SHIFT, LOWER SHIFT, INTERVAL SHIFT INVARIANCE

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ABSTRACT. This is an advanced draft of section 2 of the monograph Invariant Maximality under preparation. This treats the Invariance in Invariant Maximality (without the Maximality), in some basic core mathematical contexts. The semialgebraic context is most interesting and we regard it as a form of Combinatorial Semialgebraic Geometry.

1. INTRODUCTION

Invariant Maximality combines the two fundamental notions of Maximality and Invariance in the context of subsets of Q^k, where Q is the set of rational numbers.

Here we discuss relevant invariance notions on their own, without maximality. These invariance notions are variants of

what we call shift invariance. We have initiated a systematic study of shift invariance and variants in the context of Z^k, $\mathfrak{R}^k$ where the sets are arbitrary, piecewise linear, and semialgebraic. We treat three notions of invariance: Shift Invariance, Lower Shift Invariance, and Interval Shift Invariance.

All of the results for the $\mathfrak{R}^k$ hold for any R^k, where R is any subfield of $\mathfrak{R}$ (e.g., the rationals or the algebraic real numbers). This is taken up in section 5.

The status of our results where R is an arbitrary ordered field (i.e., not necessarily Archimedean) is more delicate. We will not take this matter up here.

In Invariant Maximality, in its present form, we have only been using Lower Shift Invariance for arbitrary subsets of Q^k. Here we do not involve maximality. Isolating the Invariance has proved rewarding with its connections to familiar core mathematical phenomena. This also has led to a new result in semialgebraic geometry which, in simplest form, is as follows.

for every semialgebraic $f: \mathfrak{R}^2 \to \mathfrak{R}$ there exists $a < b$ such that the image of f on $(a,b)^2$ is not $\mathfrak{R}$

This fails for dimension 3 (and higher) by using $1/(x-z) + 1/(y-z)$, which is onto $\mathfrak{R}$ over every $(a,b)^3$, $a < b$. (Return default 0 where undefined).

It is interesting to note that this is reminiscent of our seemingly unrelated purely combinatorial Thin Set Theorem

for every $f:Z^k \to Z$ there exists infinite $E \subseteq Z$ such that the image of f on E^k is not Z

which starts to get nontrivial at $k = 2$. This may suggest a general theme of non surjectivity. See [Fr99],[Fr00],[CP20],[LP21].

We regard the investigation here as a kind of basic Combinatorial Semialgebraic Geometry. A particularly interesting development in Combinatorial Semialgebraic Geometry is the notable Vapnik-Chervonenkis Property - see [VD98], chapter 5.

DEFINITION 1.1. $N,Z^+,Z,Q,\mathfrak{R}$ is the set of nonnegative integers, positive integers, integers, rationals, reals, respectively. We

use i,j,k,n,m,r,s,t, with and without subscripts and superscripts, for positive integers unless otherwise indicated. We use x,y,z,w,u,v with and without subscripts and superscripts, for real numbers unless otherwise indicated. We use J with and without subscripts and superscripts for intervals of reals of nonzero finite length unless otherwise indicated. For $x \in \mathfrak{R}^k$ and $i \leq k$, we sometimes write x[i] for the i-th term (coordinate) of x. For $E \subseteq \mathfrak{R}$, $E^{k<}, E^{k>}, E^{k=} = \{x \in E^k$: x is strictly increasing}, $\{x \in E^k$: x is strictly decreasing}, $\{x \in E^k$: x is constant}, respectively. Continuous always means pointwise continuous. x,y $\in \mathfrak{R}^k$ are order equivalent if and only if for all $i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. The order types for $\mathfrak{R}^k$ are the equivalence classes under order equivalence. $K = \{0,1\}^\infty$, viewed as the set of all $f:Z^+ \to \{0,1\}$ or the set of all infinite bit sequences. For $k \geq 2$, $E \subseteq \mathfrak{R}^k$, $x \in \mathfrak{R}$, $E_x = \{y \in \mathfrak{R}^{k-1}: (x,y) \in E\}$. For $k \geq 2$, $f:J^k \to \mathfrak{R}$, $x \in \mathfrak{R}$, $f_x:J^{k-1} \to \mathfrak{R}$ is given by $f_x(y) = f(x,y)$. For $A \subseteq N$ and $i \leq |A|$, A[i] is the i-th least element of A.

DEFINITION 1.2. $S \subseteq \mathfrak{R}^k$ is piecewise linear if and only if $S = \{x \in \mathfrak{R}^k: \varphi\}$ where φ is a propositional (Boolean) combination of linear inequalities with real coefficients. $S \subseteq Z^k$ is semialgebraic if and only if $S = \{x \in Q^k: \varphi\}$ where φ is a propositional (Boolean) combination of linear inequalities with integer coefficients. Here constant coefficients are allowed.

DEFINITION 1.3. $S \subseteq \mathfrak{R}^k$ is semialgebraic if and only if $S = \{x \in \mathfrak{R}^k: \varphi\}$ where φ is a propositional (Boolean) combination of polynomial inequalities with real coefficients. $S \subseteq Z^k$ is semialgebraic if and only if $S = \{x \in Z^k: \varphi\}$ where φ is a propositional (Boolean) combination of polynomial inequalities with integer coefficients.

DEFINITION 1.4. Definitions 1.2,1.3 are adapted to R^k, R any subfield of $\mathfrak{R}$, using coefficients from R.

2. SHIFT INVARIANCE

We have both infinite shift invariance and finite shift invariance.

DEFINITION 2.1. Let $S \subseteq \mathfrak{R}^k$. S is $x_1,x_2,\ldots$ shift invariant if and only if

i. $x_1,x_2\ldots \in \mathfrak{R}$.

ii. For $y \in \{x_1,x_2,...\}^k$, membership of $y \in S$ remains the same when we replace each x_i by $x_{i+1} \in y$.

DEFINITION 2.2. Let $S \subseteq \Re^k$. S is $x_1,...,x_n$ shift invariant if and only if

i. $x_1,...,x_n \in \Re$.
ii. For $y \in \{x_1,...,x_{n-1}\}^k$, membership of $y \in S$ remains the same when we replace each x_i by x_{i+1} in y.

DEFINITION 2.3. $S \subseteq \Re^k$ is $x_1,x_2,...$ shift invariant* if and only if for all infinite subsequences $y_1,y_2,...$ of $x_1,x_2,...$, S is $y_1,y_2,...$ shift invariant. $S \subseteq \Re^k$ is $x_1,...,x_n$ shift invariant* if and only if for all subsequences $y_1,...,y_{n'}$ of $x_1,...,x_n$, S is $y_1,...,y_{n'}$ shift invariant.

These said replacements are made in the coordinates of the elements of the y's.

2.1. ARBITRARY

For arbitrary subsets, we can readily use the Infinite Ramsey Theorem for infinite shift invariance, and the Finite Ramsey Theorem for finite shift invariance.

INFINITE RAMSEY THEOREM. In every partition of the k element subsets of Z into finitely many pieces, some infinite subset of Z has all of its k element subsets lying in the same piece.

FINITE RAMSEY THEOREM. Let t >> k,r,m. In every partition of the k element subsets of {0,...,t} into r pieces, some m element subset if {0,...,t} has all of its k element subsets lying in the same piece.

See [Ra30] and [GRS91]. The >> is rather exotic and is roughly a k fold iterated exponential as shown in [GRS91].

It is particular convenient to use well known strengthened Ramsey Theorems using multiple sets order equivalence and infinite set given in advance in the infinite case.

INFINITE RAMSEY THEOREM'. For all $S_1,...,S_n \subseteq Z^k$ and infinite $V \subseteq Z$ there exists infinite $E \subseteq V$ such that for all $i \leq n$ and order equivalent $x,y \in E^k$, $x \in S_i \leftrightarrow y \in S_i$.

FINITE RAMSEY THEOREM'. Let t >> k,r,n. For all $S_1,...,S_n \subseteq \{0,...,t\}^k$ there exists r element $E \subseteq \{0,...,t\}$ such that for all $i \leq n$ and order equivalent $x,y \in E^k$, $x \in S_i \leftrightarrow y \in S_i$.

Order types are handled by partitioning the k-tuples from Z according to their order type. There are only finitely many order types of elements of Z^k.

THEOREM 2.1.1. Every $S_1,...,S_n \subseteq \mathfrak{R}^k$ is $x_1,x_2,...$ shift invariant* for some common $x_1 < x_2 < ... \in Z$.

Proof: Let $S_1,...,S_n \subseteq \mathfrak{R}^k$. By Infinite Ramsey Theorem', let $E \subseteq N$ be infinite such that for all $i \leq n$ and order equivalent $x,y \in E^k$, $x \in S_i \leftrightarrow y \in S_i$. Then clearly $S_1,...,S_n$ is $E_1,E_2,...$ shift invariant since for any $x \in E^k$, the order type of x remains the same if we shift x (adding 1 to subscripts). This also holds for infinite subsequences of $E_1,E_2,...$. QED

For finite shift invariance, we only need the Finite Ramsey Theorem':

THEOREM 2.1.2. Every $S_1,...,S_r \subseteq \mathfrak{R}^k$ is $x_1,...,x_n$ shift invariant* for some common $x_1 < ... < x_n \in Z$.

Actually, the Finite Ramsey Theorem' gives more information.

THEOREM 2.1.3. Let t >> k,n,r. Every $S_1,...,S_f \subseteq \{0,...,t\}^k$ is $x_1,...,x_n$ shift invariant* for some common $x_1 < ... < x_n \leq t$.

The >> here involves iterated exponentials as does the Finite Ramsey Theorem('), even for shift invariance. See [Fr10], [DLR95].

2.2. PIECEWISE LINEAR

LEMMA 2.2.1. Let $T(x_1,...,x_k) = c_1x_1+...+c_kx_k+d$ be given, $c_1,...,c_k,d \in \mathfrak{R}$, $c_k > 0$. Then for all sufficiently large $r \in \mathfrak{R}$, $T(r,r^2,...,r^k)$ is positive. If $c_k < 0$ then for all sufficiently large $r \in \mathfrak{R}$, $T(r,r^2,...,r^k) < 0$. We do not even need these r^i to be consecutive.

Proof: Left to the reader. QED

THEOREM 2.2.2. Let $S \subseteq \Re^k$ be piecewise linear. For sufficiently large $x \in \Re$, S is $x,x^2,\ldots$ shift invariant*.

Proof: Let $S = \{x \in \Re^k: \varphi\}$ where φ is a propositional combination of linear inequalities of the form $T(x_1,\ldots,x_k) > 0$, with various leading variables. By Lemma 2.2.1, for sufficiently large x, the truth values of each $T(x,x^2,\ldots,x^k) > 0$ are the same. It follows that for sufficiently large x, the truth values of the propositional combination are the same. Hence S is $x,x^2,\ldots$ shift invariant.

In Lemma 2.2.1, we can skip powers of x under T. Hence we have invariant* and not just invariant. QED

Note that with piecewise linear, we have shift invariance(*) with explicitly given numbers. Also we gain nothing by giving a multiple form with piecewise linear $S_1,\ldots,S_n \subseteq \Re^k$ as that trivially follows from the single set form Theorem 2.2.2.

2.3. SEMIALGEBRAIC

DEFINITION 2.3.1. A monomial is a product of nonzero powers of variables, with a nonzero coefficient in front, where the one or more variables have strictly increasing subscripts. Two monomials are unlike if and only if they differ after the nonzero constant coefficient in front is removed. For monomials M,M', we write M <* M' if and only if M,M' are unlike, and
i. the last variable in M has lower subscript than the last variable in M', or

ii. M,M' have the same last variable and the exponent of the last variable in M is less than the last variable in M'.
The leading monomial in a nonconstant polynomial (sum of zero or more unlike monomials and a constant) is the monomial that is greatest under <*.

LEMMA 2.3.1. Let $P(x_1,\ldots,x_k)$ be a polynomial with real coefficients. Let $x,c \in \Re$ be sufficiently large. If the sign of the leading monomial is positive then $P(x,x^c,x^{c\hat{}2},\ldots,x^{c\hat{}(k-1)}) > 0$. If negative then < 0. We do not even need these $x^{c\hat{}i}$ to be consecutive.

Proof: We compare M and any lesser M'. Let j be greatest such that x_j appears in P. There is a greater power of x_j in M, but perhaps greater powers of the $x_{j'}$, $j' < j$, in M'. All of these powers (exponents) are definite (tied to P) and so much less

than c. Hence M dominates |M'|. Same for any definite sum of these lesser M'. QED

THEOREM 2.3.2. Let $S \subseteq \Re^k$ be semialgebraic. S is $x,x^c,x^{c^2},\ldots$ shift invariant* for sufficiently large x,c.

Proof: Similar to the proof of Theorem 2.2.2. Let $S = \{x \in \Re^k: \varphi\}$ where φ is a propositional combination of polynomial inequalities $P(x_1,\ldots,x_k) > 0$. By Lemma 2.3.2, for sufficiently large x,c, the truth values of $P(x,x^c,x^{c^2},\ldots,x^{c^{(k-1)}}) > 0$ are the same. It follows that for sufficiently large r,c, S is $x,x^c,x^{c^2},\ldots$ shift invariant. Since we don't need consecutivity in Lemma 2.3.1, we have shift invariance*. QED

Note that with semialgebraic, we have shift invariance(*) with explicitly given numbers. Also, we gain no generalilty by giving a multiple form with semialgebraic $S_1,\ldots,S_n \subseteq \Re^k$.

3. LOWER SHIFT INVARIANCE

In lower shift invariance, we strengthen shift invariance by allowing lower parameters.

DEFINITION 3.1. Let $S \subseteq \Re^k$. S is $p;x_1,x_2,\ldots$ lower shift invariant if and only if
i. $0 < p \leq x_1 < x_2 < \ldots \in \Re$.
ii. For $y \in ([0,p) \cup \{x_1,x_2,\ldots\})^k$, membership of y in S remains the same when we replace each x_i by x_{i+1} in y.

DEFINITION 3.2. Let $S \subseteq \Re^k$. S is $p;x_1,\ldots,x_n$ lower shift invariant if and only if
i. $0 < p \leq x_1 < \ldots < x_n \in \Re$.
ii. For $y \in ([0,p) \cup \{x_1,\ldots,x_{n-1}\})^k$, membership of $y \in S$ remains the same when we replace each x_i by x_{i+1} in y.

DEFINITION 3.3. Let $S \subseteq \Re^k$. S is $p;x_1,x_2,\ldots$ lower shift invariant* if and only if for all infinite subsequences $y_1,y_2,\ldots$ of $x_1,x_2,\ldots$, S is $p;y_1,y_2,\ldots$ lower shift invariant.

DEFINITION 3.4. Let $S \subseteq \Re^k$. S is $p;x_1,\ldots,x_n$ lower shift invariant* if and only if for all subsequences $y_1,\ldots,y_{n'}$ of $x_1,\ldots,x_n$, S is $p;y_1,\ldots,y_{n'}$ lower shift invariant.

We are working on $Z,\Re$. In the above, the sets are in $\Re$ and the $p,x_1,x_2,\ldots$ are in $\Re$. This is the master case $\Re,\Re$. There are four possibilities using $Z,\Re$. For one of these, $Z,\Re$, we have lower shift invariance* trivially, and so we don't see it in sections 3.1 - 3.3:

THEOREM 3.1. Every $S_1,\ldots,S_n \subseteq Z^k$ is $x_1;x_2,\ldots$ lower shift invariant* for some common $x_1,x_2,\ldots$ in $\Re$.

Proof: Choose $x_1 < x_2 < \ldots \in \Re\backslash Z$. If we use some x_i then we get automatic non membership in S. Otherwise there is nothing to prove. QED

Substantial positive results are always stated with multiple sets (although not if this adds no generality). Some of the positive results are particularly strong using $\Re,Z$ and not just $\Re,\Re$.

For negative results, we often use $\Re,\Re$. Often we just have the negative result for $\Re,Z$ only. For each negative result we use just a single set. Z,Z is the purely discrete combinatorial case, and involves no semialgebraic geometry.

The cases $x_1;x_1,x_2,\ldots$ lower shift invariance(*) and $x_1;x_1,\ldots,x_n$ lower shift invariance(*) are given special attention. $x_1;x_1,x_2,\ldots$ lower shift invariance(*) and $x_1;x_1,\ldots,x_n$ lower shift invariance(*) also drive sections 6,7.

These same remarks apply equally well to the Interval Shift Invariance(*) of section 4. There the cases $(a,x_1);x_1,x_2,\ldots$ interval shift invariance(*) and $(a,x_1);x_1,\ldots,x_n$ interval shift invariance(*) are also given special attention.

3.1. ARBITRARY

THEOREM 3.1.1. Every $S_1,\ldots,S_n \subseteq \Re$ is $x_1;x_1,x_2,\ldots$ lower shift invariant* for some common $x_1,x_2,\ldots \in Z$.

Proof: Let $S_1,\ldots,S_n \subseteq \Re$. Choose $0 < x_1 < x_2 < \ldots \in Z$ such that for all $i \leq n$, S_i contains or is disjoint from $\{x_1,x_2,\ldots\}$. QED

DEFINITION 3.1.1. For $\alpha \in K$ and infinite $A \subseteq Z^+$, define $\alpha\langle A\rangle \in K$ as follows. $\alpha\langle A\rangle(i) = \alpha(A[i])$.

THEOREM 3.1.2. There is a low level Borel function $f:\Re \to \Re$ which is surjective on every J.

Proof: Write every real uniquely in the form $n + .a_1a_2...$ where $n \in Z$ and $a_1,a_2,... \in \{0,1\}$ and $a_1a_2...$ is not eventually 1. Let $W_1,W_2...$ be a partition of Z^+ into infinitely many infinite pieces.

Define $f:\Re \to \Re$ as follows. Let $x \in \Re$, and let $\alpha \in K$ be the corresponding $a_1a_2...$. Let $b_1,b_2,...$ in $\{0,1\}$ be such that $b_i = 1$ if $\alpha<W_i>$ is eventually 1; 0 if $\alpha<W_i>$ is not eventually 1. Define $f(x) = n + .b_1b_2...$. Clearly any $b_1b_2...$ can be so obtained from some $a_1a_2...$, getting the same $b_1b_2...$ no matter how we make a finite change in the given $a_1a_2,...$. QED

QUESTION: What can we say about the complexity of such $f:\Re \to \Re$ here?

THEOREM 3.1.3. Some $S \subseteq \Re^2$ is not $p;x_1,x_2$ lower shift invariant, for any p,x_1,x_2. This holds in every $\Re^k$, $k \geq 2$.

Proof: Let $S = \{(x,y) \in \Re^2: f(x) = y\}$, where f is given by Lemma 3.1.2. Let S be $p;x_1,x_2$ lower shift invariant. Let $f(q) = x_1$, where $0 \leq q < p$. Then $(q,x_1) \in S$ but $(q,x_2) \notin S$. The second claim is immediate. QED

THEOREM 3.1.4. Let $p \in Z^+$. Every $S_1,...,S_n \subseteq Z^k$ is $p;x_1,x_2,...$ lower shift invariant* for some p in Z^+ and common $x_1,x_2,... \in Z^+$.

Proof: Let $S_1,...,S_n \subseteq Z^k$ and $p \in Z^+$. We use Infinite Ramsey Theorem'. For each $i \leq p$ and choice of s-tuples from $\{0,...,\min(A)\}$, $s < k$, in various of k positions, we form the set of (k-s)-tuples for the remaining positions that put the k-tuple in S_i. This yields a large number of sets $T_1,...,T_b$ of various dimensions <k. (We can add dummy arguments to consolidate these dimensions). We also use the $S_1,...,S_n$ unmodified corresponding to the degenerate case s = 0. We now apply the Infinite Ramsey Theorem' to obtain the required $x_1 < x_2 < ...$. QED

$x_1;x_1,x_2,...$ lower shift invariance(*) and even $x_1;x_1,x_2$(*) lower shift invariance is particularly interesting and strong in other contexts. We focus on this in sections *6*,7.

3.2. PIECEWISE LINEAR

Theorem 3.1.1 takes care of dimension 1. For dimension 2 (and higher), we first treat $x_1;x_1,x_2,\ldots$.

THEOREM 3.2.1. $S = \{(x,y) \in \Re^2: y = x/2\}$ is not $x_1;x_1,x_2$ lower shift invariant for any x_1,x_2. $S = \{(y_1,\ldots,y_k) \in \Re^k: y_2 = y_1/2\}$ is not $x_1;x_1,x_2$ lower shift invariant. $S = \{(y_1,\ldots,y_k) \in Z^k: y_2 = y_1+1\}$ is not $x_1;x_1,x_2$ lower shift invariant, $x_1,x_2 \in Z$.

Proof: Let S be $p;x_1,x_2$ lower shift invariant. Then $(x_1/2,x_1) \in S$ but $(x_1/2,x_2) \notin S$. The remaining claims are proved the same way. QED

LEMMA 3.2.2. Let $T(x_1,\ldots,x_k,y_1,\ldots,y_n) = b_1x_1+\ldots+b_nx_n+c_1y_1+\ldots+c_my_m+d$ be given, $b_1,\ldots,b_n,c_1,\ldots,c_m,d \in \Re$, $c_m > 0$. Then for all sufficiently small $q_1,\ldots,q_n > 0 \in \Re$ and sufficiently large $r \in \Re$, $T(q_1,\ldots,q_n,r,r^2,\ldots,r^k) > 0$.

Proof: Left to the reader. QED

THEOREM 3.2.3. Let $S \subseteq \Re^k$ be piecewise linear. For sufficiently small reals $p > 0$ and sufficiently large $x \in \Re$, S is $p;x,x^2\ldots$ lower shift invariant*.

Proof: Let $S = \{x \in \Re^k: \varphi\}$ where φ is a propositional combination of inequalities of the form $T(x_1,\ldots,x_k) > 0$. By Lemma 3.2.2, for sufficiently small $p > 0$ and sufficiently large $x \in \Re$, the following holds. For any fixed $0 \leq q_1,\ldots,q_i < p$, the truth values of various inequalities $T(q_1,\ldots,q_i,x_1,x_2,\ldots,x_{k-i}) > 0$ in φ are the same. It follows that for sufficiently small reals $q > 0$ and sufficiently large $x \in \Re$, S is $q;x,x^2,\ldots$ lower shift invariant.

In Lemma 3.2.2, we can skip powers of r with the same reasoning. Hence we have lower shift invariant*. QED

3.3. SEMIALGEBRAIC

THEOREM 3.3.1. $S = \{(x,y) \in \Re^2: xy = 1\}$ is not $p;x_1,x_2,\ldots$ lower shift invariant for any p, and any $x_1,x_2\ldots \in Z$. $S = \{(y_1,\ldots,y_k) \in \Re^k: y_1y_2 = 1$ is not $p;x_1,x_2,\ldots$ lower shift invariant, for any p, and any $x_1,x_2,\ldots$ in Z.

Proof: Let S be $p;x_1,x_2,...$ lower shift invariant. Let $1/x_i < p$. Then $(1/x_i,x_i) \in S$ but $(1/x_i,x_{i+1}) \notin S$. The remaining claims are proved the same way. QED

The case $p;x_1,x_2,...$ where the x's are not required to be integers, and $p;x_1,...,x_n$ where the x's are required to be integers, are very different and interesting. First a counterexample in dimension 3.

LEMMA 3.3.2. $1/x - 1/y$ is surjective on any nonempty $(0,p)$. (Use default 0 where undefined). $\{(x,y,z) \in \Re^3: 1/x - 1/y = z\}$ is semialgebraic.

Proof: Over $(0,p)$, $1/x$ and $1/y$ can be any two sufficiently large real numbers. Therefore their difference can be any real number.

For the second claim, note that for all $x,y,z \in \Re^3$, $1/x - 1/y = z \leftrightarrow y-x = xyz \wedge xy \neq 0$. QED

THEOREM 3.3.3. $S = \{(x,y,z) \in \Re^3: 1/x - 1/y = z\}$ is not $p;x_1,x_2$ lower shift invariant for any p,x_1,x_2. $S = \{(y_1,...,y_k) \in \Re^k: 1/y_1 - 1/y_2 = y_3\}$ is not $p;x_1,x_2$ lower shift invariant for any p,x_1,x_2. S is semialgebraic.

Proof: Let S be $p;x_1,x_2$ lower shift invariant. By Lemma 3.3.2, let $1/x - 1/y = x_1$, $x,y \in [0,p)$. Then $(x,y,x_1) \in S$ but $(x,y,x_2) \notin S$. The remaining claims are proved in the same way. The last claim is from Lemma 3.3.2. QED

The interesting case is dimension 2. Here we will obtain $p;x_1,...,x_n$ shift invariance where $x_1,...,x_n \in Z$ and $p;x_1,x_2,...$. We make free use of basic semialgebraic geometry treated in logic as o-minimality over the ordered field of real numbers.

DEFINITION 3.3.1. $f:\Re \to \Re$ is simple on interval $J \subseteq \Re$ if and only if f is continuous, and strictly increasing or strictly decreasing or constant on J.

LEMMA 3.3.4. Let $f:\Re \to \Re$ be simple on interval $J \subseteq \Re$. Then f is simple on every subinterval $J' \subseteq J$.

Proof: Immediate. QED

LEMMA 3.3.5. Let $f_1,...,f_n:\Re \to \Re$ be semialgebraic. There exists $x_1 < ... < x_n$ such that each f_i is simple on $(-\infty,x_1),(x_n,\infty)$ and each (x_i,x_{i+1}), $1 \leq i < n$.

Proof: Standard from semialgebraic geometry. with one function, this is the Monotonicity Theorem on page 3 in [VD98]. It is easily extended to multiple functions. QED

LEMMA 3.3.6. Let $f:\Re \to \Re$ be semialgebraic. There exists $p > 0$ such that f is simple on $(0,p]$.

Proof: It is clear that for any $x_1 < ... < x_n \in \Re$, there exists q > 0 such that (0,q) is a subinterval of $(-\infty,x_1),(x_n,\infty)$, or some (x_i,x_{i+1}). Now set $p = q/2$. QED

LEMMA 3.3.7. Let $f:\Re \to \Re$ be semialgebraic. There exists $p > 0$ such that f is continuous on (0,p) and exactly one of the following holds.
i. f is constant on (0,p].
ii. f is strictly increasing on (0,p] with finite limit at 0.
iii. f is strictly increasing on (0,p] with limit $-\infty$ at 0.
iv. f is strictly decreasing on (0,p] with finite limit at 0.
v. f is strictly decreasing on (0,p] with limit ∞ at 0.

Proof: Use Lemma 3.3.6 to obtain $p > 0$ such that f is simple on (0,p]. QED

LEMMA 3.3.8. Let $f:\Re \to \Re$ be semialgebraic. For all m there exists $q > 0$ and $J \subseteq (0,\infty)$ of length m such that f omits J on [0,q]. We can require that the J all have the same left endpoints.

Proof: Let f be as given. Apply Lemma 3.3.7. We verify this for all of the five cases in Lemma 3.3.7. In cases i,ii,iii, f is bounded above on (0,p] by f(p) and so we are omitting $(|f(p)+1,|f(p)|+m)$ on (0,p]. In case iv, let the finite limit be b. we are omitting $(|b|+1,|b|+m)$ on (0,p]. Thus in cases i-iv, we just use $q = p$.

In case v, let m be given. let $0 < q < p$ be such that $f(q) > m+1$. Then f omits $(1,m+1)$ on [0,q], and we use common left endpoint 1. QED

LEMMA 3.3.9. Let $f_1,...,f_n:\Re \to \Re$ be semialgebraic and let m be given. There exists $q > 0$ and $J \subseteq [0,\infty)$ of length m such that

each f_i omits J on [0,q]. We can require that the J used for all f_i have the same left endpoints (even with different f_i).

Proof: Let $f_1,...,f_n$ be as given and m be given. By let $y_1,...,y_n$ be the common left endpoints used in Lemma 3.3.8 for $f_1,...,f_n$. We now use $\max(y_1,...,y_n)$ for the common left endpoint used across all f_i. Now apply Lemma 3.3.8 taking m there as $\max(q_1,...,q_n)+m$ here. We get $q_1,...,q_n > 0$ and $J_1,...,J_n$ of length $m+\max(q_1,...,q_n)$ with left endpoints $y_1,...,y_n$ such that each f_i omits J_i on $[0,q_i]$. Set $q = \min(q_1,...,q_n)$ and use $(\max(q_1,...,q_n),\max(q_1,...,q_n)+m)$ for the common J here, of length m. QED

LEMMA 3.3.10. Let S c $\Re^2$ be semialgebraic where each S_x is finite. Then there exists $n \geq 0$ such that each S_x has at most n elements.

Proof: Standard semialgebraic geometry. This is called the Finiteness Lemma page 46, in [VD98]. QED

DEFINITION 3.3.2. Let $E \subseteq \Re$ be semialgebraic. sing(E) is the set of reals x such that no open interval containing x is contained or disjoint from E. We view sing(E) as the set of singularities of E.

LEMMA 3.3.11. Let $E \subseteq \Re$ be semialgebraic. sing($\Re$) is finite.

Proof: Standard semialgebraic geometry. sing(E) is semialgebraic (semialgebraic geometry, quantifier elimination). Suppose sing(E) is infinite. Then sing(R) contains some J and J contains some J' which is contained or disjoint from E. This violates that $J' \subseteq$ sing(E). QED

LEMMA 3.3.12. Let $S \subseteq \Re^2$ be semialgebraic. There exists $p > 0$ and $n \geq 0$ such that for all $0 < q < p$, $|\text{sing}(S_q)| = n$.

Proof: Let $T = \{(x,y): y \in \text{sing}(S_x)\}$. By Lemma 3.3.10, every T_x is finite. By Lemmas 3.3.10 and 3.3.11, let $m \geq 0$ be such that every $|\text{sing}(S_x)| \leq m$. Now define $f:\Re \to \Re$ by $f(x) = |\text{sing}(S_x)|$. Then f is semialgebraic and by Lemma 3.3.5, f is constant on some nonempty (0,p). QED

LEMMA 3.3.13. Let $S_1,...,S_n \subseteq \Re^2$ be semialgebraic. For all m there exists $p > 0$ and an interval $J \subseteq (0,\infty)$ of length m that is disjoint from every $\text{sing}((S_i)_q)$, $0 < q < p$.

Proof: Let $S_1,...,S_n$ be as given. By Lemma 3.3.12, let $p > 0$ and $t_1,...,t_n$ be such that for all $0 < q < p$, $|sing((S_i)_q)| = t_i$. For $i \leq n$ and $j \leq \max(t_1,...,t_n)$, define $f_{ij}:(0,p) \rightarrow \Re$ by $f_{ij}(x)$ is the j-th least element of $sing((S_i)_x)$ if $j \leq t_i$; 0 otherwise. Then the f_{ij} are semialgebraic. Let m be given. By Lemma 3.3.9, let $p > 0$ be such that each f_{ij} omits some interval $J \subseteq (0,\infty)$ of length m, on $[0,p]$. QED

THEOREM 3.3.14. Every semialgebraic $S_1,...,S_r \subseteq \Re^2$ is $p;x_1,...,x_n$ lower shift invariant*, for some common $p,x_1,...,x_n$ where $x_1,...,x_n \in Z$.

Proof: Let $S_1,...,S_r$ be as given. We apply Lemma 3.3.13 to $S_1,...,S_r$ with m sufficiently large relative to this data and n.

We obtain $p > 0$ and $J \subseteq (0,\infty)$ of length m which omits each $\alpha((S_i)_q)$, $0 \leq q < p$. Because of the omitting, we see that the truth values

LEMMA 3.3.15. Let $f:\Re \rightarrow \Re$ be semialgebraic. For all m there exists $q > 0$ and $J \subseteq (0,\infty)$ of length m such that f omits J on $[0,q]$. We can require that the J all have the same left endpoints.

Proof: Let f be as given. Apply Lemma 3.3.7. We verify this for all of the five cases in Lemma 3.3.7. In cases i,ii,iii, f is bounded above on $(0,p]$ by $f(p)$ and so we are omitting $(|f(p)+1,|f(p)|+m)$ on $(0,p]$. In case iv, let the finite limit be b. we are omitting $(|b|+1,|b|+m)$ on $(0,p]$. Thus in cases i-iv, we just use $q = p$.

In case v, let m be given. let $0 < q < p$ be such that $f(q) > m+1$. Then f omits $(1,m+1)$ on $[0,q]$, and we use common left endpoint 1. QED

THEOREM 3.3.16. Every semialgebraic $S_1,...,S_n \subseteq \Re^2$ is $p;x_1,x_2,...$ lower shift invariant*, for some $p,x_1,x_2,...$.

Proof: Let $S_1,...,S_n$ be as given. Let $S_1',...,S_n'$ be given by $S_i'(x,y) \leftrightarrow S_i(y,x)$. We apply Lemma 3.3.14 to $S_1,...,S_n,S_1',...,S_r'$.

We obtain $p > 0$ and $J \subseteq (0,\infty)$ of length m which omits each $sing((S_i)_q)$, $sing((S_i')_q)$, $0 \leq q < p$. Because of the omitting, we see that the truth values of $S_i(q,j)$ and $S_i(j,q)$, depend on i,j and not on $0 \leq q < p$. Here we only need J of length m = 1, and choose $y_1 < y_2 < \ldots$ from J. This gives the desired $p;y_1,y_2,\ldots$ but only for pairs of arguments q,y_i and y_i,q. For pairs of arguments $q,q' < p$ there is nothing to prove.

But we don't have the required invariance for all pairs from $y_1,y_2,\ldots$. So we apply the Infinite Ramsey Theorem' for ordered pairs obtaining subsequence $x_1,x_2,\ldots$ of $y_1,y_2,\ldots$ so that each S_i is $p,x_1,x_2,\ldots$ lower shift invariant. QED

4. INTERVAL SHIFT INVARIANCE

DEFINITION 4.1. Let $S \subseteq \Re^k$. S is $J;x_1,x_2,\ldots$ interval shift invariant if and only if
i. $0 \leq \inf(J) < \sup(J) <= x_1 < x_2 < \ldots \in \Re$.
ii. For $y \in (J \cup \{x_1,x_2,\ldots\})^k$, membership of y in S remains the same when we replace each x_i by x_{i+1} in y.

DEFINITION 4.2. Let $S \subseteq \Re^k$. S is $J;x_1,\ldots x_n$ interval shift invariant if and only if
i. $0 \leq \inf(J) < \sup(J) \leq x_1 < x_2 < \ldots \in \Re$.
ii. For $y \in (J \cup \{x_1,\ldots,x_{n-1}\})^k$, membership of y in S remains the same when we replace each x_i by x_{i+1} in y.

DEFINITION 4.3. Let $S \subseteq \Re^k$. S is $J;x_1,x_2,\ldots$ interval shift invariant* if and only if for all infinite subsequences $y_1,y_2,\ldots$ of $x_1,x_2,\ldots$, S is $J;y_1,y_2,\ldots$ interval shift invariant.

DEFINITION 4.4. Let $S \subseteq \Re^k$. S is $J;x_1,\ldots,x_n$ interval shift invariant* if and only if for all subsequences $y_1,\ldots,y_{n'}$ of $x_1,\ldots,x_n$, S is $J;y_1,\ldots,y_{n'}$ interval shift invariant.

Obviously $p;x_1,x_2,\ldots$ lower shift invariant if and only if $[0,p);x_1,x_2,\ldots$ interval shift invariant, and $p;x_1,x_2,\ldots$ lower shift invariant* if and only if $[0,p);x_1,x_2,\ldots$ interval shift invariant*. Also $p;x_1,\ldots,x_n$ lower shift invariant implies $[0,p);x_1,\ldots,x_n$ interval shift invariant, and $p;x_1,x_2,\ldots$ lower shift invariant* implies $[0,p);x_1,x_2,\ldots$ interval shift invariant*. This avoids some duplication of work.

4.1. ARBITRARY

THEOREM 4.1.1. Every $S_1,...,S_n \subseteq \Re$ is $[0,x_1);x_1,x_2,...$ interval shift invariant* for some common $x_1,x_2,... \in Z$.

Proof: By Theorem 3.1.1. QED

THEOREM 4.1.2. Some $S \subseteq \Re^2$ has no interval shift invariant $J;x_1,x_2$, for any J,x_1,x_2. This holds in every $\Re^k$, $k \geq 2$. However, for $S \subseteq Z^k$, see Theorem 3.1.5.

Proof: See proof of Theorem 3.1.3 using Lemma 3.1.2. QED

4.2. PIECEWISE LINEAR

Theorem 4.1.1 takes care of dimension 1. For dimension 2 (and higher), we already have the affirmative Theorem 3.2.3 with the stronger lower shift invariance.

Interval shift invariant $(a,x_1);x_1,x_2,...$ and $(a,x_1);x_1,...,x_n$ is not yet covered in dimension 2. The case $a = 0$ is covered negatively by Theorem 3.2.1 very strongly. Both will be covered affirmatively in section 4.3 (for semialgebraic and not just piecewise linear).

THEOREM 4.2.1. $S = \{(x,y) \in \Re^2: y < x+(1/2)\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant for any a, and any $x_1,x_2 \in Z$. $S = \{(y_1,...,y_k) \in \Re^k: y_2 < y_1+(1/2)\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant for any a, and any $x_1,x_2 \in Z$. $S = \{(y_1,...,y_k) \in Z^k: y_2 = y_1+1\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant for any $a,x_1,x_2 \in Z$.

Proof: Let S be $(a,x_1);x_1,x_2$ interval shift invariant, $x_1,x_2 \in Z$. Then $(x_1+(1/2),x_1) \notin S$ but $(x_1+(1/2),x_2) \notin S$ because $x_1 < x_2$ are positive integers. The remaining claims are proved the same way. QED

THEOREM 4.2.2. $S = \{(x,y,z) \in \Re^3: x-y = y-z\}$ is not $[a,x_1);x_1,x_2$ interval shift invariant for any a,x_1,x_2. For $k \geq 3$, $S = \{(x_1,...,x_k) \in \Re^k, x_1-x_2 = x_2 -x_3\}$ is not $[a,x_1);x_1,x_2$ interval shift invariant for any a,x_1,x_2. For $k \geq 3$, $S = \{(x_1,...,x_k) \in Z^k: x_1 = x_2 = x_3-1\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant for any $a,x_1,x_2 \in Z$.

Proof: Suppose $S = \{(x_1,...,x_k) \in \Re^3: x_1-x_2 = x_2-x_3\}$ is $(a,x_1);x_1,x_2$ interval shift invariant. Then $a - (a+x_1)/2 =$

$(a+x_1)/2 - x_1$, but $a - (a+x_1)/2 \neq (a+x_1)/2 - x_2$. The third claim is proved in the same way. Suppose $S = \{(x_1,...,x_k) \in Z^k: x_1 = x_2 = x_3-1\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant, $a,x_1,x_2 \in Z$. Then $x_1-1 = x_1-1 = x_1-1$, but $x_1-1 = x_1-1 \neq x_2$. QED

4.3. SEMIALGEBRAIC IN DIM 2

We already have Theorems 3.3.15 and 3.3.16 for lower shift invariance, with some $p < x_1$. However, $(a,x_1);x_1,x_2$ interval shift invariance with $x_1,x_2 \in Z$ is impossible even for piecewise linear, by Theorem 4.2.1. We now drop the integer requirement and obtain affirmative results.

LEMMA 4.3.1. Let $f_1,...,f_n: \mathfrak{R} \to \mathfrak{R}$ be semialgebraic, where each $f_i(x) > x$. There exists $0 < a < b$ such that each f_i maps $[a,b]$ into $[b,\infty)$. There exists $0 < a < b < c$ such that each f_i maps $[a,b] \to [c,\infty)$.

Proof: Using Monotonicity, let $0 < a < d$ be such that each f_i is simple on $[a,d]$. Among the strictly increasing and constant f_i on $[a,d]$, let c be the least $f_i(a)$. If f_i is strictly decreasing on $[a,d]$ then $f_i(a) > d$. Let $b = \min(c,d)$. Let $a \leq x \leq b$. If f_i is strictly increasing or constant, then $x \in [a,d]$ and so $f_i(x) \geq f_i(a) \geq c \geq b$. If f_i is strictly decreasing, then $f_i(x) > d \geq b$. This establishes the first claim. The second follows immediately by reducing b. QED

THEOREM 4.3.2. Every semialgebraic $S_1,...,S_n \subseteq \mathfrak{R}^2$ is $(p,x_1);x_1,x_2,...$ interval shift invariant* for some common $p,x_1,x_2,...$.

Proof: Let $S_1,...,S_n$ be as given. Let $S_{n+1},...,S_{2n}$ be given by $S_{n+i}(x,y)$ iff $S_i(y,x)$. Let $f_1,...,f_{2n}: \mathfrak{R} \to \mathfrak{R}$ be given by $f_i(x)$ is the least element of $\text{sing}((S_i)_x)$ greater than x if it exists; x+1 otherwise. By Lemma 4.3.1, let $0 < a < b < c$ be such that each f_i maps $(a,b] \to [c,\infty)$. For all $x \in (a,b)$, the truth value of $(x,y) \in S_i(x,y),S_i(y,x)$ is constant for $y \in (x,c)$. By the Infinite Ramsey Theorem' for ordered pairs, choose $x_1 < x_2 < ...$ from (a,b) where for all $i \leq n$, order equivalent ordered pairs from $\{x_1,x_2,...\}$ all lie in or all lie outside S_i. Then $S_1,...,S_n$ is $(a,x_1);x_1,x_2,...$ interval shift invariant*. QED

4.4. SEMIALGEBRAIC $f:J^2 \to \mathfrak{R}$

A function is semialgebraic if and only if its graph is a semialgebraic set. It is sometimes convenient to display the left and right endpoints of nonempty finite open intervals in the reals with lower case letters, J = [(a,b)], a < b, with four endpoint possibilities. We write $J^{2<}, J^{2>}, J^{2=}$ for $\{(x,y) \in J^2: x < y\}, \{(x,y) \in J^2: x > y\}, \{(x,y) \in J^2: x = y\}$, respectively.

We separate out important information about semialgebraic $f:J^2 \to \Re$ which we use in section 4.5. There are new results here, which in a weak form reads: there exists $J' \subseteq J$ such that f is not surjective on J'^2. This is strongly false in dimensions ≥ 3 as we now see.

THEOREM 4.4.1. $1/(x-z) - 1/(y-z)$ maps any J^3 onto $\Re$. For all J, the function $f:J^3 \to \Re$, given by $f(x,y,z) = 1/(x-z) - 1/(y-z)$ if $x,y,z \in J \wedge x \neq z \wedge y \neq z$; 0 otherwise, is semialgebraic, and has range $\Re$.

Proof: Let J be given. Let z be in the interior of J. Then 1(x-z) and 1/(y-z) can each be any arbitrarily large real. Therefore their difference can be any real. The $f:J^2 \to \Re$ obviously has range $\Re$ by the first claim. The graph of the f is $\{(x,y,z,w) \in \Re^4: (1/(x-z) - 1/(y-z) = w \wedge x,y,z \in J \wedge x \neq z \wedge y \neq z) \vee ((x = z \vee y = z) \wedge w = 0)\}$. To see that this graph is semialgebraic, it suffices to prove that $\{(x,y,z,w) \in \Re^4: 1/(x-z) - 1/(y-z) = w\}$ is semialgebraic. But for all $x,y,z,w \in \Re$, $1/(x-z) - 1/(y-z) = w \leftrightarrow (y-z)-(x-z) = (x-z)(y-z)w \wedge x-z \neq 0 \wedge y-z \neq 0$. QED

LEMMA 4.4.2. Let $f:(a,b)^{2>} \to \Re$ be semialgebraic. For all $x \in (a,b)$ there exists $y \in (a,x)$ such that f_x is continuous on [y,x), and exactly one of the following holds.
1. f_x is constant on [y,x).
2. f_x is strictly increasing on [y,x) with finite limit at x.
3. f_x is strictly increasing on [y,x) with limit infinity at x.
4. f_x is strictly decreasing on [y,x) with finite limit at x.
5. f_x is strictly decreasing on [y,x) with limit -infinity at x.

Proof: From the Monotonicity Theorem Lemma 3.3.5, analogously to our proof of Lemma 3.3.7. QED

LEMMA 4.4.3. Let $f:(a,b)^{2>} \to \Re$ be semialgebraic. There exists a < c < b such that the following holds. For all $x \in (a,c)$ there exists $y \in (a,x)$ such that f_x is continuous on [y,x), and exactly one of the following holds independently of the choice of x.

1. f_x is constant on [y,x).
2. f_x is strictly increasing on [y,x) with finite limit at x.
3. f_x is strictly increasing on [y,x) with limit infinity at x.
4. f_x is strictly decreasing on [y,x) with finite limit at x.
5. f_x is strictly decreasing on [y,x) with limit -infinity at x.

Proof: From Lemma 4.4.2 we obtain the corresponding semialgebraic function from $(a,b) \rightarrow \{1,2,3,4,5\}$. This function must be constant near a. QED

Note that the choice of y in Lemma 4.4.3 may depend on x.

LEMMA 4.4.4. Let $E \subseteq \mathfrak{R}^2$ be semialgebraic. There exists semialgebraic $f: \mathfrak{R} \rightarrow \mathfrak{R}$ such that for all E(x,y) we have E(x,f(x)).

Proof: This is well known. See [VD91], p. 94. For each x in dom(E) let F(x) be the maximal subinterval (may have endpoints) contained in E_x which is closest to 0. If F(x) is bounded then take f(x) to be its midpoint. If $F(x) = \mathfrak{R}$ take f(x) = 0. If F(x) is (a,∞) or $[a,\infty)$, take f(x) = a+1. If F(x) is $(-\infty,a)$ or $(-\infty,a]$ take f(x) = a-1. F and therefore f is semialgebraic. QED

LEMMA 4.4.5. Let $f_1,\ldots,f_n:(a,b) \rightarrow \mathfrak{R}$ where each $f_i(x) < x$. There exists $a < c < d < b$ such that each f_i is $<c$ on [c,d].

Proof: Let $f_1,\ldots,f_n$ be as given. Let $a < c < d < b$ where each f_i is continuous and strictly increasing or strictly decreasing on [c,d]. Move d close to c. QED

LEMMA 4.4.6. Let $f:(a,b)^{2>} \rightarrow \mathfrak{R}$ be semialgebraic. There exists $a < c < d < b$ such that the following holds. For all $x \in (c,d)$, f_x is continuous on [c,x), and exactly one of the following holds independently of the choice of x.
1. f_x is constant on [c,x).
2. f_x is strictly increasing on [c,x) with finite limit at x.
3. f_x is strictly increasing on [c,x) with limit infinity at x.
4. f_x is strictly decreasing on [c,x) with finite limit at x.

5. f_x is strictly decreasing on [c,x) with limit -infinity at x.

Proof: We apply Lemma 4.4.3 to f obtaining $a < e < b$ such that for all $x \in (a,e)$ there exists $y \in (a,x)$ such that f_x is continuous on $[y,x)$ and $i \leq 5$ holds, i independently of the choice of x. The $y < x$ can be taken to be a semialgebraic function of the $x \in (a,e)$ by Lemma 4.4.4. Now apply Lemma 4.4.5 to obtain the required $a < c < d < e$. QED

LEMMA 4.4.7. Let $f_1,...,f_n:(a,b)^{2>} \to \Re$ be semialgebraic. There exists $a < c < d < b$ such that the following holds. For all $x \in (c,d)$ and $i \leq n$, $(f_i)_x$ is continuous on $[c,x)$, and exactly one of the following holds, depending on i, and independently of the choice of x.

1. $(f_i)_x$ is constant on $[c,x)$.

2. $(f_i)_x$ is strictly increasing on $[c,x)$ with finite limit at x.

3. $(f_i)_x$ is strictly increasing on $[c,x)$ with limit infinity at x.

4. $(f_i)_x$ is strictly decreasing on $[c,x)$ with finite limit at x.

5. $(f_i)_x$ is strictly decreasing on $[c,x)$ with limit -infinity at x.

Proof: Let $f_1,...,f_n$ be as given. Apply Lemma 4.4.6 to f_1 obtaining $a < c < d < b$. Then apply Lemma 4.4.6 to $f_2|(c,d)^{2>}$ obtaining $c < c' < d' < d$. Continue in this way through all of $f_1,...,f_n$. QED

LEMMA 4.4.8. Let $f_1,...,f_n:(a,b)^{2>} \to \Re$ be semialgebraic. There exists $e > 0$ such that for all m, each f_i omits $(e,e+m)$ on some common $J^{>2} \subseteq (a,b)^{>2}$.

Proof: Let $f_1,...,f_n$ be as given. Let $a < c < d < b$ be as given by Lemma 4.4.7. Let $g_1,...,g_i$ enumerate the f_j's which have clause 1,2 in Lemma 4.4.7. For each g_j we define the semialgebraic function $G_j(x)$ = the finite limit of $(g_j)_x$ on $[c,x)$ at x. Choose $J_1 \subseteq [c,d]$ so that all of the G_j are bounded above on J_1. Because of the increasing in 1,2, this provides an upper bound B_1 for $f_1,...,f_n$ on $J_1^{2>} = [a_1,b_1]^{2>}$.

Let $h_1,...,h_i$ enumerate the f_j's which have clause 4,5 in Lemma 4.4.7. (This i may be different from the i in the previous paragraph). Let $J_2 \subseteq [a_1,b_1]$ be such that the semialgebraic functions $H_j(x) = f_j(a_1,x)$ are bounded above by some B_2 on some $J_2 = (a_2,b_2) \subseteq J_1$. By the decreasing in 4,5, $h_1,...,h_i$ are bounded above by B_2 on $J_2^{2>}$.

Let $F_1,...,F_i$ enumerate the f_j's which have clause 3 in Lemma 4.4.7. (This i may be different from the i in the previous paragraphs). Fix m. Let $F^*(x) < x$ be such that each $F_j(x,F^*(x)) > m$, defined on $x \in J_2$, and semialgebraic. By Lemma 4.4.5 let $[a_3,b_3] \subseteq J_2$ be such that F^* is $< a_3$ on $[a_3,b_3]$. By increasing, $F_1,...,F_i$ are $> m$ on $[a_3,b_3]^{2>}$.

Set $e = |B_1|+|B_2|+1$. Let m be given. In the previous paragraph set the m there to be e+m here, and set $J = (a_3,b_3)$. Then $f_1,...,f_n$ omits (e,e+m) on $J^{2>}$. QED

THEOREM 4.4.9. Let $f_1,...,f_n:(a,b)^2 \to \Re$ be semialgebraic. There exists $e > 0$ such that for all m, each f_i omits (e,e+m) on some common $J \subseteq (a,b)^2$.

Proof: Let the f's be as given. Let $g_1,...,g_n:(a,b)^{2>} \to \Re$ be given by $g_i(x,y) = f_i(y,x)$. Let $h_i,...,h_n:(a,b)^{2>} \to \Re$ be given by $h_i(x,y) = f_i(x,x)$. Now apply Lemma 4.4.9 to $f_1,...,f_n,g_1,...,g_n,h_1,...,h_n:(a,b)^{2>} \to \Re$. Let $e > 0$ where for all m, each f_i,g_i,h_i omits (e,e+m) on some common $J \subseteq (a,b)^{>2}$. Hence for all m, each f_i omits (e,e+m) on some common $J \subseteq (a,b)^2$. QED

4.5. SEMIALGEBRAIC IN DIM ≥ 3

We already have Theorem 4.2.2 in dim 3, where we needed only piecewise linear and not semialgebraic.

We begin by using Theorem 4.4.1 to dispense with dimension 4 and greater.

THEOREM 4.5.1. $S = \{(x,y,z,w) \in \Re^4: w = 1/(x-z) - 1/(y-z)\}$ is not $J;x_1,x_2$ interval shift invariant, for any J,x_1,x_2. For $k \geq 4$, $S = \{(y_1,...,y_k) \in \Re^k: y_4 = 1/(y_1-y_3) - 1/(y_2-y_3)\}$ is not $J;x_1,x_2$ interval shift invariant, for any J,x_1,x_2, However, for $S \subseteq Z^k$, see Theorem 3.1.5.

Proof: Let S be $J;x_1,x_2$ interval shift invariant. By Theorem 4.4.1, let $x,y,z \in J$ where $x_1 = 1/(x-z) - 1/(y-z)$. Then $(x_1,x,y,z) \in S$ but $(x_2,x,y,z) \notin S$. The other claims are analogous. QED

So it remains to work exclusively in dimension 3.

LEMMA 4.5.2. Let $S_1,...,S_n \subseteq \Re^3$ be semialgebraic. For all m there exists p > 0 and an interval $J \subseteq (0,\infty)$ of length m that omits every $\text{sing}((S_i)_{ab})$, $0 < a,b < p$.

Proof: This is obtained from Theorem 4.4.11 the same way Lemma 3.3.13 was obtained from Lemma 3.3.9. QED

LEMMA 4.5.3. Let $A \subseteq (1,\infty)$ be infinite, $J \subseteq (0,1)$, $k \geq 2$, $\varepsilon > 0$, and F be a function from $A^{k<}$ into subintervals of J, of length ε. There exists infinite $B \subseteq A$ such that the intersection of the $F(x)$, $x \in B^k$, is an interval of length $\leq \varepsilon/2$.

Proof: Let A,ε,J,k,F be as given. We apply the Infinite Ramsey Theorem' to $A^{2k<}$. We obtain infinite $x_1 < x_2 < ...$ in A such that the following holds.

Let $i_1 < ... < i_{2k}$ and $F(x[i_1],...,x[i_k]) = (a,b)$ and $F(x[i_{k+1}],...,x[i_{2k}] = (c,d)$. Then exactly one of the following holds, independently of the choice of $i_1,...,i_{2k}$.

1. $(a,b) = (c,d)$.

2. $(a,b) \cap (c,d) = \emptyset$.

3. $a + \varepsilon/2 \leq c < b$
4. $a < c < a + \varepsilon/2$
5. $c + \varepsilon/2 \leq a < d$
6. $c < a < c + \varepsilon/2$

This strongly uses that $(a,b),(c,d)$ have length ε. With case 1 we see that F is constant so there is nothing to prove. With case 2, we get an infinite set of pairwise disjoint intervals of length ε in J which is impossible. With case 3, we get an infinite sequence of intervals where the left endpoints successively increase by at least ε, in J, which is impossible. With case 5, we get an infinite sequence of intervals where the left endpoints successively decrease by at least ε, in J, which is impossible.

We come to case 4. Let $F(x[1],...,x[k]) = (a,b)$. Let $k < i_1 < ... < i_k$. Then $F(x[i_1],...,x[i_k]) \supseteq \{x[1] + \varepsilon/2,b)$, and we can use $B = A\backslash\{x[1],...,x[k]\}$. Case 6 is handled analogously.

Now let $A \subseteq (1,\infty)$ be infinite, $J \subseteq (0,1)$, $k \geq 2$, $\varepsilon > 0$, and F be a function from A^k into subintervals of J of length ε. Let $\alpha_1,\ldots,\alpha_t$ list the order types of k-tuples from $\mathfrak{R}$. Thus F is the disjoint union of t functions in the obvious way, each of which can be rewritten as a function G_j from $\mathfrak{R}^{k<}$ where F_j and G_j have the same range. So we can apply the above to G_1 obtaining infinite $B_1 \subseteq A$ where the intersection contains an interval J_1 of length $\leq \varepsilon/2$. Then we can apply the above to G_2 obtaining infinite $B_2 \subseteq B_1$ where the intersection contains an interval J_2 of length $\leq \varepsilon/4$. We continue in this way through all of $G_1,\ldots,G_t$. QED

DEFINITION 4.5.1. In single parameter interval shift invariance, we use zero or one element from the J in the definition of interval shift invariance.

LEMMA 4.5.5. Multidimensional Finiteness. Let $S \subseteq \mathfrak{R}^n \times \mathfrak{R}^m$ be semialgebraic, where each S_x is finite. There exists t such that each $|S_x| \leq t$.

Proof: See [VD98], page 60, Corollary 3.7. QED

There exists t such that for all $x \in \mathfrak{R}^{k-1}$, $(S_1)_x,\ldots,(S_n)_x$ all have at most t singularities. I.e., each $|\text{sing}((S_i)_x)| \leq t$.

THEOREM 4.5.6. Let $A \subseteq [1,\infty)$ be infinite of order type ω. Every semialgebraic $S_1,\ldots,S_n \subseteq \mathfrak{R}^k$ is $J;x_1,x_2,\ldots$ single parameter interval shift invariant* for some J and some $x_1,x_2,\ldots \in A$.

Proof: Let $A,S_1,\ldots,S_n$ be as given. By Lemma 4.5.5, let t be such that for all $x \in \mathfrak{R}^{k-1}$, each $|\text{sing}((S_i)_x) \cap (0,1)| \leq t$. Let F map A^{k-1} into intervals where for $x \in E^{k-1}$, F(x) is the rightmost open subinterval of (0,1) of greatest length which avoids $\cup_{i\leq n}\text{sing}((S_i)_x)\cap(0,1)$. Clearly every F(x) has length $\geq 1/nt$. Now apply Lemma 4.5.3 to obtain $B \subseteq A$ with J there as $\cup_{i\leq n}\text{sing}((S_i)_x)\cap(0,1)$. Set J here to be J there and set $x_1,x_2,\ldots$ to enumerate the elements of B. QED

THEOREM 4.5.7. Every semialgebraic $S_1,\ldots,S_n \subseteq \mathfrak{R}^3$ is $J;x_1,x_2\ldots$ interval shift invariant* for some common $J;x_1,x_2,\ldots$. For all t, we can alternatively use $x_1,\ldots,x_t \in Z$.

Proof: Let $S_1,\ldots,S_n$ be as given. For the first claim, we first apply Theorem 4.4.9 as follows. The three cases with two

parameters are indicated by $(S_i)_{aby}$, $(S_i)_{ayb}$, $(S_i)_{yab}$, $a,b \in J$. We use 3n semialgebraic functions $f_{i1},f_{i2},f_{i3}:J^2 \to \mathfrak{R}$ where $f_{i1}(a,b),f_{i2}(a,b),f_{i3}(a,b)$ is the greatest singularity of the 1 dimensional set given by S_i with ab fixed in the three ways. (If there is no singularity then use 0 as default). We get a fixed left endpoint e such that for all m, we can choose $J' \subseteq J$ so that these 3n semialgebraic functions omit (e,e+m). For the first claim we now choose $y_1 < y_2 < \ldots$ from (e,e+1) and apply Theorem 4.4.12 to take care of the single parameter interval shift invariance*, obtaining subsequence $x_1,x_2,\ldots$ of $y_1,y_2,\ldots$.

For the second claim, we first apply Theorem 4.5.6 with $A = Z^+$. We take care of the (at most) single parameter part with infinite $B \subseteq A$. Then we apply Theorem 4.4.9 to take care of the two parameter part with left endpoint e. Here we choose m so that $|B \cap (e,e+m)| = t$. QED

5. IN SUBFIELDS OF $\mathfrak{R}$

We now look at sections 2-4 for subfields R of $\mathfrak{R}$. Throughout this section we fix one subfield R of $\mathfrak{R}$. The intervals in R are the sets $[(a,b)] \cap R$, where $a < b$ are from R. The piecewise linear and semialgebraic subsets of the R^k are defined as for $\mathfrak{R}^k$ except that we require that the coefficients in the linear functions and polynomials be elements of R. Then all of the theorems of sections 2-4 have a clear meaning for R except for Theorem 2.3.2 where we need to avoid exponentiation, and Theorem 3.1.2 where we avoid Borel.

Most of the theorems of sections 2-4 are proved with almost no additional work. With few exceptions, the positive results for R are proved from the original positive version for $\mathfrak{R}$ by taking the given $S_1,\ldots,S_n \subseteq R^k$ and using "liftings" $S_1^*,\ldots,S_n^* \subseteq \mathfrak{R}^k$ obtained by using the same linear functions and polynomials with coefficients from R. We do not claim that these liftings from R to $\mathfrak{R}$ are unique. Fortunately, this does not cause us any difficulties.

We do however encounter a difficulty with Theorem 4.4.9' where we encounter the lifting of the graph of a semialgebraic $f:R^k \to R$. The problem is that this might not be the graph of a semialgebraic $g:\mathfrak{R}^k \to \mathfrak{R}$. To address this problem we add the hypothesis that R is real closed. This is the only place in this section where we need anything more than R being a field.

In shift invariance, lower shift invariance, and interval shift invariance, we require that $x_1,x_2,... \in R$, $p \in R$, and J is an interval in R.

THEOREM 2.1.1.' Every $S_1,...,S_n \subseteq R^k$ is $x_1,x_2,...$ shift invariant* for some common $x_1 < x_2 < ... \in Z$.

THEOREM 2.1.2.' Every $S_1,...,S_r \subseteq R^k$ is $x_1,...,x_n$ shift invariant* for some common $x_1 < ... < x_n \in Z$.

THEOREM 2.2.2.' Let $S \subseteq R^k$ be piecewise linear. For sufficiently large $x \in R$, S is $x,x^2,...$ shift invariant*.

THEOREM 2.3.2.' Let $S \subseteq R^k$ be semialgebraic. S is $x,x^c,x^{c^2},...$ shift invariant* for sufficiently large x,c with $x \in R$ and $c \in Z$.

THEOREM 3.1.1.' Every $S_1,...,S_n \subseteq R$ is $x_1;x_1,x_2,...$ lower shift invariant* for some common $x_1,x_2,... \in Z$.

THEOREM 3.1.2.' There is a function $f:R \to R$ which is surjective on every J. In fact, this is true for any infinite linear ordering R whose nontrivial intervals have the same cardinality as R.

Proof: R obviously has the cardinality condition. Now let R merely have the cardinality condition. Let x_α, $\alpha < |R|$, be a transfinite enumeration of R without repetition. Let J_α, $\alpha < |R|$, be a transfinite enumeration of the nonempty open intervals in R with endpoints in R, where every J is repeated |R| times. We now construct $f:R \to R$ in |R| stages. Suppose we have defined f at various elements of R, in stages $\beta < \alpha$. We now define f at a certain element of R. We let y be the first element of J_α for which we have not yet defined f(y). Define f(y) to be the first $z \in R$ not yet a value of f at any element of J_α. QED

THEOREM 3.1.3.' Some $S \subseteq R^2$ is not $p;x_1,x_2$ lower shift invariant, for any p,x_1,x_2. This holds in every R^k, $k \geq 2$.

Proof: Use Theorem 3.1.2' in the same way that the proof of Theorem 3.1.3 uses Theorem 3.1.2. QED

THEOREM 3.2.1.' $S = \{(x,y) \in R^2: y = x/2\}$ is not $x_1;x_1,x_2$ lower shift invariant for any x_1,x_2. $S = \{(y_1,...,y_k) \in R^k: y_2 = y_1/2\}$ is not $x_1;x_1,x_2$ lower shift invariant.

THEOREM 3.2.3.' Let $S \subseteq R^k$ be piecewise linear. For sufficiently small $p > 0$ in R and sufficiently large $x \in R$, S is $p;x,x^2...$ lower shift invariant*.

THEOREM 3.3.1.' $S = \{(x,y) \in R^2: y = 1/x\}$ is not $p;x_1,x_2,...$ lower shift invariant for any p, and any $x_1,x_2... \in Z$. $S = \{(y_1,...,y_k) \in R^k: y_2 = 1/y_1\}$ is not $p;x_1,x_2,...$ lower shift invariant, for any p, and any $x_1,x_2,... \in Z$.

THEOREM 3.3.3.' $S = \{(x,y,z) \in R^3: 1/x - 1/y = z\}$ is not $p;x_1,x_2$ lower shift invariant for any p,x_1,x_2. $S = \{(y_1,...,y_k) \in R^k: 1/y_1 - 1/y_2 = y_3\}$ is not $p;x_1,x_2$ lower shift invariant for any p,x_1,x_2.

THEOREM 3.3.14.' Every semialgebraic $S_1,...,S_r \subseteq R^2$ is $p;x_1,...,x_n$ lower shift invariant*, for some common $p,x_1,...,x_n$ where $x_1,...,x_n \in Z$.

THEOREM 3.3.16.' Every semialgebraic $S_1,...,S_r \subseteq R^2$ is $p;x_1,x_2,...$ lower shift invariant*, for some $p,x_1,x_2,...$.

Proof: Let $S_1,...,S_r \subseteq R^2$ be semialgebraic. Let $S_1^*,...,S_r^* \subseteq \mathfrak{R}^2$ be respective liftings. We follow the proof of Theorem 3.3.16, obtaining $p > 0$ and $J \subseteq (0,\infty)$, and_until we arrive at "choose $y_1 < y_2 < ...$ from J". We instead choose $y_1 < y_2 < ... \in J \cap R$. We continue as in the proof of Theorem 3.3.16 obtaining $x_1,x_2,... \in \mathfrak{R}$. But now we know that $x_1,x_2,... \in R$. QED

THEOREM 4.1.1.' Every $S_1,...,S_n \subseteq R$ is $[0,x_1);x_1,x_2,...$ interval shift invariant* for some common $x_1,x_2,... \in Z$.

THEOREM 4.1.2.' Some $S \subseteq R^2$ has no interval shift invariant $J;x_1,x_2$, for any J,x_1,x_2. This holds in every R^k, $k \geq 2$. However, for $S \subseteq Z^k$, see Theorem 3.1.5.

Proof: Use Lemma 3.1.2'. QED

THEOREM 4.2.1.' $S = \{(x,y) \in R^2: y < x+(1/2)\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant for any a, and any $x_1,x_2 \in Z$. S =

$\{(y_1,...,y_k) \in R^k: y_2 < y_1+(1/2)\}$ is not $(a,x_1);x_1,x_2$ interval shift invariant for any a, and any $x_1,x_2 \in Z$.

THEOREM 4.2.2.' $S = \{(x,y,z) \in R^3: x-y = y-z\}$ is not $[a,x_1);x_1,x_2$ interval shift invariant for any a,x_1,x_2. For $k \geq 3$, $S = \{(x_1,...,x_k) \in R^k, x_1-x_2 = x_2 - x_3\}$ is not $[a,x_1);x_1,x_2$ interval shift invariant for any a,x_1,x_2.

THEOREM 4.3.2.' Every semialgebraic $S_1,...,S_n \subseteq R^2$ is $(p,x_1);x_1,x_2,...$ interval shift invariant* for some common $p,x_1,x_2,...$.

Proof: See the proof of Theorem 3.3.16'. QED

THEOREM 4.4.1.' $1/(x-z) - 1/(y-z)$ maps any J^3 onto R, J an interval in R. For all intervals J in R, the function $f:J^3 \to R$, given by $f(x,y,z) = 1/(x-z) - 1/(y-z)$ if $x,y,z \in J \wedge x \neq z \wedge y \neq z$; 0 otherwise, is semialgebraic, and has range R.

THEOREM 4.4.9.' Assume R is real closed. Let $f_1,...,f_n:(a,b)^2 \to R$ be semialgebraic. There exists $e > 0$ such that for all m, each f_i omits $(e,e+m)$ on some common $J \subseteq (a,b)^2$.

Proof: Let R be real closed, and let $f_1,...,f_n$ be as given. Then $graph(f_1),...,graph(f_n) \subseteq R^3$ are semialgebraic. Let $graph(f_1)^*,...,graph(f_n)^*$ c $\Re^3$ be respective liftings. Since R is an elementary substructure of $\Re$ (as ordered fields), $graph(f_1)^*,...,graph(f_n)^*$ are graphs of semialgebraic functions from $(a,b)^2 \to \Re$. Now apply Theorem 4.4.9. QED

THEOREM 4.5.1.' $S = \{(x,y,z,w) \in R^4: w = 1/(x-z) - 1/(y-z)\}$ is not $J;x_1,x_2$ interval shift invariant, for any J,x_1,x_2. For $k \geq 4$, $S = \{(y_1,...,y_k) \in R^k: y_4 = 1/(y_1-y_3) - 1/(y_2-y_3)\}$ is not $J;x_1,x_2$ interval shift invariant, for any J,x_1,x_2,

THEOREM 4.5.6.' Let $A \subseteq [1,\infty)$ be infinite of order type omega. Every semialgebraic $S_1,...,S_n \subseteq R^k$ is $J;x_1,x_2,...$ single parameter interval shift invariant* for some J and some $x_1,x_2,... \in A$.

Proof: Let A and $S_1,...,S_n$ be as given. Let $S_1^*,...,S_n^* \subseteq \Re^k$ be respective liftings. By Theorem 4.5.6, let $S_1^*,...,S_n^*$ be $J;x_1,x_2,...$ single parameter interval shift invariant* where $x_1,x_2,... \in A$. Then $S_1,...,S_n$ is $J\cap\Re;x_1,x_2,...$ single parameter interval shift invariant*. QED

THEOREM 4.5.7.' Every semialgebraic $S_1,...,S_n \subseteq R^3$ is $J;x_1,x_2...$ interval shift invariant* for some common $J;x_1,x_2,...$ for some $J,x_1,x_2...$. For all t, we can alternatively use some $x_1,...,x_t \in Z$.

Proof: Let $S_1,...,S_n$ c R^3 be $J;x_1,x_2,...$ interval shift invariant. Let $S_1^*,...,S_n^* \subseteq \mathfrak{R}^3$ by (a,b);x1,x2,... interval shift invariant. We follow the proof of Theorem 4.5.7 till we get to "choose $y_1 < y_2 < ...$ from (e,e+1)", and instead use "choose $y_1 < y_2 < ...$ from (e,e+1) $\cap$ R".

Let t be given. Use the second claim of Theorem 4.5.7 to obtain that $S_1^*,...,S_n^*$ is J;x1,...,xt for some J and $x_1,...,x_t \in Z$. QED

6. IN ORDINALS

In the context of ordinals, $<x_1;x_1,...,x_n$ and $<x_1,x_1,x_2,...$ lower shift invariance(*) are equivalent to fundamental properties in large cardinal theory. Specifically in and around the Stationary Ramsey Property hierarchy, referred to as SRP. See Appendix A regarding the SRP hierarchy of large cardinals. Here is some background information before we present Definition 4.5 and Theorem 4.2 below.

Minimalist versions of the SRP hierarchy were introduced and treated in [Fr01], based on regressive functions on ordinals. A novelty there was the avoidance of using closed unbounded sets of ordinals. The following are the most relevant forms of "k-subtlety" taken from [Fr01]. The first version, k-subtle, is the primary form used in the literature. It uses regressive functions into sets. The second version uses regressive functions into ordinals, which is less powerful. The third version removes the closed unbounded set C. The fourth version just uses the shift, which is what our lower shift invariance is based on.

Definitions 6.1-6.4 and Theorem 6.1 are taken from Appendix A, where supporting definitions can be found.

DEFINITION 6.1. Let $k \geq 1$. Limit ordinal λ is k-subtle if and only if
i) λ is a limit ordinal;
ii) For all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \rightarrow S(\lambda)$, there exists an f-homogenous $A \in S_{k+1}(C)$.

It is easily seen that λ must be a cardinal, and therefore we always write k-subtle cardinal. 0-subtle is taken to be regular and uncountable, the least of which is ω_1.

DEFINITION 6.2. Let $k \geq 1$. λ is k-subtle' if and only if

i) λ is a limit ordinal;
ii) For all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \to \lambda$, there exists $E \in S_{k+1}(C)$ such that f is constant on $S_k(E)$.

It is easily seen that again lambda must be a cardinal.

DEFINITION 6.3. α is purely k-subtle if and only if
i) α is an ordinal;
ii) For all regressive $f:S_k(\alpha) \to \alpha$, there exists $A \in S_{k+1}(\alpha\backslash\{0,1\})$ such that f is constant on $S_k(A)$.

DEFINITION 6.4. α is k-large if and only if
i) α is an ordinal;
ii) For all regressive $f: \alpha^k \to \alpha$, there exists $1 < \beta_1 < \ldots < \beta_{k+1}$ such that $f(\beta_1,\ldots,\beta_k) = f(\beta_2,\ldots,\beta_{k+1})$.

Note that purely k-subtle and k-large are closed upwards, and so are not even necessarily limit ordinals.

THEOREM 6.1. (ZFC) Let $k \geq 2$. The least (k-1)-subtle cardinal, k-subtle′ cardinal, purely k-subtle ordinal, k-large ordinal are the same (if any of the four exists). There exists a (k-1)-subtle cardinal if and only if there exists a k-subtle′ cardinal if and only if there exists a purely k-subtle ordinal if and only if there exists a k-large ordinal.

Proof: From Corollary 2.17, [Fr01]. QED

Lower Shift Invariance (LSI) takes the following form on ordinals.

DEFINITION 6.5. α has LSI[k,∞] if and only if for all $S \subseteq \alpha^k$ there exists $\beta_1,\beta_2,\ldots$ such that S is $\beta_1;\beta_1,\beta_2,\ldots$ lower shift invariant. α has LSI[k,n] if and only if for all $S \to \alpha^k$ there exists $\beta_1,\ldots,\beta_n$ such that S is $\beta_1;\beta_1,\ldots,\beta_n$ lower shift

invariant. With LSI[k,∞]*,LSI[k,n]*, we can skip terms (the β's).

LEMMA 6.2. (ZFC) Let $k \geq 3$. If λ is (k-2)-subtle then λ has LSI[k,∞]*.

Proof: Let λ,k be as given. By [Fr01], Lemma 1.6 (see Appendix A),

**1) for all closed unbounded $C \subseteq \lambda$ and regressive $f:S_{k-2}(\lambda) \to S(\lambda)$,
there exists an f-homogenous $A \subseteq C$ of type ω**

This implies

**2) For all closed unbounded $C \subseteq \lambda$ and regressive $f:S_{k-1}(\lambda) \to \lambda$,
there exists an f-homogenous $A \subseteq C$ of type ω**

This is proved in the same way that Theorem 4.1 of [Fr01] is proved in [Fr01] - that k-subtle implies (equivalent to) (k+1)-subtle'.

For LSI[k,∞], let $S \subseteq \lambda^k$. Let C be the closed unbounded set of infinite cardinals < λ. Make the obvious finitely many regressive functions on $S_{k-1}(C)$ into λ arising from each order type of (k-1)-tuples from C, and use standard finite sequence coding of values to create a single regressive function $f:S_{k-1}(C) \to \lambda$. Apply 2) to obtain f-homogenous $\{\beta_1 < \beta_2 < \ldots\}$. Then S is $\beta_1;\beta_1,\beta_2,\ldots$ lower shift invariant. QED

LEMMA 6.3. (ZFC) Let $k \geq 3$. If λ has LSI[k,k+1] then λ is (k-2)-subtle.

Proof: Let k,λ be as given. By Theorem 4.1, if λ is (k-1)-large then λ is (k-2)-subtle. Hence it suffices to show that λ is (k-1)-large.

Let $f:\lambda^{k-1} \to \lambda$ be regressive. Let $S \subseteq \lambda^k$ be the graph of f. Then S is $\beta_1;\beta_1,\ldots,\beta_k$ lower shift invariance. Then for all $\gamma < \beta_1$, $(\beta_1,\ldots,\beta_{k-1},\gamma) \in S \leftrightarrow (\beta_2,\ldots,\beta_k,\gamma) \in S$. Hence $f(\beta_1,\ldots,\beta_k) = g(\beta_2,\ldots,\beta_{k+1})$. QED

THEOREM 6.4. (ZFC) Let $k \geq 3$. The least α with LSI[k,k+1], LSI[k,∞]* are both the same as the least (k-2)-subtle cardinal.

The existence of α with LSI[k,k+1], LSI[k,∞]* are each equivalent to the existence of a (k-2)-subtle cardinal.

Proof: From Lemmas 6.2 and 6.3. QED

7. IN LINEAR ORDERINGS

We extend lower shift invariance to linear orderings as follows.

DEFINITION 7.1. Let (X,<) be a linear ordering without endpoints. (X,<) has LSI[k,∞] if and only if for all $S \subseteq X^k$ there exists $x_1,x_2,\ldots$ such that S is $x_1;x_1,\ldots$ lower shift invariant. X has LSI[k,n] if and only if for all $S \subseteq X^k$ there exists $x_1,\ldots,x_n$ such that S is $x_1;x_1,\ldots,x_n$ lower shift invariant. With LSI[k,∞]*,LSI[k,n]*, we can skip terms (the x's).

In [Fr01] we investigated these same kind of combinatorial properties not only for ordinals as discussed in section 4, but also for linear orderings without endpoints. Specifically we made the following definition.

DEFINITION 7.2. We say that a linear ordering (X,<) is k-critical if and only if it has no endpoints, and: for all regressive $f:X^k \to X$, there exists $b_1 < \ldots < b_{k+1}$ such that $f(b_1,\ldots,b_k) = f(b_2,\ldots,b_{k+1})$.

THEOREM 7.1. (ZFC) Let $k \geq 2$. The least cardinality of a k-critical linear ordering is the least (k-1)-subtle cardinal.

Proof: From Corollary 2.17 of [Fr01]. QED

THEOREM 7.2. (ZFC) Let $k \geq 3$.The least cardinality of an (X,<) with LSI[k,k+1], LSI[k,∞]* are both the same as the least (k-2)-subtle cardinal. The existence of (X,<) with LSI[k,k+1], LSI[k,∞]* are each equivalent to the existence of a (k-2)-subtle cardinal.

Proof: It is easy to see that -omega + λ has LSI[k,∞]* if λ is a (k-2)-subtle cardinal using Theorem 4.4. Now suppose (X,<) has LSI[k,k+1]. Let $f:X^{k-1} \to X$ be regressive. Apply LSI[k,k+1] to the graph of f to see that (X,<) is (k-1)-critical. By Theorem 5.1, there is a (k-2)-subtle cardinal. QED

In our Tangible Incompleteness, we work on the rational interval Q[0,n]. We use the following case of lower shift invariance:

$S \subseteq Q[0,n]^k$ is 1;1,...,n lower shift invariant

Of course not every $S \subseteq Q[0,n]^k$ is so invariant. But in Tangible Incompleteness we prove using large cardinals that we can always find such an $S \subseteq Q[0,n]^k$ obeying certain properties.

APPENDIX A
THE STATIONARY RAMSEY PROPERTY

All results in this section are taken from [Fr01]. All of these results, with the exception of Theorem 9.1.1, iv $\leftrightarrow$ v $\rightarrow$ vi, are credited in [Fr01] to James Baumgartner. Below, λ always denotes a limit ordinal.

DEFINITION A.1. We say that $C \subseteq \lambda$ is unbounded if and only for all $\alpha < \lambda$ there exists $\beta \in C$ such that $\beta \geq \alpha$.

DEFINITION A.2. We say that $C \subseteq \lambda$ is closed if and only if for all limit ordinals $x < \lambda$, if the sup of the elements of C below x is x, then $x \in C$.

DEFINITION A.3. We say that $A \subseteq \lambda$ is stationary if and only if it intersects every closed unbounded subset of λ.

DEFINITION A.4. For sets A, let S(A) be the set of all subsets of A. For integers $k \geq 1$, let $S_k(A)$ be the set of all k element subsets of A.

DEFINITION A.5. Let $k \geq 1$. We say that λ has the k-SRP if and only if for every $f:S_k(\lambda) \rightarrow \{0,1\}$, there exists a stationary $E \subseteq \lambda$ such that f is constant on $S_k(E)$. Here SRP stands for "stationary Ramsey property."

The k-SRP is a particularly simple large cardinal property. To put it in perspective, the existence of an ordinal with the 2-SRP is stronger than the existence of higher order indescribable cardinals, which is stronger than the existence of weakly compact cardinals, which is stronger than the existence of cardinals which are, for all k, strongly k-Mahlo (see Theorem A.1 below, and [Fr01], Lemma 1.11).

Our main results are stated in terms of the stationary Ramsey property. In particular, we use the following extensions of ZFC based on the SRP.

DEFINITION A.6. SRP^+ = ZFC + "for all k there exists an ordinal with the k-SRP". SRP = ZFC + {there exists an ordinal with the k-SRP}$_k$. We also use SRP[k] for the formal system ZFC + $(\exists\lambda)(\lambda$ has the k-SRP).

For technical reasons, we will need to consider some large cardinal properties that rely on regressive functions.

DEFINITION A.7. $f:S_k(\lambda) \to \lambda$ is regressive if and only if for all $A \in S_k(\lambda)$, if $\min(A) > 0$ then $f(A) < \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, $f(B) = f(C)$.

DEFINITION A.8. We say that $f:S_k(\lambda) \to S(\lambda)$ is regressive if and only if for all $A \in S_k(\lambda)$, $f(A) \subseteq \min(A)$. (We take $\min(\emptyset) = 0$, and so $f(\emptyset) = \emptyset$). We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, we have $f(B) \cap \min(B \cup C) = f(C) \cap \min(B \cup C)$.

DEFINITION A.9. Let $k \geq 1$. We say that α is purely k-subtle if and only if

i) α is an ordinal;

ii) For all regressive $f:S_k(\alpha) \to \alpha$, there exists $A \in S_{k+1}(\alpha\backslash\{0,1\})$ such that f is constant on $S_k(A)$.

DEFINITION A.10. We say that λ is k-subtle if and only if for all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \in S_{k+1}(C)$.

DEFINITION A.11. We say that λ is k-almost ineffable if and only if for all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \subseteq \lambda$ of cardinality λ.

DEFINITION A.12. We say that λ is k-ineffable if and only if for all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous stationary $A \subseteq \lambda$.

THEOREM A.1. Let $k \geq 2$. Each of the following implies the next, over ZFC.
i. there exists an ordinal with the k-SRP.
ii. there exists a (k-1)-ineffable ordinal.
iii. there exists a (k-1)-almost ineffable ordinal.
iv. there exists a (k-1)-subtle ordinal.
v. there exists a purely k-subtle ordinal.
vi. there exists an ordinal with the (k-1)-SRP.
Furthermore, i,ii are equivalent, and iv,v are equivalent. There are no other equivalences. ZFC proves that the least ordinal with properties i - vi (whichever exist) form a decreasing (≥) sequence of uncountable cardinals, with equality between i,ii, equality between iv,v, and strict inequality for the remaining consecutive pairs.

Proof: i ↔ ii is from [Fr01], Theorem 1.28, iv ↔ v is from [Fr01], Corollary 2.17. The strict implications ii → iii → iv → vi are from [Fr01], Theorem 1.28. Same references apply for comparing the least ordinals. QED

There is an important additional fact about k-subtle that is proved in [Fr01], Lemma 1.6.

THEOREM A.2. Let λ be k-subtle and $f:S_k(\lambda) \rightarrow S(\lambda)$ be regressive, where $C \subseteq \lambda$ is closed unbounded. There exists f-homogenous $E \subseteq C$ of every cardinality $< \lambda$.

DEFINITION A.13. We follow the convention that for integers $p \leq 0$, a p-subtle, p-almost ineffable, p-ineffable ordinal is a limit ordinal, and that the ordinals that are 0-subtle, 0-almost ineffable, 0-ineffable, or have the 0-SRP, are exactly the limit ordinals. An ordinal is called subtle, almost ineffable, ineffable, if and only if it is 1-subtle, 1-almost ineffable, 1-ineffable, respectively.

APPENDIX B
ASSOCIATED FORMAL SYSTEMS

SRP[k] ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), for fixed k. Appendix A.

SRP ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. Appendix A.

SRP^+ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). Appendix A.

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