FINITE INVARIANT CHOICE

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Abstract. INV/CHOICE: Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:Q[0,k]^n \to Q[0,k]^n$. INV/CHOICE/Π^0_2: Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:GEN(m,f,0,...,k)^n \to Q[0,k]^n$. INV/CHOICE/$\Pi^0_1$: Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:GEN(m,f,0,...,k)^n \to Q[0,k]^n$ with field contained in $N^n/kn(k+1)^{(n+1)\wedge(m+2)}$. INV/CHOICE is provably equivalent to Con(SRP) over WKL_0. INV/CHOICE/Π^0_2 and INV/CHOICE/Π^0_1 are provably equivalent to Con(SRP) over PRA.

1. INTRODUCTION

We rely on [Fr24der],[Fr24rev] where our lead statement in Invariant Choice, INV/CHOICE, is treated. INV/CHOICE and its weakening, Restricted INV/CHOICE, are shown to be provably equivalent to Con(SRP) over WKL_0, in [Fr24der],[Fr24rev]. See [Fr24rev], Theorem 12.2.

This paper features the following three mathematical statements: The remaining statement, Restricted INV/CHOICE, is used for the technical development.

INV/CHOICE. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:Q[0,k]^n \to Q[0,k]^n$.

INV/CHOICE/Π^0_2. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k] shift invariant free choice function $f:GEN(m,f,0,...,k)^n \to Q[0,k]^n$.

INV/CHOICE/Π^0_1. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k] shift invariant free choice function $f:GEN(m,f,0,...,k)^n \to Q[0,k]n$ with field contained in $N/kn(k+1)^{(n+1)\char`^(m+2)}$.

We now review the definitions used in INV/CHOICE. We define GEN in section 3.

DEFINITION 1.1. We use a,b,c,d,e,i,j,k,m,n,r,s,t, with and without subscripts and superscripts, for positive integers unless indicated otherwise. We use p,q with and without subscripts and superscripts, for rational numbers unless indicated otherwise. We use Q,N,Z for the set of all rationals, nonnegative integers, and integers. We use interval notation Q[(a,b)], where a,b are extended rationals. [k] = {1,...,k}.

DEFINITION 1.2. $x,y \in Q^k$ are order equivalent if and only if for all $1 \le i,j \le k$, $x_i < x_j \leftrightarrow y_i < y_j$. $T \subseteq Q[0,k]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[0,k]^k$, $x \in T \leftrightarrow y \in T$. R is an order invariant relation on $Q[0,k]^k$ if and only if R is an order invariant subset of $Q[0,k]^{2k}$.

DEFINITION 1.3. Let R be a relation on X. S is R free if and only if $S \subseteq X$ and for all distinct $x,y \in S$, $x \neg R\ y$. S is a maximal R free set if and only if S is R free and S is not a proper subset of any R free set.

DEFINITION 1.4. $S \subseteq Q[0,k]^n$ is 1;1,...,k lower shift invariant if and only if for all $x \in (Q[0,1) \cup \{1,...,k-1\})^n$, $x \in S \leftrightarrow x' \in S$, where x' results from x by replacing 1,...,k-1 respectively by 2,...,k.

DEFINITION 1.5. $f::A \to B$ if and only if f is a partial function from A into B. I.e., $dom(f) \subseteq A$ and $rng(f) \subseteq B$.

DEFINITION 1.6. Let R be a relation on X. A choice function for R is a function $f::X \to X$ such that for all $x \in dom(f)$, $x\ R\ f(x)$. A free choice function for R is a choice function whose range is R free. We often identify functions with their graphs as in "f is 1;1,...,k lower shift invariant".

We derive INV/CHOICE/$\prod^0_2$ from INV/CHOICE, and INV/CHOICE/$\prod^0_1$ from INV/CHOICE/$\prod^0_2$. For the reversal of INV/CHOICE/$\prod^0_2$, we show that it implies Restricted iNV/CHOICE, which itself is reversed in [Fr24rev], Theorem 12.2.

RESTRICTED INV/CHOICE. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant choice function $f:E^n \rightarrow E^n$, $\{0,...,k\} \subseteq E \subseteq Q[0,k]$.

Note that INV/CHOICE/$\prod^0_2$ is explicitly $\prod^0_2$. It is explicitly $\prod^0_1$ modulo quantifier elimination for (Q,<.+). Also note that INV/CHOICE/$\prod^0_1$ is explicitly $\prod^0_1$.

Our results show that INV/CHOICE/$\prod^0_1$,INV/CHOICE/$\prod^0_2$ are provably equivalent to Con(SRP) over PRA.

2. DERIVATION

DEFINITION 2.1. Let $f::Q[0,k]^n \rightarrow Q[0,k]^n$. Note that f has n coordinate functions. GEN(m,f,0,...,k) is the set of values of defined terms in the coordinate functions of f and constants 0,...,k, where there are at most m occurrences of coordinate functions.

LEMMA 2.1. (EFA) Let $f:GEN(m,f,0,...,k)^n \rightarrow Q[0,k]^n$.
i. f is finite.
ii. all of the terms in the coordinate functions of f and constants 0,...,k with at most m occurrences of coordinate functions are defined.
iii. For all $1 \leq i < m$, $f:GEN(i,f,0,...,k)^n \rightarrow GEN(n(in+1),f,0,...,k)^n$.

Proof: i is immediate since $GEN(m,k,0,...,k)^n$ is obviously finite. For ii, let t be a relevant term which is undefined, and has the smallest number of occurrences of coordinate functions, and with ≤m occurrences of coordinate functions. Write t = $f_i(t_1,...,t_n)$. Then $t_1,...,t_n$ are defined and lie in GEN(m,k,f,0,...,k). Hence $f_i(t_1,...,t_n)$ is defined. This is a contradiction.

For iii, let $p_1,...,p_n \in GEN(i,k,f,0,...,k)$. Then $f(p_1,...,p_n) = (f_1(p_1,...,p_n),...,f_n(p_1,...,p_n))$ is defined and uses at most n(in+1) occurrences of coordinate functions of f. QED

INV/CHOICE/Π^0_2. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:GEN(m,f,0,...,k)^n \rightarrow Q[0,k]^n$.

THEOREM 2.2. (RCA_0) INV/CHOICE implies INV/CHOICE/Π^0_2.

Proof: Assume INV/CHOICE. For INV/CHOICE/Π^0_2, let R be a reflexive symmetric order invariant relation on $Q[0,k]^n$ and m be given. Let f be a lower [k] shift invariant free choice function whose 0-definable multivariate relations are lower [k] shift invariant. We show that $f|GEN(m,f,0,...,k)^n$ is lower [k] shift invariant.

Let $(p_1,...,p_n,q_1,...,q_n)$ lie in (the graph of) $f|GEN(m,f,0,...,k)^n$. Then $p_1,...,p_n \in GEN(m,f,0,...,k)$ and $f(p_1,...,p_n) = (q_1,...,q_n)$. Furthermore assume that $(p_1,...,p_n,q_1,...,q_n) \in (Q[0,1) \cup \{1,...,k-1\})^{2n}$. Let $p_1',...,p_n',q_1',...,q_n'$ be the result of replacing 0,...,k-1 by 1,...,k in $p_1,...,p_n,q_1,...,q_n$.

We need to show that $(p_1',...,p_n',q_1',...,q_n')$ lies in (the graph of) $f|GEN(m,f,0,...,k)^n$.

We have $(p_1',...,p_n') \in GEN(m,f,0,...,k)^n$ since each p_i' is either p_i or is among 1,...,k. Also $f(p_1',...,p_n') = (q_1',...,q_n')$ since f itself is 1;1,...,k lower shift invariant. QED

It is easy to see that INVCHOICE/Π^0_2 is explicitly Π^0_1 modulo quantifier elimination for (Q,<,+). The outermost quantifiers are k,n,m. The formula A(k,n,m) is a complex first order formula over (Q,<,0,...,k). The various order invariant relations R on $Q[0,k]^n$ are listed by their quantifier free definitions. Reflexive symmetric for each is an antecedent first order over (Q,<,0,...,k). The conclusion for each asserts existence of a 1;1,...,k lower shift invariant free choice function on $GEN(m,f,0,...,k)^n$. But this is expressible by a long finite string of existential quantifiers over Q[0,k] of definite exponential length, followed by a quantified statement involving only constants 0,...,k, expressing that the string of existential quantifiers is organized in a specific way to form a finite function f whose domain is $GEN(m,f,0,...,k)^n$. Also requiring that it is a [k] shift invariant choice function for R.

However there is a much more direct way to put INV/CHOICE/Π^0_2 in explicitly Π^0_1 form.

INV/CHOICE/Π^0_1. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:GEN(m,f,0,...,k)^n \to Q[0,k]^n$ with field contained in $N^n/kn(k+1)^{(n+1)\hat{}(m+2)}$.

LEMMA 2.3. For $f:Q[0,k]^n \to Q[0,k]^n$, $|GEN(f,m,0,...,k)| \leq (k+1)^{2n(n+1)\hat{}(m-1)}$. The field of f restricted to $GEN(f,m,0,...,k)^n$ has at most $n(k+1)^{(n+1)\hat{}(m+2)}$ elements.

Proof: We need to compute the number of relevant terms of depth $\leq$ i where $i \geq 1$. For i = 1, this is $k+1+n(k+1)^n \leq (k+1)^{2n}$. Suppose that $i \geq 1$ and the number of relevant terms of depth $\leq$ i is at most r. Then the number of such of depth $\leq$ i+1 is at most $r + nr^n \leq (n+1)r^n \leq r^{n+1}$ since $r \geq n+1$.

So with i = 1 we have $(k+1)^{2n}$. With i = 2 we have $(k+1)^{2n(n+1)}$, and so forth to i = m yielding $(k+1)^{2n(n+1)\hat{}(m-1))}$.

For the last claim, for each coordinate function we need to raise to the n-th power, yielding $\leq (k+1)^{(n+1)\hat{}(m+2)}$. There are n coordinate functions, yielding $\leq n(k+1)^{(n+1)\hat{}(m+2)}$. QED

THEOREM 2.4. (RCA_0) INV/CHOICE/Π^0_2 implies INV/CHOICE/Π^0_1.

Proof: Assume INV/CHOICE/Π^0_2. For INV/CHOICE/Π^0_1, let R be a reflexive symmetric order invariant relation on $Q[0,k]^n$. Let $f:GEN(f,m,0,...,k)^n \to Q[0,k]^n$ be a 1;1,...,k lower shift invariant choice function for R, $|fld(f)| < n(k+1)^{(n+1)\hat{}(m+2)}$. Since $0,...,k \in fld(f)$, let h be a one-one order preserving map from fld(f) into {0,1/t,2/t,...,kt/k} which is the identity on {0,...,k}, where $t = kn(k+1)^{(n+1)\hat{}(m+2)}$. Then h sends f onto some $g:GEN(m,g,0,...,k)^n \to Q[0,k]^n$ with field contained in $N/kn(k+1)^{(n+1)\hat{}(m+2)}$, acting on arguments and values. Because of the preservation properties of h, g remains a free choice function and 1;1,...,k lower shift invariant. QED

3. REVERSAL

RESTRICTED INV/CHOICE. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a 1;1,...,k lower shift invariant free choice function $f:E^n \to E^n$, $\{0,...,k\} \subseteq E \subseteq Q[0,k]$.

THEOREM 3.1. Restricted INV/CHOICE is provably equivalent to Con(SRP) over WKL_0.

Proof: From [Fr24der],[Fr24rev], Theorem 2.2. QED

We now derive Restricted INV/CHOICE from INV/CHOICE/Π^0_2 over WKL_0. It is very convenient to use the machinery of nonstandard models of EFA. First some technical logical preliminaries used to control the base theory used for this derivation.

LEMMA 3.2. Let A be a Π^0_1 sentence.
i. RCA_0 proves A $\rightarrow$ Con(EFA + A).
ii. RCA_0 proves A $\rightarrow$ Con(EFA + A + $\neg$Con(EFA + A)).
iii. RCA_0 proves A $\rightarrow$ Con(EFA + A + $\neg I\Sigma_1$).
iv. There is a single instance of induction, IND, such that RCA_0 + A proves Con(EFA + A + $\neg$ IND).

Proof: First replace bounded quantification in A by propositional combinations of equations appropriately within EFA resulting in A* which is purely universal in 0,1,+,•,exp. For i, suppose EFA proves $\neg$A*. By Herbrand's Theorem, which is available in RCA_0, there is a tautological disjunction of substitution instances witnessing $\neg$ A*, so that we have $\neg$ A*, and therefore $\neg$ A.

For ii, apply the second incompleteness theorem to EFA + A.

For iii, note that $I\Sigma_1$ proves A implies Con(EFA + A).

For iv, note that $I\Sigma_1$ is logically equivalent to a single instance of induction over RCA_0. QED

THEOREM 3.3. (ACA_0) INV/CHOICE/Π^0_2 implies Restricted INV/CHOICE.

Proof: Over EFA, put INVCHOICE/Π^0_2 in Π^0_1 form using quantifier elimination for (Q,<,+), and then replace bounded quantification by equations. Call the result A.

Assume A. For Restricted INV/CHOICE, let k,n be standard and R be a reflexive symmetric order invariant relation on $Q[0,k]^n$, R given standardly in terms of k,n. Now Con(EFA + A) by Lemma 3.2i. Using WKL_0, let M be a countable nonstandard model of EFA + A. Let m be a nonstandard integer in M. Let $f:GEN(m,f,0,...,k)^n \rightarrow Q[0,k]^n$ be a 1;1,...,k lower shift invariant free choice function for R in the sense of M. By Lemma 2.1, for all $1 \leq i < m$, $f:GEN(i,f,0,...,k)^n \rightarrow q[0,k]^n$ with field contained in $GEN(n(in+1),f,0,...,k)^n$.

Now let E be the union of the GEN(i,f,0,...,k), i standard, in the sense of M. We work in (Q[0,k],<,E,f|E^n), in the sense of M. Then f|E^n mapping E^n into E^n is a free choice function for R and is 1;1,...,k lower shift invariant, all in the sense of M, Since Q[0,k] is a countable dense linear ordering with distinguished elements 0,...,k in the sense of M, we can make an isomorphism from this $(Q[0,k],<,E,f|E^n)^M$ onto some (Q[0,k],<,E',(f|E^n)'), mapping 0,...,k in M to 0,...,k. Clearly (f|E^n)' is a 1;1,...,k lower shift invariant free choice function for R from E'^n into E'^n. This witnesses Restricted INV/CHOICE.

This whole argument is conducted in WKL_0 except for the construction of E. That is where we use ACA_0. QED

THEOREM 3.4. (WKL_0) INV/CHOICE/Π^0_2 implies Restricted INV/CHOICE.

Proof: Assume INV/CHOICE/Π^0_2. Let A be as in the proof of Theorem 4.3. For Restricted INV/CHOICE, let k,n be standard and R be a reflexive symmetric order invariant relation on $Q[0,k]^n$, R given standardly in terms of k,n. Now Con(EFA + A + ¬IND). By WKL_0 and Lemma 4.2iv, let M be a countable model of EFA + A + ¬IND with a satisfaction relation. Then M has a definable cut C. In the proof of Theorem 4.3, we defined E as a union along the standard integers. Here we define E as a union along the cut C. Actually by the nature of Lemma 3.1iii, we need to instead use the cut C' ⊆ C of points in C whose square lies in C. Once we have this union along C', we repeat the proof of Theorem 3.3. QED

THEOREM 3.5. Restricted INV/CHOICE,INV/CHOICE,INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2, are provably equivalent to Con(SRP) over WKL_0. INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2 are provably equivalent to Con(SRP) over PRA.

Proof: We have shown the following over WKL_0.

1. INV/CHOICE → INV/CHOICE/Π^0_2. Theorem 3.2.
2. INV/CHOICE/Π^0_2 → INV/CHOICE/Π^0_1. Theorem 3.4.
3. INV/CHOICE/Π^0_1 → INV/CHOICE/Π^0_2. Trivial
4. INV/CHOICE/Π^0_2 → Restricted INV/CHOICE. Theorem 4.4.
5. Restricted INV/CHOICE,INV/CHOICE are each equivalent to Con(SRP). [Fr24der],[Fr24rev].

This establishes the first claim.

For the second claim, WKL_0 proves INV/CHOICE/Π^0_1 $\leftrightarrow$ INV/CHOICE/Π^0_2 $\leftrightarrow$ Con(SRP). EFA proves that each of these three sentences are Π^0_1, using that EFA proves quantifier elimination for (Q,<). Hence by the conservation of WKL_0 over PRA for Π^0_2 sentences, we have that INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2 are provably equivalent to Con(SRP) over PRA. QED

THEOREM 3.6. Restricted INV/CHOICE,Restricted,INV/CHOICE,INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2 are provable in SRP^+ but not in any consistent fragment of SRP that proves WKL_0 (as formalized in set theory about V(ω+1)). INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2 are not provable in any consistent fragment of SRP that proves PRA (as formalized in set theory about V(ω)).

Proof: Restricted INV/CHOICE,INV/CHOICE,INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2 are provable in SRP^+ by the first claim of Theorem 4.5. Let T be a consistent fragment of SRP that proves WKL_0 in the sense indicated. Let $T_0 \subseteq$ T be finite and prove WKL_0 (formalized as indicated) and any of Restricted INV/CHOICE,INV/CHOICE,INV/CHOICE/Π^0_1,INV/CHOICE/Π^0_2. By Theorem 4.5, T_0 proves Con(SRP). Since T_0 is a finite fragment of SRP proving WKL_0 (as formalized), T_0 proves its own consistency and is subject to the second incompleteness theorem. Hence T_0 is inconsistent, which is a contradiction.

Suppose INV/CHOICE/Π^0_2 is provable in a consistent fragment T of SRP that proves PRA (formalized as indicated). By the same argument using the second incompleteness theorem, we obtain a contradiction. QED

5. MIDPOINT GENERATION

Midpoint generation refers to the construction GEN(m,f,mid,0,...,k) where we generate not only from the coordinate functions of f and the constants 0,...,k, but also the midpoint function mid(p,q) = (p+q)/2.

INV/CHOICE/Pi02/MID. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k] shift invariant free choice function f:GEN(m,f,mid,0,...,k)^n $\rightarrow$ $Q[0,k]^n$.

It is considerably simpler to work with INV/CHOICE/Π^0_2/MID than with INV/CHOICE/Π^0_2. This is because

1. The derivation of INV/CHOICE/Π^0_2/MID from INV/CHOICE is essentially the same.
2. A simple modification of the reversal, Theorem 4.3, shows how INV/CHOICE/Π^0_2/MID implies INV/CHOICE.

Thus there is no need to consider Restricted INV/CHOICE, and thus we bypass section 12 of [Fr24rev].

1 is left to the reader. For 2, we obtain Restricted INV/CHOICE with the set {0,...,k} $\subseteq$ E $\subseteq$ Q[0,k] being dense because of the closure under mid. Then we obtain INV/CHOICE using any order isomorphism from E onto Q[0,k] which is the identity on 0,...,k.

INV/CHOICE/Π^0_1/MID is formulated with a straightforward adjustment of the estimate kn(k+1)$^{(n+1)\wedge(m+2)}$ which we leave to the reader.

Thus we have shown the following without using restricted INV/Choice.

THEOREM 5.1. INV/CHOICE/Π^0_2/MID, INV/CHOICE/Π^0_1/MID are provably equivalent to Con(SRP) over PRA.

REFERENCES

[Fr24rev] H. Friedman, Invariant Maximality/Choice Reversals.

[Fr24der] H. Friedman, Invariant Maximality/Choice Derivations.