philosophical and foundational import of G2 is already present with G2/1-Con, we propose that G2/1-con be revisited with the same deep intensity as has G2. We call for a proof of G2 by transparent diagonalization. We then present two

proofs of G2. The first proof is an attempt at transparent diagonalization with limited success. The second proof puts all of the diagonalization related ideas into a basic familiar situation in recursion theory that is a particularly transparent diagonalization of its own. This is the construction of what we call a remarkable set and an EFA effectively remarkable set. Then we take any EFA effectively remarkable set and apply its remarkability to a naturally closely associated set and derive G2 now without any semblance of diagonalization. We have retained [Fr21] in the list of references, because there are ideas we don't discuss here that may have some future importance.

1. G1. Gödel's first incompleteness theorem.
2. G2. Gödel's second incompleteness theorem.
3. Proof of G2/1-Con by transparent diagonalization.
4. Proof of G2 by transparent diagonalization?
5. Proof of G2 by remarkable sets.

1. G1. GÖDEL'S FIRST INCOMPLETENESS THEOREM

By a theory we will usually mean a theory T in the usual PC(=), (predicate calculus with equality), which comes with a designated language (of constant, relation, and function symbols). Sometimes it is important to use many sorted logic.

The most common way to formulate G1 is to assert that any theory T with an "adequacy condition" has a "wildness property". There are several important kinds of adequacy conditions and wildness properties.

Common adequacy conditions on a theory T (with multiple choices):
a. T is consistent.
b. T is (finitely axiomatized, axiomatized by finitely many axiom schemes, recursively axiomatized).
c. T interprets a given theory K, (finitely axiomatized, axiomatized by finitely many axiom schemes, recursively axiomatized).
d. T is consistent with an interpretation of a given theory K, in the same language as T, (finitely axiomatized, axiomatized by finitely many axiom schemes, recursively axiomatized).
e. The language of T is or extends a given language, and T proves a certain theory K (finitely axiomatized, axiomatized by finitely many axiom schemes, recursively axiomatized).

Common wildness properties of a theory T (with multiple choices):
A. T is incomplete in the sense that there is a sentence in the language of T that is neither provable nor refutable in T.
B. T is essentially incomplete in the sense that no consistent extension of T by finitely many sentences is complete.
C. The set of theorems of (T, any finite extension of T, any recursive extension of T) is (complete r.e., not recursive, not primitive recursive, not elementary recursive, not polytime computable).
D. The set of theorems of T and the set of refutables of T are recursively inseparable.
E. Assuming the language of T is or extends a given language, A-D restricted to sentences in a given sublanguage.

There are likely some other interesting adequacy conditions and wildness properties that should be considered in such a systematic investigation.

TEMPLATE FOR G1. Let T obey a chosen one or more (parts) of a-e. Then T has a chosen one or more (parts) of properties A-E.

SYSTEMATIC G1 INVESTIGATION. Determine relationships between various instances of the Template for G1, including their correctness for various K.

The most elemental form of G1 involving only the most rudimentary of notions, is arguably the following.

PURE G1 (finite). There is a consistent finitely axiomatized theory K such that any consistent finitely axiomatized theory T interpreting K is incomplete.

Robinson's Q is most commonly used for pure G1, as well as its many natural "variants" in the sense of being mutually interpretable with Q. There is no known natural system K for this pure G1 that does not interpret Q.

PURE G1 (schematic). There is a consistent theory K with finitely many axiom schemes, such that any consistent theory T axiomatized with finitely many axiom schemes, interpreting K, is incomplete.

Here there is a natural infinitely axiomatized system R, interpretable in Q, but where Q is not interpretable in R, that we can use. But we would like to say that R is "very recursive". However, as "schematic" as the system R looks, it is not officially given by finitely many axiom schemes. So we need to either expand the notion of axiom scheme to allow R, or we need to modify R to fit into the usual notion of schemes. This should be investigated.

What is missing is insight into the special status of Q and R and perhaps variants of Q and R, for G1.

Furthermore, as we vary the wildness properties we seek for T, how does that affect the choices of K that we can use in the adequacy conditions?

There is also an attractive simplicity investigation here. There are some reasonably natural measures of the complexity of presentations of finitely axiomatized axiom systems in PC(=). E,g., one can count the number of occurrences of symbols other than parentheses and commas, each occurrence of a variable counted as 1. We can seek information on the smallest complexity of a K supporting Pure G1 or other instances of the G1 Template. The language of arithmetic would not be a good choice for this. The language of set theory would be much better, through the system AS of Adjunctive Set Theory, as well as theories of strings. AS,Q are mutually interpretable.

CONJECTURE. Any finitely axiomatized system K usable for pure G1 and variants of G1, of complexity at most that of AS, interprets AS.

The most common languages used to formulate versions of G1 are arithmetic (with and without exponentiation, with and without primitive recursive function symbols, with and without <), set theory with membership, and string theory concatenation. There are some important special classes of formulas, most notably Π^0_1, Σ^0_1, and Σ^0_1 using polynomial equations. Here G1 meets Hilbert's Tenth Problem. See. e.g., [Je16].

2. G2. GÖDEL'S SECOND INCOMPLETENESS THEOREM

This section is a reworking of section 2 of [Fr21].

We use the systems PA (Peano arithmetic), PRA (primitive recursive arithmetic), SEFA (superexponential arithmetic), EFA (exponential function arithmetic), PFA (polynomial function arithmetic). The super exponential is the iterated exponential. SEFA, EFA, PRA are also known as $I\Sigma_0$(superexp), $I\Sigma_0$(exp), $I\Sigma_0$, respectively.

We formulate a purely model theoretic form of G2 which we call No Interpretation G2/PA. Its more natural formulation uses many sorted logic.

NO INTERPRETATION G2/PA. NIG2(PA). No consistent many sorted theory that proves PA is interpretable in any of its theorems in the language of PA.

Here we are requiring that one of the sorts is the arithmetic sort with the primitives 0,S,+,• included.

We can't directly apply NIG2(PA) to a system without this arithmetic sort, but NIG2(PA) can be used in obvious ways to be much more general. Or we can make NIG2(PA) more general. this is easily fixed by the following formulation.

NO INTERPRETATION G2/PA*. NIG2(PA*). No consistent theory proving an interpretation of PA is interpretable in any of its theorems about the interpretation.

Here an interpretation is taken to be a first order definition of a structure in the language of PA. the equivalence of NIG2(PA) and NIG2(PA*) is left to the reader.

Proof of NIG2(PA): Let T be as given. Let T prove PA + φ, φ a sentence in L(PA), where T is interpretable in φ. We show that T is inconsistent.

It is well known that $\varphi \rightarrow \mathrm{Con}(\varphi)$ is provable in PA using partial truth definitions and cut elimination. Let T' be a finite fragment of T which proves
i. EFA (which is finitely axiomatized).
ii. $\varphi \rightarrow \mathrm{Con}(\varphi)$ (which is provable in PA).
iii. φ (since T proves φ).
Therefore T' proves Con(φ). Since T' $\subseteq$ T, T' is interpretable in φ, and by i above, Con(T') makes sense. From this interpretation, EFA proves $\mathrm{Con}(\varphi) \rightarrow \mathrm{Con}(T')$. Hence T' proves Con(T'), and therefore by G2, T' $\subseteq$ T is inconsistent. QED

NO INTERPRETATION G2/($I\Sigma_n$). NIG2($I\Sigma_n$). Let $n \geq 1$. No consistent many sorted theory that proves $I\Sigma_n$ is interpretable in any of its Σ^0_{n+2} theorems.

Proof: Let T,n be as given. Let T prove $I\Sigma_n + \varphi$, φ in Σ^0_{n+2}, where T is interpretable in φ. We show that T is inconsistent.

It is known that $\varphi \to Con(\varphi)$ is provable in $I\Sigma_n$ using partial truth definitions and cut elimination, as proved in [Le83] (also see [Be97], [Be05]). Let T' be a finite fragment of T which proves

i. EFA (which is provable in $I\Sigma_n$)

ii. $\varphi \to Con(\varphi)$ (which is provable in $I\Sigma_n$)

iii. φ (since T proves φ)

Therefore T' proves $Con(\varphi)$. Since $T' \subseteq T$, T' is interpretable in φ, and by i above, Con(T') makes sense. From this interpretation, EFA proves $Con(\varphi) \to Con(T')$. Hence T' proves Con(T'), and therefore by G2, $T' \subseteq T$ is inconsistent. QED

NO INTERPRETATION G2/PRA. NIG2(PRA). No consistent many sorted theory that proves PRA is interpretable in any of its Π^0_1 theorems in L(PRA).

Proof: Let T be as given. Let T prove φ, φ being Π^0_1 in L(PRA), where T is interpretable in φ. We show that T is inconsistent.

It is known that $\varphi \to Con(\varphi)$ is provable in PRA since we have Herbrand's theorem available in PRA and induction applied to bounded formulas in the primitive recursive function symbols used in φ. Use of Herbrand here involves iteration of the underlying functions, afforded by PRA. Let T' be a finite fragment of T which proves

i. EFA (which is provable in PRA)

ii. $\varphi \to Con(\varphi)$ (which is provable in PRA)

iii. φ (since T proves φ)

Therefore T' proves $Con(\varphi)$. Since $T' \subseteq T$, T' is interpretable in φ, and by i above, Con(T') makes sense. From this interpretation, EFA proves $Con(\varphi) \to Con(T')$. Hence T' proves Con(T'), and therefore by G2, $T' \subseteq T$ is inconsistent. QED

SEFA is super exponential function arithmetic, and EFA is exponential function arithmetic, as we have been using.

NO INTERPRETATION G2/EFA. NIG2(EFA). No consistent many sorted theory that proves SEFA is interpretable in any of its Π^0_1 theorems in L(EFA).

THEOREM 2.4. No consistent extension T of SEFA, in any language, is interpretable in any Π^0_1 theorem of T in L(EFA).

Proof: Let T be as given. Let T prove φ, φ being Π^0_1 in L(EFA), where T is interpretable in φ. We show that T is inconsistent.

It is known that φ → Con(φ) is provable in SEFA. To see this, assume φ is refutable, and apply Herbrand's theorem, available in SEFA. This creates indefinite iterations of addition and multiplication and exponentiation, and the associated truth definitions are handled appropriately by SEFA. Let T' be a finite fragment of T which proves
i. EFA (which is provable in SEFA)
ii. φ → Con(φ) (which is provable in SEFA)
iii. φ (since T proves φ)
Therefore T' proves Con(φ). Since T' ⊆ T, T' is interpretable in φ, and by i above, Con(T') makes sense. From this interpretation, EFA proves Con(φ) → Con(T'). Hence T' proves Con(T'), and therefore by G2, T' ⊆ T is inconsistent. QED

THEOREM 2.5. No consistent extension T of EFA, in any language, is interpretable in any Π^0_1 theorem of T in L(PFA).

Proof: Let T be as given. Let T prove φ, φ being Π^0_1 in L(PFA), where T is interpretable in φ. We show that T is inconsistent.

It is known that φ → WCon(φ) is provable in EFA, where WCon is the weakened form of Con also referred to as cut free consistency. Since we have Herbrand's theorem available in EFA for specific complexity, and we can use it here with indefinite iteration of addition and multiplication, we obtain φ → WCon(φ) in EFA. Let T' be a finite fragment of T which proves
i. EFA
ii. φ → WCon(φ) (which is provable in EFA)
iii φ (since T proves PH)
Therefore T' proves WCon(PH). Since T' ⊆ T, T' is interpretable in φ, and by I above, Con(T') makes sense. From this interpretation, EFA proves WCon(φ) → WCon(T'). Hence T' proves WCon(T'), and therefore by G2, T' ⊆ T is inconsistent. NOTE: G2 is well known to hold for WCon. QED

To justify the name "No Interpretation G2" we now consider the relationship between Theorems 2.1 - 2.5 and certain forms of G2. These relationships need to be established without using any of the techniques involved in proving G2.

Note that we have derived Theorems 2.1 - 2.5 from G2 applied to finitely axiomatized theories extending EFA.

THEOREM 2.6. Theorems 2.1 - 2.5 imply G2 for r.e. presented theories in any language, where the axioms extend PA, PA[n], PRA, SEFA, EFA, respectively.

Proof: Suppose Theorem 2.1 – 2,5 and let T be a consistent r.e. presented extension of PA, PA[n], PRA, SEFA, EFA, respectively. For G2, let T prove Con(T), where Con(T) is formulated as a Π^0_1 sentence in L(PFA). Now in each of the five cases, T is interpretable in Con(T) with some infrastructure needed to properly use Con(T). EFA easily serves as this infrastructure. So using Theorems 2.1 - 2.5, we see that T is inconsistent, establishing G2, where the r.e. axioms extend PA, PA[n], PRA, SEFA, EFA, respectively, in each case. QED

We now characterize the Con statement for finitely axiomatized theories (single sentences). We first characterize the Con statement up to PA provable equivalence.

THEOREM 2.7. For all sentences A, A,Con(A) obey the following property P(A,Con(A)): For all arithmetic B, PA + B interprets A if and only if PA + B proves Con(A). For all sentences A, Con(A) is the unique arithmetic sentence with P(A,Con(A)) up to PA provable equivalence.

Proof: Let A be a sentence and B be an arithmetic sentence. If PA + B proves Con(A) then obviously PA + B interprets A via the formalized completeness theorem. Now suppose PA + B interprets A. Let $I\Sigma n$ + B interpret A. Then EFA proves Con($I\Sigma_n$) + B) $\rightarrow$ Con(A). Now PA + B proves Con($I\Sigma_n$] + B) by formalized cut elimination and truth definition. Hence PA + B proves Con(A).

Now let C be an arithmetic sentence such that P(A,C). I.e., for all arithmetic sentences B, PA + B interprets A if and only if PA + B proves C. Then by P(A,Con(A)), we have that for all arithmetic sentences B,

*) PA + B proves C if and only if PA + B proves Con(A).

Setting B = C in *), we get PA proves C $\rightarrow$ Con(A). By setting B = Con(A) in *), we get PA proves Con(A) $\rightarrow$ C. Hence PA proves C $\leftrightarrow$ Con(A). QED

Next we characterize the Con statement up to PRA provable equivalence.

THEOREM 2.8. For all sentences A, A,Con(A) obey the following property P(A,Con(A)): For all Π^0_1 sentences B in L(PRA), PRA + B interprets A if and only if PRA + B proves Con(A). For all sentences A, Con(A) is the unique Π^0_1 sentence in L(PRA) with P(A,Con(A) up to PRA provable equivalence.

Proof: Let A be a sentence and B be Π^0_1 in L(PRA). If PRA + B proves Con(A) then obviously PRA + B interprets A via the formalized completeness theorem. Now suppose PRA + B interprets A. Let PRA' + B interpret A, where PRA' is a finite fragment of PRA. Then EFA proves Con(PRA' + B) → Con(A). Now PRA + B proves Con(PRA' + B) by formalized cut elimination and truth definition, using that B is Π^0_1 in L(PRA). Hence PRA + B proves Con(A).

Now let C be a Π^0_1 sentence in L(PRA) such that for all Π^0_1 B in L(PRA), PRA + B interprets A if and only if PRA + B proves C. Then for all Π^0_1 B in L(PRA), PRA + B proves C if and only if PRA + B proves Con(A). Setting B = C we get PRA proves C → Con(A), and by setting B = Con(A), we get PRA proves Con(A) → C. Hence PRA proves C ↔ Con(A).

Now let C be a Π^0_1 sentence in L(PRA) such that P(A,C). I.e., for all Π^0_1 sentences B, PRA + B interprets A if and only if PRA + B proves C. Then by P(A,Con(A)), we have that for all Π^0_1 sentences B,

*) PA + B proves C if and only if PA + B proves Con(A).

Setting B = C in *), we get PRA proves C → Con(A). By setting B = Con(A) in *), we get PRA proves Con(A) → C. Hence PRA proves C ↔ Con(A). QED

3. PROOF OF G2/1-Con BY TRANSPARENT DIAGONALIZATION

The prime example of what we call Transparent Diagonalization is the usual proof by Cantor that the set of infinite sequences of 0's and 1's cannot be countable. This diagonalization argument is more direct and straightforward than the diagonalization/self reference argument used in Gödel's original proofs of G1,G2. Those original proofs using the self reference lemma are still considered rather mysterious in light of, for example, Barkley

Rosser's use of it in the Gödel/Rosser theorem. To this day we don't have a good understanding of what Rosser sentences are like under "natural" numberings. For a "usual" numbering, we don't know whether any two Rosser sentences are equivalent, and also how the Rosser sentences compare when we use different "natural" numberings. See [GS79], [Bu08] for some background information.

A somewhat well known proof of a modified form of G2 can be proved using an utterly straightforward Transparent Diagonalization.

DEFINITION 3.1. T is adequate if and only if T is a finitely axiomatized theory extending EFA in many sorted logic with finitely many sorts. 1-Con(T) asserts that "every true Σ^0_1 sentence provable in T is true", formalized in the well known way using a natural enumeration of the Σ^0_1 formulas.

1-Con(T) is also referred to as Σ_1 soundness for T.

G2/1-Con. No 1-consistent adequate theory proves its own 1-consistency. I.e., if T is adequate and 1-consistent, then T does not prove 1-Con(T).

The origins of G2/1-Con, or G2 for 1-consistency are rather unclear. Lev Beklemishev has a paper in the 1980's about this, but it probably was first proved much earlier, perhaps when the notion of provably recursive functions of a theory first came into common use. That is probably in the 1950s with G. Kreisel. Some of the early proof theorists of that period are good candidates for having known about the directly straightforward proof of G2/1-Con that we sketched above. E.g., perhaps G. Kreisel.

We associate an important well known set of objects Θ(T) to adequate T.

DEFINITION 3.2. Let T be adequate. Θ(T) is the set of all provably recursive functions of T. f is a provably recursive function of T if and only if there exists e such that f = φ_e is total, and T proves "φ_e is total".

We prove the following strengthening of G2(1-Con) by Transparent Diagonalization.

G2/1-Con GROWTH. Let T be adequate and 1-consistent. Then Θ(T) is a proper subset of Θ(T + 1-Con(T)). There is an enumeration of Θ(T) by a provably recursive function (of two variables) of T + 1-Con(T).

Proof: Let T be as given. Define f(n) by looking at all partial recursive functions for which its index and a proof in T that it is everywhere defined can be found ≤ n, and returning the least nonnegative integer that is greater than all of the values these functions have at n. Since T is 1-consistent, this describes a recursive function. It is clear that this recursive function eventually strictly dominates all provably recursive functions of T. Finally, note that this recursive function is a provably recursive function of the adequate T + 1-Con(T). QED

Much of the philosophical force of G2 is already available with G2/1-Con. This indicates that it is very worthwhile to investigate G2/1-Con with the same intensity and detail as G2 has been investigated. We have chosen to simplify matters by requiring that T be finitely axiomatized.

DEFINITION 3.3. T is adequate/re if and only if T is r.e. presented extending EFA in many sorted logic with an r.e. presented set of sorts. Θ(T) is defined using the r.e. presentation.

Note that G2/1-Con for r.e. presented T extending EFA in many sorted logic whose sorts are r.e. presented, G2(1-Con) trivially follows from our G2(1-Con). Moreover,

G2/1-Con GROWTH/re. Let T be adequate/re and 1-consistent. Then Θ(T) is a proper subset of Θ(T + 1-Con(T)). There is an enumeration of Θ(T) by a provably recursive function (of two variables) of T + 1-Con(T).

Proof: Adapt the proof of G2/1-Con Growth to adequate/re T. QED

Much of the foundational import of G2 is already present with G2/1-Con. This suggests that G2/1-Con might be profitably investigated with the same intensity as G2 has. We make some preliminary explorations.

THEOREM 3.1. Let T be adequate/re. The following are equivalent.
i. Θ(T) is not the set of all recursive functions.
ii. Θ(T) has a recursive enumeration.
iii. T is 1-consistent.

Proof: iii $\rightarrow$ ii $\rightarrow$ i by G2/1-Con Growth/re. It now suffices to prove i $\rightarrow$ iii. Suppose T is not 1-consistent. Let $(\exists n)(R(n))$ be provable in T and false, R Δ_0. Let $f:\omega \rightarrow \omega$ be a recursive function with index e. Change e to the natural index e′ for computing $g(n) = f(n)$ if $f(n)$ is computed in a number of steps m such that $\neg(\exists n \leq m)(R(n))$; 0 otherwise. Then T proves $\varphi_{e'}$ is total and the actual $\varphi_{e'} = g$ is recursive, being the same as f (using that $(\exists n)(R(n))$ is false). So f is a provably recursive function of T. QED

THEOREM 3.2. Let T be adequate/re and 1-consistent. Then T + not 1-Con(T) is 1-consistent.

Proof: Let T be as given. Let T + not 1-Con(T) prove $(\exists n)(R(n))$, R Δ_0. Then T prove $(\exists n)(R(n)) \vee$ 1-Con(T). Now 1-Con(T) is a Π^0_2 sentence. Hence this disjunction is a Π^0_2 sentence. Therefore since T is 1-consistent, this disjunction is true. Since 1-Con(T) is false, $(\exists n)(R(n))$ is true. QED

4. PROOF OF G2 BY TRANSPARENT DIAGONALIZATION?

The proof in section 3 of G2/1-con is totally straightforward and totally satisfying, with no mystery of any kind. We now make an attempt to do this for G2 with some limited success.

Fix an adequate theory T (Definition 3.1). We assume that T is consistent.

Cantor built an infinite sequence of 0's and 1's that differs from every infinite sequence of 0's and 1's somewhere, assuming that there are only *countably many infinite sequences of 0's and 1's*. Then we are forced to discard that assumption and arrive at Cantor's Theorem for infinite sequences of 0's and 1's.

Here we want to similarly construct an r.e. subset of ω which differs from every r.e. subset of ω somewhere, with some assumption on T that we are forced to discard. Henceforth r.e. set will always mean r.e. subset of ω.

A standard r.e. enumeration of the r.e. sets by W_n gives a numerical name to every r.e. set. This is useful for possible paradoxical constructions involving r.e. subsets of ω.

We start with the following construction of an r.e. set that differs from every r.e. set somewhere. We want to define A which differs from every W_n somewhere. In particular we want to define A which differs from every W_n at n (following Cantor). This amounts to defining A this way:

$$n \in A \text{ if and only if } n \notin W_n$$

The problem with this definition is that A is not going to be r.e. So we can try to fix this in the most obvious way:

1) $n \in A$ if and only if T proves $n^* \notin W_{e^*}$.

Does this make A differ from every r.e. set somewhere? In particularly, does it make A differ from every W_n at n? For that we would need, for all n,

2) T proves $n^* \notin W_{n^*}$ if and only if $n \notin W_n$ (want)

However, we only get the forward direction of 2), for all n. We are stuck for the reverse direction of 2).

But of course, we really only need that A differs from We at e, where A = We, to get our contradiction. (I.e., $e \in A \leftrightarrow e \notin A$). In other words, it suffices to show

3) T proves $e^* \notin W_{e^*}$ if and only if $e \notin W_e$ (want, with $A = W_e$)

to obtain a contradiction - and then look back to see where we went wrong. So we need to determine the truth values of the antecedent and the consequent in 3).

Well if $e \in$ We then T proves $e^* \in W_{e^*}$, in which case T does not prove $e^* \notin W_{e^*}$. The reason is that T is consistent. So we have shown

4) $e \notin W_e$ (have)

But examining the way we established 4), we just used Con(T), and so we have

5) EFA + Con(T) proves $e \notin W_e$ (have)

Since *T proves Con(T)*, we finally have

6) T proves $e \notin W_e$

which establishes 3) using 4). So we have now obtained the desired contradiction! What went wrong? Well *T proves Con(T)* is the culprit. Hence T does not prove Con(T). Thus we have proved G2.

I like the proof of G2/1-con better. However, can a claim be made that this proof is perhaps “fully motivated” in a sense that other proofs are not?

5. PROOF OF G2 VIA REMARKABLE SETS

We finally turn to a slightly novel proof of G2 that can be construed as being suggestively organized - rather than radically new.

The idea is to use the notion of REMARKABLE SET to push all of the work that can be construed as diagonalization or mysterious into recursion theory. Actually it is rather invisible also as recursion theory, almost unnoticeable. So what diagonalization remains is particularly friendly.

DEFINITION 5.1. A is remarkable if and only if A is an r.e. subset of ω which agrees somewhere with every r.e. subset of ω. I.e., for every r.e. set B, there exists e such that $e \in A \leftrightarrow e \in B$.

It is very easy to see that this notion looks to be intriguing, but is really rather pedestrian. For what does it mean to NOT be remarkable? Just that A is r.e. and disagrees everywhere with some r.e. set. But that just means that A is r.e. with an r.e. complement. I.e., we have shown the following.

THEOREM 5.1. A is remarkable if and only if A is r.e. and not recursive.

Now we introduce a natural strengthening of remarkable using the weak system EFA of exponential function arithmetic. Other weak systems can be used.

Coming back to the definition of remarkable, it is a very common move in mathematics to take a notion, which asserts existence, and simply ask that one be very explicit about an example. Thus we are led quickly to the following notion.

DEIFNITION 5.2. A is EFA remarkable if and only if for all r.e. sets B, there exists e such that EFA proves that A,B agree at e.

Here we just use EFA = exponential function arithmetic, as a convenient way of making things very explicit.

THEOREM 5.3. There is an explicitly remarkable set A.

Proof: This kind of thing is very much present in recursion theory where one has extra effectivity. We can use a familiar natural complete r.e. set A. We can effectively find a place of agreement for any r.e. set B from the r.e. index of B. NAMELY THE INDEX OF B! So this is NOT EVEN REALLY A DIAGONAL ARGUMENT. Set $A = \{e: e \in W_e\}$. Let $B = W_r$. Then $r \in A \leftrightarrow r \in B$, which is obviously provable in EFA. QED

So the only real hint of a diagonal argument so far is just the definition of $A = \{e: e \in W_e\}$, a very familiar construction in elementary recursion theory.

We now prove G2 by starting with any EFA remarkable A, not just the special $\{e: e \in We\}$, forming an obviously interesting and natural set B related to A, apply EFA remarkability to A,B, and then argue without any trace of diagonalization or mystery.

THEOREM 5.4. G2.

Proof: Let T be adequate (Definition 3.1), and consistent. Also assume T proves Con(T). We obtain a contradiction.

Let A be EFA remarkable. If we could apply EFA remarkable to A and $\{e: e \notin A\}$ then we would have an obvious contradiction (as these two sets agree nowhere). But we can't since $\{e: e \notin A\}$ is not r.e. So instead we apply EFA remarkability to A and {e: T proves $e^* \notin A$}, which is r.e.

By the EFA remarkability of A, fix n such that

1) $n^* \in A \leftrightarrow$ 'T proves $n^* \notin A$'

is provable in EFA. Arguing in T, if $n^* \in A$ then T proves $n^* \in A$, and also T proves $n^* \notin A$, using 1). Therefore T is inconsistent. Thus in T, we have proved $n^* \in A \rightarrow$ T is inconsistent, and so by hypothesis, T proves $n^* \notin A$. Then by 1), T proves $n^* \in A$. Hence T is inconsistent, which is again a contradiction. QED

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