TANGIBLE INCOMPLETENESS ABSTRACT
invariant maximality
by

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ABSTRACT. We begin with a summary of the prior lead statements of Invariant Maximality, in the Introduction. These are infinite statements in the rationals that are implicitly Π^0_1 sentences provably equivalent to Con(SRP) over WKL_0. We then use the upper shift to give a different example again at the SRP level. We then strengthen it to the level of HUGE. We also present explicitly finite forms of all of these infinite statements. Thus far, only the lead infinite statements at the level of SRP have an existing manuscript proving their equivalence with Con(SRP) over WKL_0. See [Fr2].

1. LEAD STATEMENTS IN INVARIANT MAXIMALITY

Our two lead statements for Invariant Maximality are as follows.

PROPOSITION A. Every subset of $Q[0,n]^k$ has a $0;1\to...\to n$ invariant maximal emulation.

PROPOSITION B. Every order invariant graph on $Q[0,n]^k$ has a $0;1\to...\to n$ invariant maximal clique.

We also have the following refinements.

PROPOSITION C. Every subset of $Q[0,n]^k$ has a $0;1\to...\to n$,tail invariant maximal emulation.

PROPOSITION D. Every order invariant graph on $Q[0,n]^k$ has a $0;1\rightarrow\ldots\rightarrow n$,tail invariant maximal clique.

Although obviously a bit more technical (with "tail"), we expect the latter two to play an important role in Invariant Maximality. Aside from the reversals being considerably easier, we expect strong control over the strength for fixed small k,n.

As we have emphasized elsewhere, the given subset in Propositions A,C can be taken to be finite (or even bounded by a double exponential in k) without change. This remark will also apply to the statements in section 2.

Here are the supporting definitions, which have already appeared in a number of places.

DEFINITION 1.1. S is an emulation of $E \subseteq Q[0,n]^k$ if and only if $S \subseteq Q[0,n]^k$ and every element of S^2 is order equivalent to an element of E^2. S is a maximal emulation of $E \subseteq Q[0,n]^k$ if and only if S is an emulation of $E \subseteq Q[0,n]^k$ which is not a proper subset of any emulation of $E \subseteq Q[0,n]^k$.

DEFINITION 1.2. A graph on a set V of vertices is (V,E) where $E \subseteq V^2$ is irreflexive and symmetric. E is the adjacency relation. An order invariant graph on $Q[0,n]^k$ is a graph on $Q[0,n]^k$ where the adjacency relation is an order invariant subset of $Q[0,n]^{2k}$.

DEFINITION 1.3. $S \subseteq Q[0,n]^k$ is $q_1\rightarrow\ldots\rightarrow q_m$ invariant if and only if $q_1,\ldots,q_m \in Q[0,n]$ and for all $x \in \{q_1,\ldots,q_{m-1}\}^k$, membership in S remains the same if we replace $q_1,\ldots,q_{m-1}$ by $q_2,\ldots,q_m$, respectively. $S \subseteq Q[0,n]^k$ is $p;q_1\rightarrow\ldots\rightarrow q_m$ invariant if and only if $p,q_1,\ldots,q_m \in Q[0,n]$ and for all $x \in (\{q_1,\ldots,q_{m-1}\} \cup Q[p,q_1))^k$, membership in S remains the same if we replace $q_1,\ldots,q_{m-1}$ by $q_2,\ldots,q_m$, respectively.

DEFINITION 1.4. S c $Q[0,n]^k$ is $p;q_1\rightarrow\ldots\rightarrow q_m$,tail invariant if and only if for all $1 \leq i \leq m-1$, $S \subseteq Q[0,n]^k$ is $p;q_i\rightarrow\ldots\rightarrow q_m$ invariant.

The equivalence of these statements with Con(SRP) is more or less now available in manuscripts. Most recently see [Fr2] for Proposition B. The equivalence of Propositions A,B and of C,D in the Overview manuscript. the proofs from Con(SRP) are in the Derivation manuscript.

In section 2 we present a Proposition based on the upper shift, which are fundamentally different than Propositions A-D and still competitive. We also present a version with the upper shift which corresponds to HUGE.

In section 3 we present explicitly finite forms of these infinite Propositions in sections 1,2. These are all explicitly Π^0_2, and with basic a priori bounds, they become explicitly Π^0_1.

2. INFINITE INVARIANT EMULATIONS IN Q: SRP,HUGE

We have presented our lead infinite statements in Q of level SRP in section 1, for cliques and emulations in $Q[0,n]^k$. Here we present some infinite statements based on a weakened form of maximal emulations in Q^k, and a strengthened form of invariance using the upper shift operation instead of $0;1\rightarrow...\rightarrow n$ invariance.

DEFINITION 2.1. Let $S \subseteq Q^k$. span(S) is the least k-cube $E^k \supseteq S \cup \{0\}^k$. fld(S) is the set of all coordinates of all elements of S. $S|\leq p = S \cap (-\infty,p]^k$. $S|<p = S \cap (-\infty,p)^k$.

DEFINITION 2.2. S is an emulation of $E \subseteq Q^k$ if and only if $S \subseteq Q^k$ and every element of S^2 is order equivalent to an element of E^2. x is E emulation blocked by S if and only if $x \in Q^k$ and $S \cup \{x\}$ is not an emulation of $E \subseteq Q^k$. S is a full emulation of $E \subseteq Q^k$ if and only if S is an emulation of $E \subseteq Q^k$ where every x in span(S)\S is E emulation blocked by S|<max(x).

Note the use of S|<max(x) instead of S here. This gives full emulations the flavor of a recursion rather than a simple maximality.

Here is some background information about full emulations.

THEOREM 2.1. Every subset of Q^k has either the full emulation $\emptyset$ or the full emulation $\{0\}^k$.

Proof: Let $E \subseteq Q^k$. Suppose there is no $(p,...,p) \in E$. Let $S = \emptyset$. Then S is an emulation of E. Also $x = (0,...,0)$ is E emulation blocked by S|<max(x = $\emptyset$ and $(0,...,0)$ is the unique element of

span(S)\S. Therefore S has the full emulation ∅. Now suppose there exists (p,...,p) ∈ E. Let S = {(0,...,0)}. Then S is an emulation of E. Also span(S)\S is empty, and so S is a full emulation of E. QED

DEFINITION 2.3. E ⊆ Q^k is topped if and only if there exists (p,...,p) in E where p is greater than all coordinates of all other elements of E.

THEOREM 2.2. Let E ⊆ Q^k be topped and 0 ∈ B ⊆ Q be well ordered. There is a unique full emulation of E with span B^k. This statement is provably equivalent to ATR_0 over RCA_0.

Proof: Let E,B be as given. Suppose there are two different full emulations S,S′ of E with span B^k. Let x ∈ S have least max such that x ∈ S ↔ x ∉ S′. Suppose x ∈ S\S′. Then x,x′ ∈ span(S) = span(S′), and since x ∉ S′, we have that S′|<max(x) ∪ {x} is not an E emulation. Hence S|<max(x) ∪ {x} is not an emulation of E. This contradicts x ∈ S.

For existence, we first modify full emulation to use $fld(S)^k$\S instead of span(S)\S. It is clear how to define by transfinite induction on B that for all b ∈ B there exists a modified full emulation f(b) of E with field B|≤b, which includes (b,...,b). The inclusion uses that E is topped. We have the usual coherence here from the previously established uniqueness. So we obtain a modified full emulation S of E with span B^k wince 0 ∈ B. In particular, (0,...,0) ∈ S. Therefore S is a full emulation of E with span B^k. QED

DEFINITION 2.4. The upper shift of x ∈ Q^k is obtained by adding 1 to all nonnegative coordinates and leaving the negative coordinates unchanged. The upper shift of S ⊆ Q^k is the set of upper shifts of its elements. We write ush(x) for the upper shift of x and ush(S) for the upper shift of S.

PROPOSITION E. Every subset of Q^k has a full emulation containing its upper shift.

THEOREM 2.3. Proposition E is provably equivalent to Con(SRP) over WKL_0.

We now extend these ideas to reach the level of HUGE. For this, we need to modify emulations to pointed emulations, a weaker notion. A nice but not major point is that we don’t have to use

0 in the setup, but it does appear in the equation in Proposition F.

DEFINITION 2.5. S is a pointed emulation of $E \subseteq Q^k$ if and only if $S \subseteq Q^k$ and for all $x,y \in S$ with $x_1,...,x_k,y_1,...,y_{k-1} < y_k$, there exists $z,w \in E$ such that (x,y),(z,w) are order equivalent. x is E pointed emulation blocked by S if and only if $x \in Q^k$ and $S \cup \{x\}$ is not a pointed emulation of $E \subseteq Q^k$. S is a full pointed emulation of $E \subseteq Q^k$ if and only if S is a pointed emulation of $E \subseteq Q^k$ where every $x \in fld(S)^k\backslash S$ is E pointed emulation blocked by S|<max(x).

THEOREM 2.4. Let 0 in $B \subseteq Q$ be well ordered. There is a unique full pointed emulation of E with field B. This statement is provably equivalent to ATR_0 over RCA_0.

If we use pointed emulations for Theorem 2.2 and Proposition E, there is no change.

PROPOSITION E'. Every subset of Q^k has a full pointed emulation containing its upper shift.

THEOREM 2.5. Proposition E' is provably equivalent to Con(SRP) over WKL_0.

PROPOSITION F. Every subset of Q^k, $k \geq 3$, has a full pointed emulation $S \supseteq ush(S)$, where $fld(ush(S))|\leq k = \{p \leq k: S(k+(1/2),0,p,...,p)\}$.

THEOREM 2.5. Proposition F is provably equivalent to Con(HUGE) over WKL_0.

Propositions E,E',F are implicitly Π^0_1 via the Completeness Theorem for first order logic. Propositions E,E' are provably equivalent to Con(SRP) over WKL_0 with some fixed small dimension k. Proposition F corresponds to the hierarchy of k-huge, k = 1,2,... .

We can view Propositions E,F in the following way:

Given $k \geq 1$ and a rational piecewise linear function $T:Q^k \to Q^k$. Every subset of Q^k has a full emulation $S \supseteq T[S]$.

Given $k \geq 1$ and rational piecewise linear functions $T_1,...,T_r,W_1,...,W_r:Q^k \to Q^k$. Every subset of Q^k has a full pointed emulation S such that each $T_i[S] \subseteq W_i[S]$.

Is there a decision procedure for the second Template above which is provably correct using a large cardinal hypothesis like $j:V(\kappa) \to V(\kappa)$ or $j:V(\kappa+1) \to V(\kappa+1)$? For the first Template, maybe Con(SRP) is enough? At least perhaps $\kappa \to \omega$ would suffice?

Of course this is very ambitious and there may not even be a decision procedure. But this Template very conveniently can drastically weakened in scope by, e.g., fixing k to be very small, and using only one or two pairs of very simply shaped piecewise linear maps even with only nonconstant coefficients 0,1.

3. FINITE EMULATIONS IN Q: SRP,HUGE

In this section 3, we give our explicitly finite forms for Propositions A,C,E,F.

DEFINITION 3.1. FS(Q) is the set of all finite subsets of Q. We use α,β,γ with or without subscripts/superscripts for nonempty finite lists of elements of FS(Q) unless otherwise indicated. We use x,y,z,w with or without subscripts/superscripts for elements of FS(Q) unless otherwise indicated. We use p,q,r,s,t with or without subscripts/superscripts for rationals unless otherwise indicated. We use k,n,m,i,j with or without subscripts/superscripts for positive integers unless otherwise indicated. $\cup\alpha \subseteq Q$ is the union of the terms of alpha.

DEFINITION 3.2. Let x,y,z,w in FS(Q). (x,y),(z,w) are similar if and only if the following holds. For the unique increasing h from $x \cup y$ onto $z \cup w$, we have h[x] = y and h[z] = w.

This just means that (x,y) and (z,w) are the same order theoretically.

DEFINITION 3.3. β is a continuation of α if and only if β extends α and every pair of terms from β is similar to some pair of terms from α.

DEFINITION 3.4. x is α continuation blocked by β if and only if β,x is not a continuation of α.

We will be using three related notions of full continuations, one for each of sections 3.1,3.2,3.3.

We emphasize that throughout this section 3, all objects are finite unless explicitly stated to be infinite.

3.1. FINITE CONTINUATIONS IN Q[0,n]: SRP

DEFINITION 3.1.1. β is a full continuation/n of α if and only if β is a continuation of α where every $x \subseteq (\cup\beta \cup \{0,...,n\}) \cap Q[0,n]^k$ that is not a term of β, is α continuation blocked by β.

Note how full continuation/n involves the space $Q[0,n]^k$ and the preferred elements 0,...,n.

THEOREM 3.1. If $\cup\alpha \subseteq Q[0,n]$, then α starts an infinite series of successive full continuations/n.

Proof: By the obvious greedy algorithm. QED

PROPOSITION G. If $\cup\alpha \subseteq Q[0,1)$ then α has a $0;1\to2\to...\to n$ invariant full continuation/n.
PROPOSITION H. If $\cup\alpha \subseteq Q[0,1)$ then α has a full continuation/n with a full continuation/n, both $0;1\to...\to n$ invariant.
PROPOSITION I. If $\cup\alpha \subseteq Q[0,1)$ then α starts a series of full continuations/n of every finite length, all $0;1\to...\to n$ invariant.
PROPOSITION J. If $\cup\alpha \subseteq Q[0,1)$ then α starts an infinite series of full continuations, all $0;1\to...\to n$ invariant.

Note that these statements are all explicitly Π^0_2. However, there are obvious a priori bounds on the numerators and denominators used (double exponential) that make the first three statements explicitly Π^0_1 and the last statement implicitly Π^0_1.

THEOREM 3.2. Propositions H,I are provably equivalent to Con(SRP) over EFA. Proposition G is provable in EFA + Con(SRP). Proposition J is provably equivalent to Con(SRP) over WKL_0.

It is natural to strengthen these statements using $0;1\to...\to n$,tail invariance. The results will remain unchanged.

3.2. FINITE ULTRA CONTINUATIONS IN Q: SRP

DEFINITION 3.2.1. β is a full continuation of α if and only if β is a continuation of α where every x ⊆ ∪β ∪ {0} that is not a term of β, is α continuation blocked by β|<max(x).

Note that here we don't use the space $Q[0,n]^k$ and 0,...,n as we did with full continuation/n. On the other hand, note the β|<max(x) instead of β.

DEFINITION 3.2.2. The upper shift of x in FS(Q) results from adding 1 to all of its nonnegative elements and leaving its negative elements alone. We say that α is negative if and only if all of the coordinates of all of its terms are negative.

Note that the upper shift of α is α if and only if α is negative.

DEFINITION 3.2.3. beta is an ultra full continuation of α if and only if β is a full continuation of α, where the upper shift of every term of α is a term of β.

Now consider the following Propositions.

PROPOSITION K. Every negative α has an ultra full continuation.
PROPOSITION L. Every negative α has an ultra full continuation with an ultra full continuation.
PROPOSITION M. Every negative α a series of ultra full continuations of every finite length.
PROPOSITION N. Every negative α has an infinite series of ultra full continuations.

Note that Propositions K,L,M are explicitly Π^0_2. Note that every set in the continuations has cardinality at most one of the sets in the initial α. Hence there is an obvious a priori bound on the lengths of the continuations. Then we can bound the numerators and denominators used, putting the statements in explicitly Π^0_1 form. Proposition N is implicitly Π^0_1 using these bounds, via the Completeness Theorem for first order logic.

THEOREM 3.2.1. Propositions L,M are provably equivalent to Con(SRP) over EFA. Proposition N is provably equivalent to Con(SRP) over WKL_0. Proposition K is provable in EFA + Con(SRP).

Note that $\alpha = \{(0,...,0)\}$ has no ultra continuation, and is not negative.

There is a fixed negative α for which Propositions L,M is provably equivalent to Con(SRP) over EFA and Proposition N is provably equivalent to Con(SRP). It is realistic to find such an α where all terms are of small cardinality (but the number of terms (sets) is not expected to be made small).

3.3. FINITE ULTRA* POINTED CONTINUATIONS IN Q: HUGE

DEFINITION 3.3.1. β is a pointed continuation of α if and only if β extends α, and every (x,y) from β with $x_1,...,x_k,y_1,...,y_{k-1} < y_k$, is similar to some (z,w) from alpha.

DEFINITION 3.3.2. x is α pointed continuation blocked by β if and only if β,x is not a pointed continuation of alpha.

DEFINITION 3.3.3. β is a full pointed continuation of α if and only if β is a pointed continuation of α where every $x \subseteq \cup\beta$ that is not a term of β is α pointed continuation blocked by $\beta|<\max(x)$.

DEFINITION 3.3.4. β is an ultra full pointed continuation of α if and only if β is a full pointed continuation of α, where the upper shift of every term of α is a term of β.

DEFINITION 3.3.5. β is an ultra* full pointed continuation of α if and only if
i. β is an ultra full pointed continuation of α.
ii. $\mathrm{ush}(\cup\beta|\leq k) = \{p \leq k: S(k+(1/2),0,p,...,p)\}$.

PROPOSITION O. Every negative α has an ultra* full pointed continuation
PROPOSITION P. Every negative α has an ultra* full pointed continuation with an ultra* full pointed continuation.
PROPOSITION Q. Every negative α starts a series of ultra* full pointed continuations of every finite length.
PROPOSITION R. Every negative α starts an infinite series of ultra* full pointed continuations.

Note that these statements are all explicitly Π^0_2. However, there are obvious a priori bounds on the numerators and denominators used (double exponential) that make the statements explicitly Π^0_1.

THEOREM 3.3.1. Propositions P,Q are provably equivalent to Con(HUGE) over EFA. Proposition O is provable in EFA + Con(HUGE). Proposition R is provably equivalent to Con(HUGS) over WKL_0.

REFERENCES

[Fr1] p-Sequential Invariance, 24 pages. Scheduled to be released in June, 2023 on website https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/

[Fr2] Infinite Tangible Incompleteness: invariant maximality reversals, 108 pages. Scheduled to be released in July, 2023 on website https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/