The Heisenberg group

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November 30, 2021

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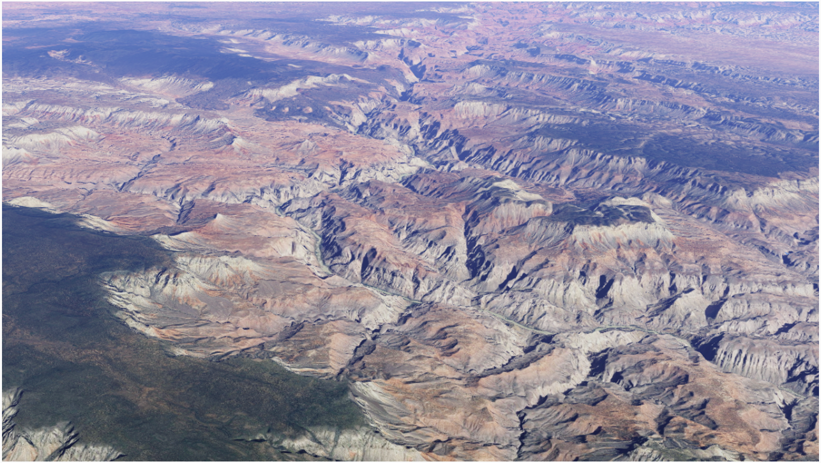

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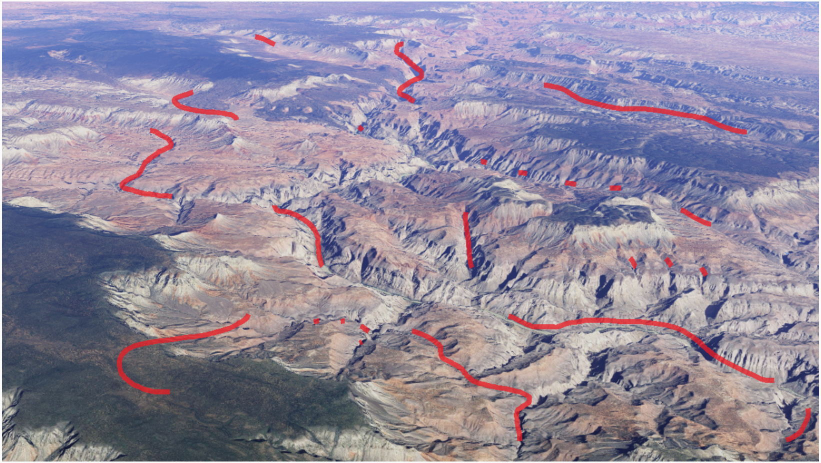

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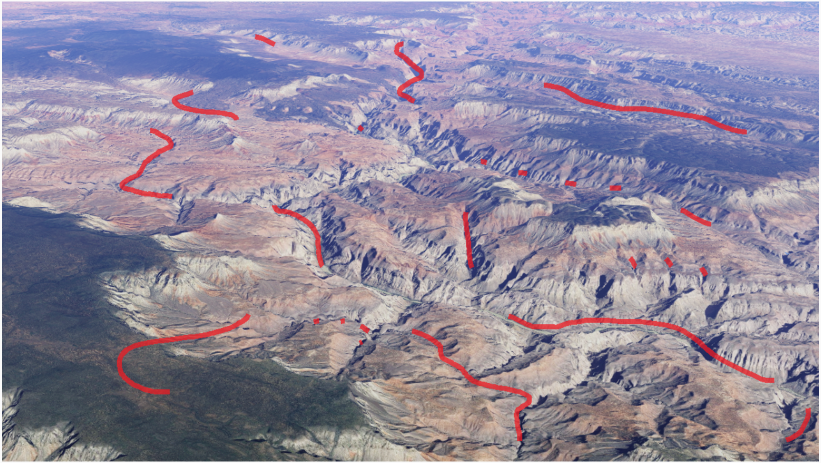

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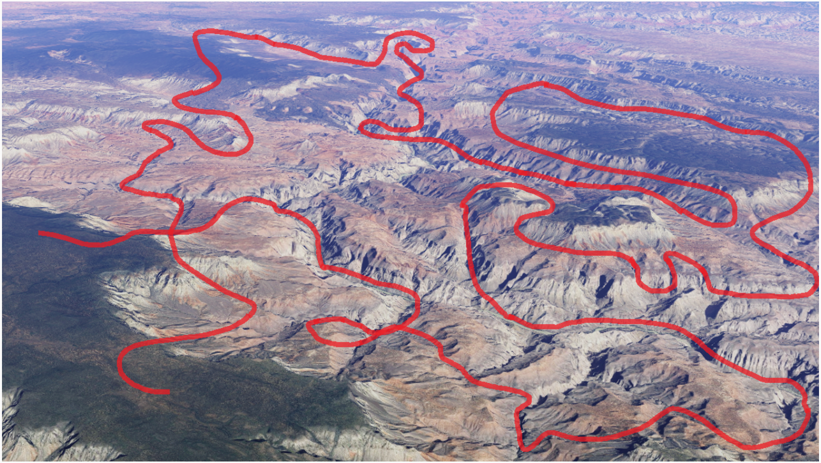

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The classical Whitney Extension Theorem

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}^k$ *is compact and* $\{f_\alpha\}_{|\alpha| \leq m}$ *are continuous real valued functions on* K*. If*

$$f_\alpha(x) = \sum_{|\beta| \leq m - |\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y)$$

for any $x, y \in K$ *and any* $|\alpha| \leq k$ *where* $\lim_{\substack{|x-y| \to 0 \\ x,y \in K}} \frac{R_\alpha(x,y)}{|x-y|^{m-|\alpha|}} = 0$*, then we can construct* $F \in C^m(K)$ *such that* $F|_K = f_0$ *and* $\partial^\alpha F|_K = f_\alpha$*.*

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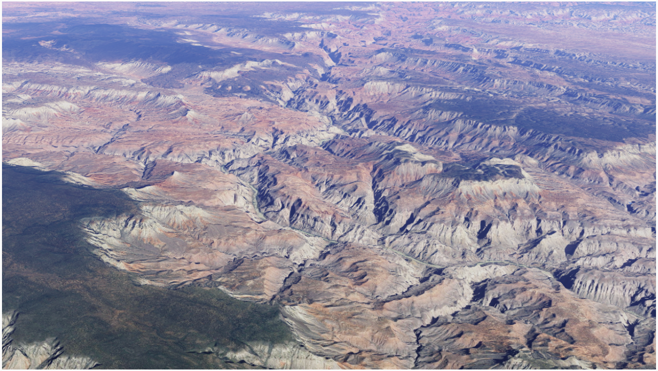
Translation: If the data lines up "nicely", then we can construct a smooth interpolation.

A different scenario

Imagine you are an fixed wing drone pilot in training.



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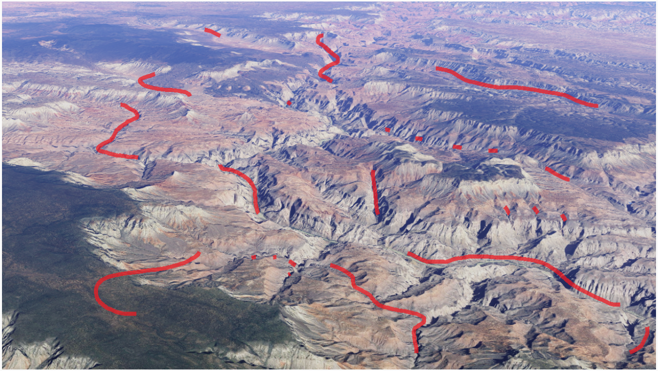

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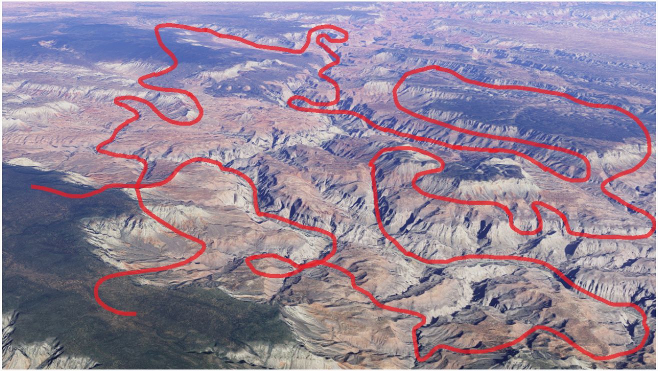

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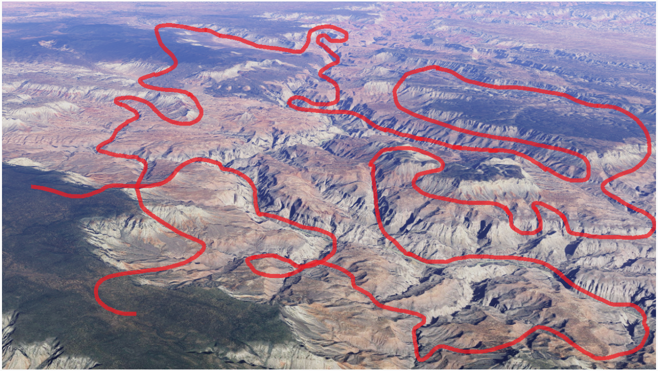

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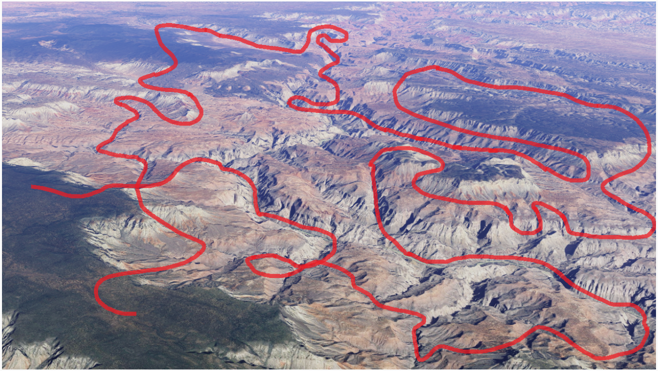

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What is different? Your motion is restricted based on the data.

Sub-Riemannian manifolds

How do you drive to the roof of a parking garage?

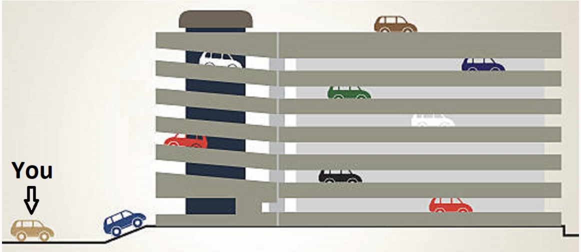

Photo credit: iStock by Getty Images

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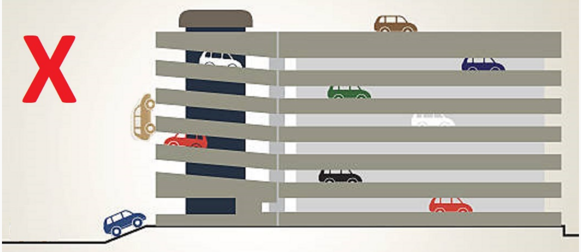

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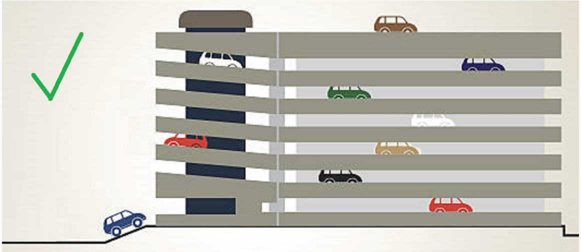

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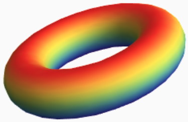

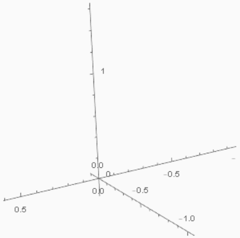

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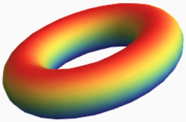

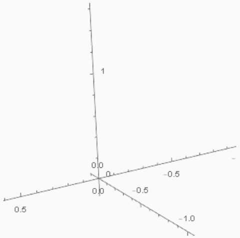

In a **sub-Riemannian manifold**, motion is restricted
to a subspace of directions at each point.

What is
the Heisenberg group

The Heisenberg group

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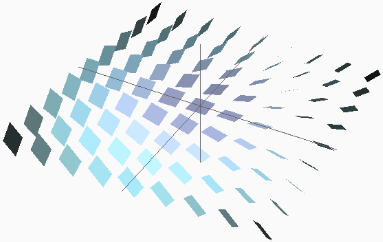

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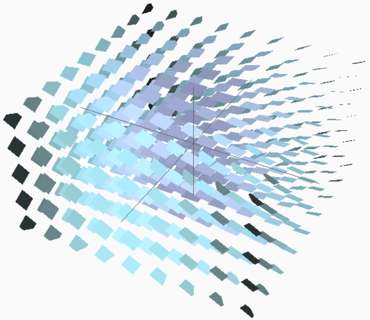

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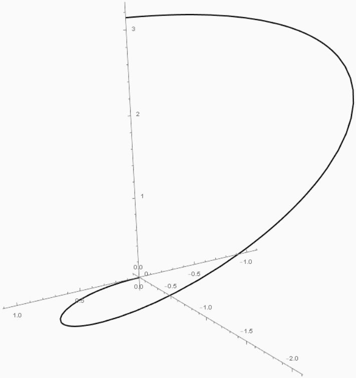

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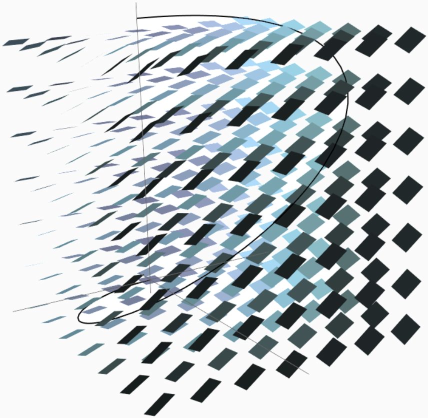

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Proposition

A curve $\gamma = (x, y, z)$ in $\mathbb{H}$ is horizontal if and only if

$$z' = 2(x'y - xy').$$

The length of a horizontal curve

Understanding the Heisenberg group

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$$|\hat{\mathbf{\imath}}| = 1 \qquad |\hat{\mathbf{\jmath}}| = 1 \qquad |\hat{\mathbf{k}}| = 1,$$

and $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$, and $\hat{\mathbf{k}}$ are the building blocks of $\mathbb{R}^3$.

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$$|X|_H = |\langle 1, 0, 2y\rangle|_H = 1^2 + 0^2 = 1, \text{ and}$$

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This is as small as possible when $x'(t)$ and $y'(t)$ are always 0.

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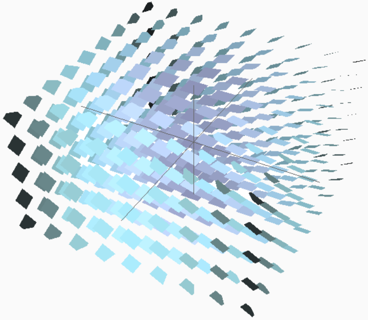
Remember: we can't move straight up!



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$$2\int_a^b (x'y - xy') = -4(\text{signed area in the } xy\text{-plane}).$$

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$$2\int_a^b (x'y - xy') = z(b) - z(a) = \text{Height of } p = \alpha.$$

According to Green's Theorem,

$$2\int_a^b (x'y - xy') = -4(\text{signed area in the } xy\text{-plane}).$$

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Geodesics

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This must be true for **any** horizontal curve from the origin to p!

Understanding the Heisenberg group

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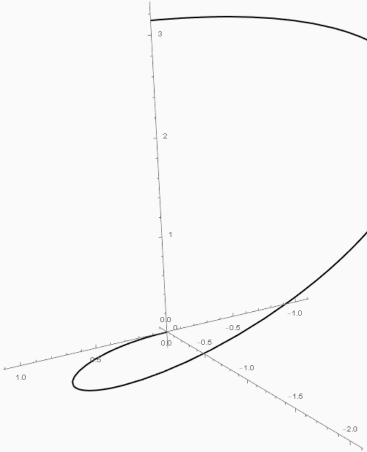
Theorem

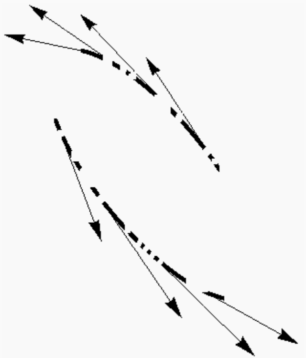
A horizontal curve from the origin to a point on the z-axis is a geodesic if and only if its projection to the xy-plane is a circle.

The Heisenberg group



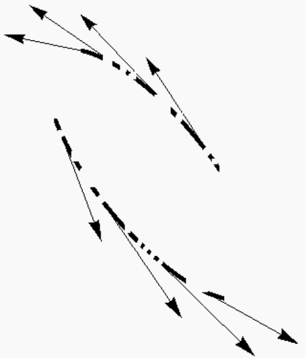
Whitney's Extension Theorem

Whitney's theorem



Recall: In 1934, Whitney described how to construct smooth interpolations of real valued data.

Whitney's theorem



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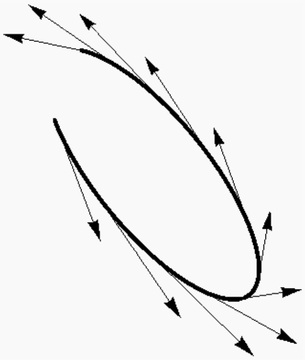

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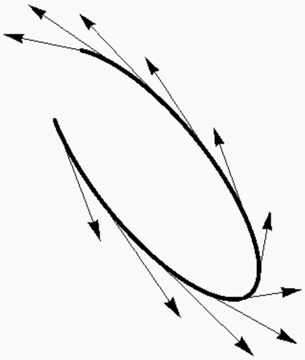

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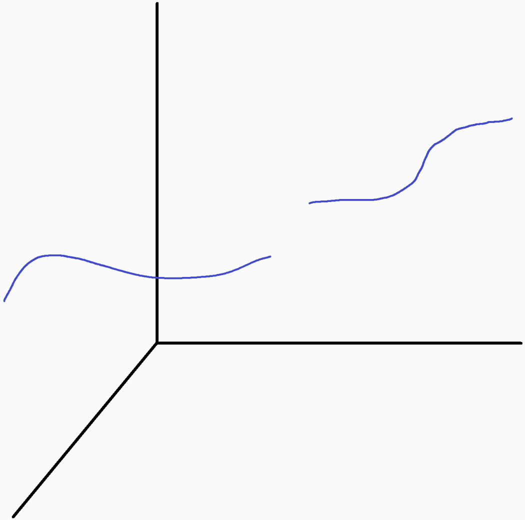
Question:
How do we fill in the gaps of a horizontal curve in $\mathbb{H}$?

Interpolating data



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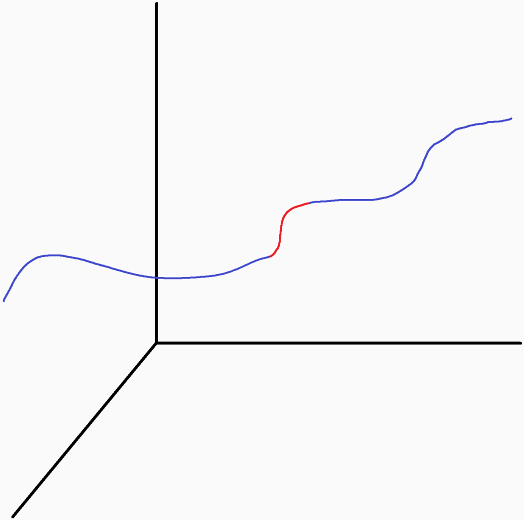

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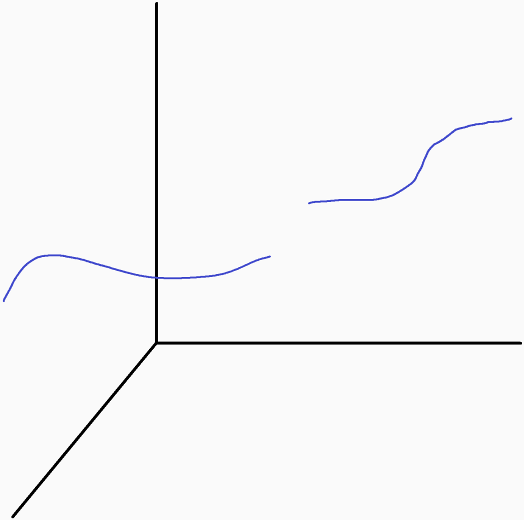

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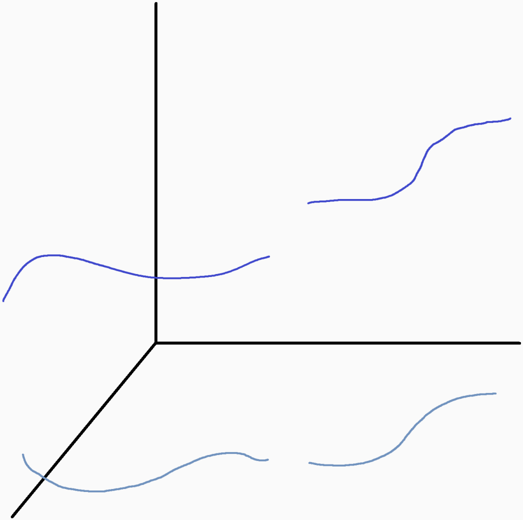

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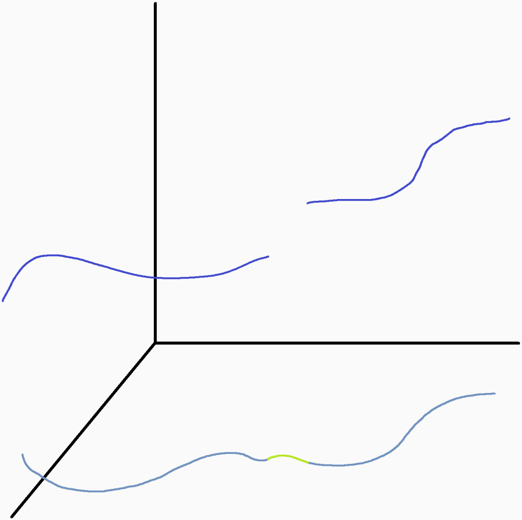

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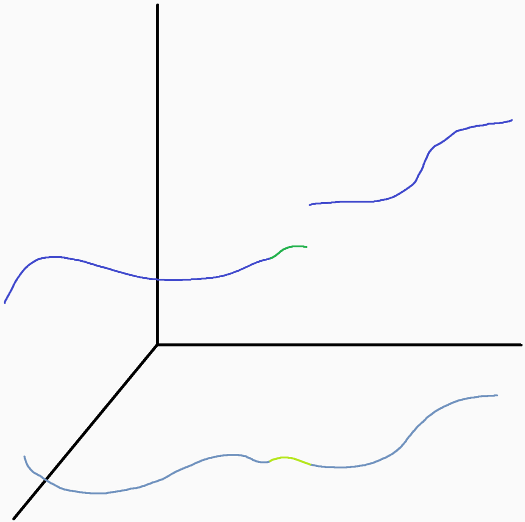

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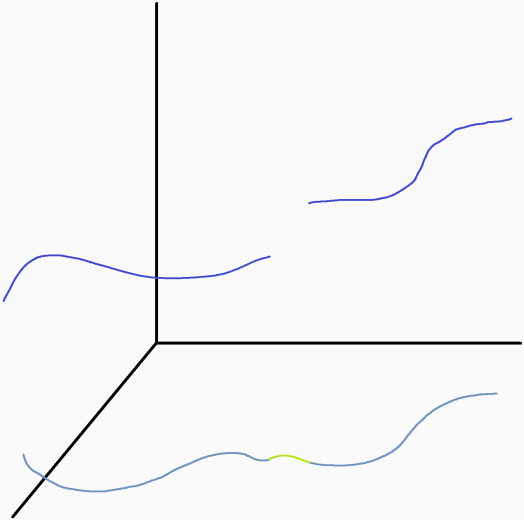

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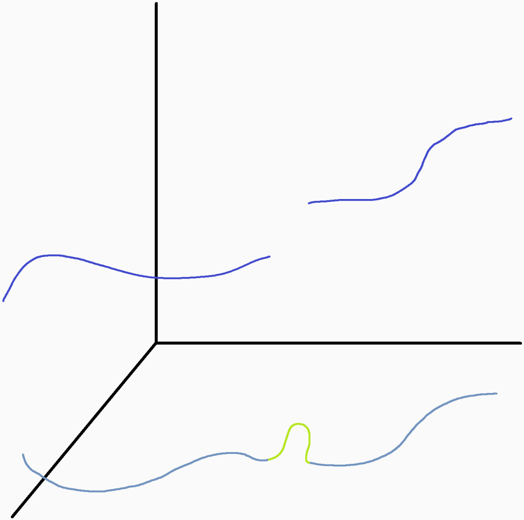

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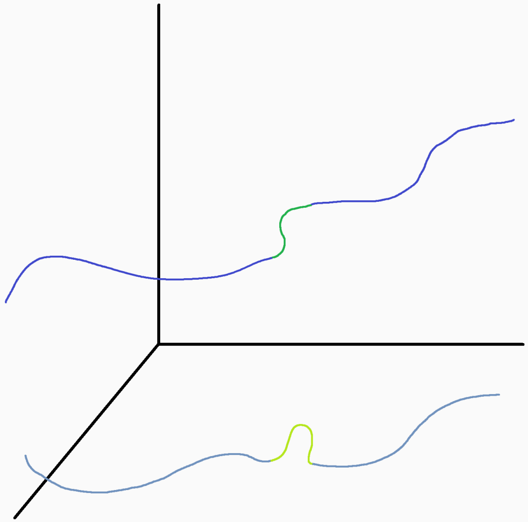

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