The Heisenberg group

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November 30, 2021

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THE OHIO STATE UNIVERSITY
MARION

A scenario

Imagine that you are a drone pilot in training.

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How can you tell if the ight plan is feasible?

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The classical Whitney Extension Theorem

Theorem (Whitney 1934)

Suppose $K \quad \mathrm{R}^k$ *is compact and* $ff \ g_{j \ j \ m}$ *are continuous real valued functions on* K*. If*

$$f \ (x) = \sum_{j \ j \ m \ j \ j} \frac{f_{\ + \ }(y)}{!}(x \quad y) \ + R \ (x;y)$$

for any $x;y \ 2 \ K$ *and any* $j \ j \quad k$ *where* $\lim_{\substack{jx \ yj \ ! \ 0 \\ x;y \, 2 \, K}} \frac{R \ (x;y)}{jx \ yj^{m \ j \ j}} = 0;$ *then we can construct* $F \ 2 \ C^m(K)$ *such that* $Fj_K = f_0$ *and* $@ \ Fj_K = f \ .$

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Suppose $K \subset \mathbb{R}^k$ is compact and $\{f_\alpha\}_{|\alpha| \le m}$ are continuous real valued functions on K. If

$$f_\alpha(x) = \sum_{|\beta| \le m - |\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x;y)$$

for any $x;y \in K$ and any $|\alpha| \le k$ where $\lim_{\substack{|x-y| \to 0 \\ x;y \in K}} \frac{R_\alpha(x;y)}{|x-y|^{m-|\alpha|}} = 0$; then we can construct $F \in C^m(K)$ such that $F|_K = f_0$ and $\partial^\alpha F|_K = f_\alpha$.

Translation: If the data lines up "nicely", then we can construct a smooth interpolation.

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A different scenario

Imagine you are an fixed wing drone pilot in training.

How can you tell if the flight plan is feasible?

What is different?

A different scenario

Imagine you are an xed wing drone pilot in training.

How can you tell if the ight plan is feasible?

What is di erent? Your motion is restricted based on the data.

Sub-Riemannian manifolds

How do you drive to the roof of a parking garage?

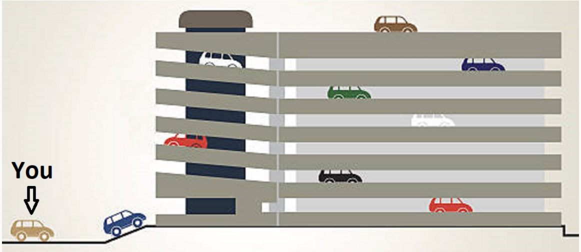

Photo credit: iStock by Getty Images

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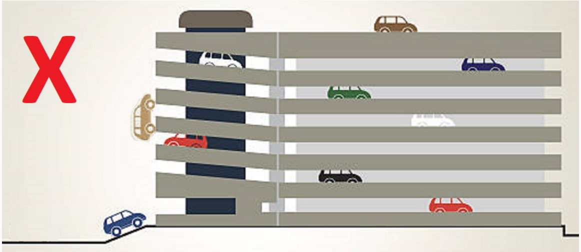

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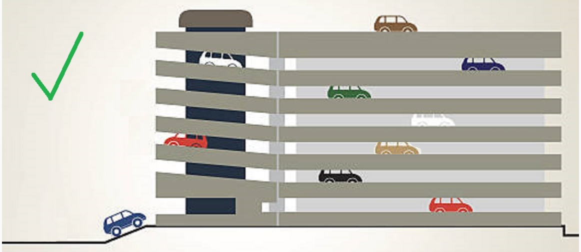

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Sub-Riemannian manifolds

In a **Riemannian manifold**, motion is allowed in any direction.

Sub-Riemannian manifolds

In a Riemannian manifold, motion is allowed in any direction.

In a sub-Riemannian manifold, motion is restricted to a subspace of directions at each point.

What is
the Heisenberg group

The Heisenberg group

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At each point (x, y, z) in $\mathbb{R}^3$, consider the vectors

$$X_{(x,y,z)} = \langle 1, 0, -2y \rangle \qquad Y_{(x,y,z)} = \langle 0, 1, 2x \rangle .$$

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Proposition

A curve $\gamma = (x, y, z)$ in H is horizontal if and only if

$$z' = 2(x'y - xy').$$

The length of a horizontal curve

Understanding the Heisenberg group

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Recall: The length of a Euclidean curve $\gamma(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ over a time interval $[a, b]$ is

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$|\gamma'(t)|$ is the magnitude (or length) of the vector $\gamma'(t)$. It has the important property that

$$|\hat{\imath}| = 1 \qquad |\hat{\jmath}| = 1 \qquad |\hat{k}| = 1,$$

and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the building blocks of $\mathbb{R}^3$.

Horizontal magnitude

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For any horizontal vector $V = \langle a, b, c\rangle$, define $|V|_H = a^2 + b^2$. Then

$$|X|_H = |\langle 1, 0, 2y\rangle|_H = 1^2 + 0^2 = 1, \text{ and}$$

$$|Y|_H = |\langle 0, 1, -2x\rangle|_H = 0^2 + 1^2 = 1.$$

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This is as small as possible when $x'(t)$ and $y'(t)$ are always 0.

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Remember: we can't move straight up!

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$$\int_a^b | \ '(t)|_H = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \text{length in the xy-plane.}$$

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This must be true for any horizontal curve from the origin to p!

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Theorem

A horizontal curve from the origin to a point on the z-axis is a geodesic if and only if its projection to the xy-plane is a circle.

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Recall: In 1934, Whitney described how to construct smooth interpolations of real valued data.

In particular, he explained how to ll in the gaps of a curve like this one in $\mathbb{R}^3$.

Question:
How do we ll in the gaps of a horizontal curve in $\mathbb{H}$?

Interpolating data

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Thank you!