POLYNOMIAL AND QUADRATIC INCOMPLETENESS

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March 26, 2024

1. Adjacent Ramsey Theory
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1. ADJACENT RAMSEY THEORY

DEFINITION 1.1. N is the set of nonnegative integers. Let $x,y \in N^k$. $x \leq_c y$ if and only if for all $1 \leq i \leq k$, $x_i \leq y_i$. x adj y if and only if x,y are strictly increasing and y is obtained from x by deleting its first term and appending a new last term.

In [Fr10], we treat the following statement where $\leq$ is the same as $\leq_c$ here.

THEOREM D, [Fr10], section 2. For all $f:N^k \to N^r$, there exist $x_1 < \ldots < x_{k+1}$ such that $f(x_1,\ldots,x_k) \leq f(x_2,\ldots,x_{k+1})$.

[Fr10] proves Theorem D in RCA_0 + ϵ_0 is well ordered, by proving Theorem D in RCA_0 + "every Π^1_1 sentence provable in ACA_0 is true".

[Fr10] derives ϵ_0 is well ordered from Theorem D over RCA_0 by constructing an infinite descending sequence through ϵ_0 effectively from any counterexample to Theorem D.

[Fr10], section 3, also considers Theorem D for various categories of functions including recursive, primitive recursive, elementary recursive, and polynomial time computable functions, with elements of N processed by their base 2 representation. We refer to these as D/rec,primrec,elem,ptime, respectively.

D/rec,primrec,elem,ptime are provably equivalent over EFA, respectively, to there is no infinite descending recursive,

primitive recursive, elementary recursive, polynomial time computable, through ϵ_0.

It is well known that "no infinite descending recursive, primitive recursive, elementary recursive, polynomial time computable, through ϵ_0" are provably equivalent, respectively, to 2-Con(PA), 1-Con(PA), 1-Con(PA), 1-Con(PA), over EFA.

THEOREM 1.1. D/rec,primrec,elem,ptime are provably equivalent, over EFA, to 2-Con(PA),1-Con(PA),1-Con(PA),1-Con(PA), respectively.

Proof: From the above. QED

Also see [HP16] which reworks some of [Fr10].

2. POLYNOMIAL/QUADRATIC INCOMPLETENESS

If we simply take D and restrict to polynomials $P:N^k \to N^r$, then we obtain a statement which is easily proved in EFA. However, we have found a very natural variant of D that is equivalent to D and where we do obtain Incompleteness in this way.

FULL/ADJ. Let $f:N^{2k} \to Z^{2k}$, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and $y \leq_c w$ such that $f(x,y) = f(z,w) = 0$.

Following [Fr10], we continue with

REC/ADJ. Let $f:N^{2k} \to Z^{2k}$ be recursive, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and $y \leq_c w$ such that $f(x,y) = f(z,w) = 0$.

PRIMREC/ADJ. Let $f:N^{2k} \to Z^{2k}$ be primitive recursive, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and $y \leq_c w$ such that $f(x,y) = f(z,w) = 0$.

ELEM/ADJ. Let $f:N^{2k} \to Z^{2k}$ be elementary recursive, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and $y \leq_c w$ such that $f(x,y) = f(z,w) = 0$.

PTIME/ADJ. Let $f:N^{2k} \to Z^{2k}$ be polynomial time computable, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and $y \leq_c w$ such that $f(x,y) = f(z,w) = 0$.

Because of the strategic choice of the FULL/ADJ, we can continue with

POLY/ADJ. Let $P:N^{2k} \to Z^{2k}$ be a polynomial with integer coefficients, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and y $\leq$c w such that $P(x,y) = P(z,w) = 0$.

QUAD/ADJ. Let $Q:N^{2k} \to Z^{2k}$ be a polynomial of degree ≤ 2 with integer coefficients, where $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$. There exists x adj z and y $\leq$c w such that $Q(x,y) = Q(z,w) = 0$.

LEMMA 2.1. FULL/ADJ is provably equivalent to Theorem D (section 2) over RCA_0. REC/ADJ is provably equivalent to D/rec over EFA.

Proof: For every (recursive) function $f:N^k \to N^k$ there exists a (recursive) function $g:N^{2k} \to Z^{2k}$ such that f is unique with $(\forall x)(g(x,f(x)) = 0)$. For every (recursive) function $f:N^{2k} \to Z^{2k}$ with $(\forall x \in N^k)(\exists y \in N^k)(f(x,y) = 0)$ there exists a (recursive) function $g:N^k \to N^k$ with $(\forall x \in N^k)(f(x,g(x)) = 0)$. QED

We use the following variant of QUAD/ADJ.

QUAD/ADJ*. Let $Q:N^k \times N^r \to Z^s$ be a polynomial of degree ≤ 2 with integer coefficients, where $(\forall x \in N^k)(\exists y \in N^r)(Q(x,y) = 0)$. There exists x adj z and y $\leq$c w such that $Q(x,y) = Q(z,w) = 0$.

LEMMA 2.2. QUAD/ADJ implies QUAD/ADJ* over EFA.

Proof: Assume QUAD/ADJ. For QUAD/ADJ*, let $Q:N^k \times N^r \to Z^s$ be a polynomial of degree ≤ 2 with integer coefficients, where $(\forall x \in N^k)(\exists y \in N^r)(Q(x,y) = 0)$. Let $t = \max(k,r,s)$. Let $Q':N^{2t} \to Z^{2t}$ be the polynomial of degree ≤ 2 with integer coefficients defined by $Q'(x_1,...,x_{2t}) = (Q(x_1,...,x_k,x_{t+1},...,x_{t+r}),0,..,0)$. Then $(\forall x \in N^t)$ $(\exists y \in N^t)(Q'(x,y) = 0)$. Let x adj z and y $\leq$c w be such that $Q'(x,y) = Q'(z,w) = 0$. Then $Q(x_1,...,x_k,y_1,...,y_r) = 0$, $Q(z_1,...,z_k,w_1,...,w_r) = 0$, $(x_1,...,x_k)$ adj $(z_1,...,z_k)$, $(y_1,...,y_r)$ $\leq$c $(w_1,...,w_r)$. QED

LEMMA 2.3. Let $S \subseteq N^k$ be r.e. There exists a polynomial $P:N^{k+m} \to Z$ with integer coefficients that for all $x \in N^k$, $x \in S \leftrightarrow (\exists y \in N^m)(P(x,y) = 0)$.

Proof: This is a standard form of the MRDP theorem solving Hilbert's Tenth Problem. QED

LEMMA 2.4. Let $P:N^k \to Z$ be a polynomial with integer coefficients. For all $x_1,\ldots,x_n \in N$ and $y \in Z$, $P(x_1,\ldots,x_n) = y \leftrightarrow (\exists z_1,\ldots,z_m \in Z)(\varphi)$, where φ is a conjunction of quadratic equations with integer coefficients in $x_1,\ldots,x_n,y,z_1,\ldots,z_m$.

Proof: This is well known. First show this for monomials P. Then show this for P. QED

LEMMA 2.5. Let $P:N^k \to Z$ be a polynomial with integer coefficients. For all $x_1,\ldots,x_n \in N$ and $y \in Z$, $P(x_1,\ldots,x_n) = y \leftrightarrow (\exists z_1,\ldots,z_m \in N)(\varphi)$, where φ is a conjunction of quadratic equations with integer coefficients in $x_1,\ldots,x_n,y,z_1,\ldots,z_m$.

Proof: This is well known. The quantifiers over Z from Lemma 2.4 must be replaced with quantifiers over N. Every integer is the difference of two nonnegative integers, so we just need to double the number m of variables in the obvious way. QED

LEMMA 2.6. Let $S \subseteq N^k$ be r.e. There exists a polynomial $Q:N^{k+r} \to Z^s$ with integer coefficients, of degree ≤ 2, such that for all $x \in N^k$, $x \in S \leftrightarrow (\exists y \in N^r)(P(x,y) = 0)$.

Proof: This is a well known variant of MRDP. It follows easily from Lemmas 2.3, 2.5, using the conjunction of s quadratic equations. QED

LEMMA 2.7. Let $f:N^k \to N^r$ be recursive. There exists a polynomial $Q:N^{k+r+s} \to Z^m$ with integer coefficients, of degree ≤ 2, such that for all $x \in N^k$, $y \in N^r$, $f(x) = y \leftrightarrow (\exists z \in N^t)(Q(x,y,z) = 0)$.

Proof: Immediate from Lemma 2.6, as $\{(x,f(x)): x \in \mathrm{dom}(f)\}$ is r.e. QED

LEMMA 2.8. QUAD/ADJ* implies D/rec over EFA.

Proof: Assume QUAD/ADJ*. For Theorem D/rec, let $f:N^k \to N^r$ be recursive. By Lemma 2.7, let $Q:N^{k+s} \to Z^{k+r+s}$ be a polynomial in integer coefficients of degree ≤ 2, where for all $x \in N^k$ and $y \in N^r$, $f(x) = y \leftrightarrow (\exists z \in N^s)(Q(x,y,z) = 0)$.

Note that $(\forall x \in N^k)(\exists\ (y,z) \text{ in } N^{r+s})(Q(x,y,z) = 0)$. Let x adj x' and $(y,z) \leq c\ (y',z')$ be such that $Q(x,y,z) = Q(x',y',z') = 0$.

Then x adj x' and y ≤c y'. Also f(x) = y and f(x') = y', since Q(x,y,z) = Q(x',y',z') = 0. QED

THEOREM 2.9. The following are provably equivalent to 2-Con(PA) over EFA. QUAD/ADJ, POLY/ADJ, REC/ADJ, D/rec.

Proof: We have shown the following over EFA.
D/rec → RED/ADJ. Lemma 2.1.
REC/ADJ → POLY/ADJ. Trivial.
POLY/ADJ → QUAD/ADJ. Trivial.
QUAD/ADJ → QUAD/ADJ*. Lemma 2.2.
QUAD/ADJ* → D/rec. Lemma 2.8.
D/rec ↔ 2-Con(PA). Theorem 1.1.

The first five lines form a loop of implications, so that the component statements are equivalent. The last line yields the claim. QED

REFERENCES

[Fr10] H. Friedman, Adjacent Ramsey Theory, August 29, 2010, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/, #65, 17 pages.

[FP16] H. Friedman and Florian Pelupessy, Independence of Ramsey theorem variants using epsilon0, Proc. Amer. Math. Soc. 144 (2016), 853-860.