Whitney's Extension Theorem for curves in the Heisenberg group

Scott Zimmerman

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The Ohio State University
zimmerman.416@osu.edu

Whitney's question

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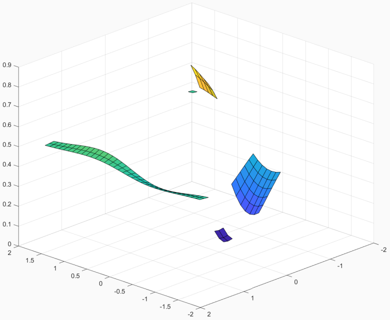

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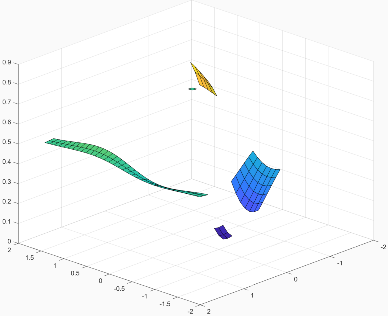

When is there a $C^{m,\omega}$ function F such that $F|_K = f$?

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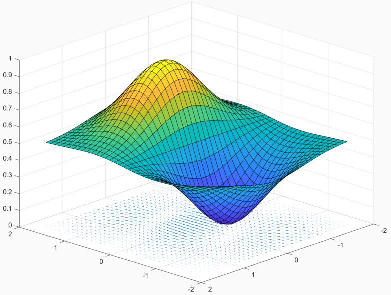

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That is,

$$|\partial^\alpha F(x) - \partial^\alpha F(y)| \le C\omega(|x-y|) \quad \forall x, y \in \mathbb{R}^n, \forall |\alpha| = m$$

for some constant $C > 0$.

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We write $\|F\|_{C^{m,\omega}}$ to denote the infimum over all possible C.

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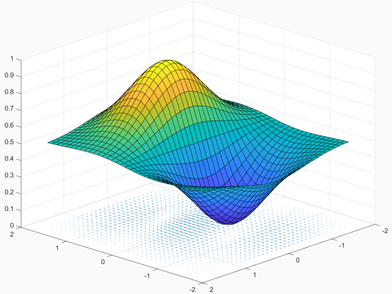

When is there a $C^{m,\omega}$ function F such that $F|_K = f$ **and** $\partial^\alpha F|_K = f_\alpha$?

Theorem (Taylor 1715)

$C^{m,\omega}$ extension of f $\implies$ Taylor's theorem holds on K

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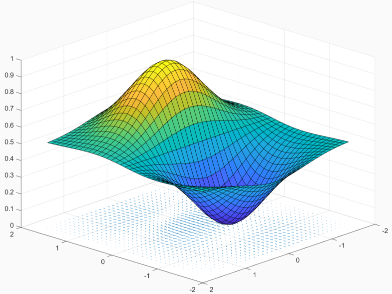

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Theorem (Whitney 1934)

$C^{m,\omega}$ *extension of* $f \iff$ *Taylor's theorem holds on* K

The classical Whitney Extension Theorem

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}^n$ is compact and $\{f_\alpha\}_{|\alpha| \le m}$ are continuous real valued functions on K. There is some $F \in C^{m,\omega}(K)$ such that $F|_K = f_0$ and $\partial^\alpha F|_K = f_\alpha$ if and only if

$$\left| f_\alpha(x) - \sum_{|\beta| \le m - |\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta \right| \le C|x-y|^{m-|\alpha|}\,\omega(|x-y|)$$

for any $x, y \in K$, any $|\alpha| \le m$, and some constant $C > 0$.

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There is a $C^{m,\omega}$ extension $F : \mathbb{R} \to \mathbb{R}$ of f if and only if,
for every choice X of $m + 2$ points in K, there is a $C^{m,\omega}$ extension $F_X : \mathbb{R} \to \mathbb{R}$ of $f|_X$ where $\|F_X\|_{C^{m,\omega}}$ is bounded uniformly.

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This formulation of Whitney's result is due to Brudnyi and Shvartsman (1994) who answered Whitney's question for $C^{1,\omega}$ maps on $\mathbb{R}^n$.

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Fefferman answered it in full (for $C^{m,\omega}$ maps on $\mathbb{R}^n$) in 2009.

A simple application

Thought experiment

Imagine that you are a drone pilot in training.

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What is di erent?

A di erent scenario

Imagine you are an xed wing drone pilot in training.

How can you tell if this ight plan is feasible?

What is di erent? Your motion is further restricted by the data.

Sub-Riemannian manifolds

How do you drive to the roof of a parking garage?

Photo credit: iStock by Getty Images

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In a sub-Riemannian manifold, motion is restricted to a subbundle of the tangent bundle.

The sub-Riemannian Heisenberg group

The Heisenberg group

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The horizontal plane H_p at $p = (x, y, z) \in \mathbb{R}^3$ is the plane spanned by

$$X_p = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}, \quad Y_p = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}.$$

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Choose the Riemannian metric which makes X, Y, and $\frac{\partial}{\partial z}$ orthonormal.

The Heisenberg group

This sub-Riemannian framework is associated with a Lie group structure.

$$(x; y; z) \quad (x^0; y^0; z^0) = (x + x^0; y + y^0; z + z^0 + 2(yx^0 \quad xy^0))$$

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$$(x,y,z)\ (x',y',z') = (x + x', y + y', z + z' + 2(yx' - xy'))$$

The left invariant vector fields are

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For each $h \neq 0$, the dilation $(x, y, z) \mapsto (hx, hy, h^2 z)$ is a group automorphism.

Also, $[X, Y] = -4Z$. All of this implies that $\mathbb{H}$ is a Carnot group.

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Curves in the Heisenberg group

Horizontal curves

Question: Which curves are horizontal?

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A curve $\gamma = (x, y, z)$ is horizontal if and only if

$$\begin{aligned}\gamma'(t) &= AX_{\gamma(t)} + BY_{\gamma(t)} \\ &= A\left(\frac{\partial}{\partial x} + 2y(t)\frac{\partial}{\partial z}\right) + B\left(\frac{\partial}{\partial y} - 2x(t)\frac{\partial}{\partial z}\right) \\ &= A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + 2(Ay(t) - Bx(t))\frac{\partial}{\partial z}.\end{aligned}$$

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Proposition

A curve $\gamma = (x; y; z)$ in H is horizontal if and only if

$$z' = 2(x'y - xy').$$

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Takeaway: A curve in the xy-plane can always be lifted to a horizontal curve by integrating this equation.

Understanding the Heisenberg group

Question: What do geodesics in the Heisenberg group look like?

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Fix a point $p = (0; 0; \quad)$ on the z-axis.

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Remember,

$$2(x'y \quad xy') = z'.$$

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$$2\int_a^b (x'y \quad xy') = z(b) \quad z(a)$$

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$$2\int_a^b (x'y \quad xy') = z(b) \quad z(a) = \text{Height of } p = \ .$$

Understanding the Heisenberg group

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$$\ell(\gamma) = \int_a^b \sqrt{\langle \gamma', \gamma' \rangle} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

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$$= \text{length in the } xy\text{-plane}$$

In summary:

We want to nd the shortest curve in the xy-plane whose signed area equals the xed constant .

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Theorem

A horizontal curve from the origin to a point on the z-axis is a geodesic if and only if its projection to the xy-plane is a circle.

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In summary:

We want to find the shortest curve in the xy-plane whose signed area equals the fixed constant.

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Takeaway:

There is a strong relationship between vertical motion and area in the xy-plane.

The Heisenberg group

Whitney extensions in the Heisenberg group

Whitney's Extension Theorem into $\mathbb{H}$

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Whitney told us when a $C^{m;!}$ extension of exists.

Question: When does a horizontal $C^{m;!}$ extension of exist?

The C^1 case

Let's rst consider Whitney's C^1 question into H (with extra data)

i.e. we have continuous maps $: K \to H$ and $_1 : K \to H$.

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Theorem (Whitney 1934)

C^1 extension of

m

uniform convergence on K of

$$\frac{(x + h) \quad (x)}{h} \to \quad _1(x)$$

Whitney's Extension Theorem into H

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Consider the following mapping : R K ! R^3. Set $_1$ 0 on K .

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Consider the following mapping : $\mathbb{R} \quad K \; ! \quad \mathbb{R}^3$. Set $_1 \quad 0$ on K.

According to Whitney's theorem, there is a C^1 extension of whose derivative is 0 on K.

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However, there is no horizontal C^1 extension!

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Whitney's Extension Theorem into H

Why not?

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$$\frac{(x+h) \qquad (x)}{h}$$

converge uniformly to ^{0} on K.

Whitney's Extension Theorem into H

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converge uniformly to ^{0} on K.

Question: Can we say something similar for horizontal C^1 curves in H?

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De nition

The Pansu derivative at $x \in \mathbb{R}$ of a horizontal curve in $\mathbb{H}$ is

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whenever this limit exists.

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Note: If is Lipschitz and this limit exists, the limit must have the form $(a, b, 0)$.

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Theorem (Taylor 1712)

C^1 extension of =) uniform convergence of di erence quotient on K

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$$\lim_{\substack{|b-a|\to 0 \\ a,b\in K}} \frac{z(b) - z(a) - 2\,(x(b)y(a) - x(a)y(b))}{|b-a|^2} = 0 .$$

A proof

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The C^m case

Now consider Whitney's C^m question into H (with extra data)

i.e. we have continuous maps $_k : K \; ! \; H$ for $0 \quad k \quad m$.

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i.e. we have continuous maps $_k : K \, ! \;\; H$ for $0 \quad k \quad m$.

Unfortunately, little is known about higher order Pansu derivatives.

The C^m case

Maybe we can build on the \area / velocity" relationship from before:

$$\lim_{\substack{|b-a|\to 0\\ a,b\in K}} \frac{z(b)-z(a)-2\,(x(b)y(a)-x(a)y(b))}{|b-a|^2} = 0\,.$$

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$$A(a,b) = z(b)-z(a)-2\int_a^b (T_a^m x)' T_a^m y - (T_a^m y)' T_a^m x$$

$$V(a,b) = (b-a)^{2m} + (b-a)^m \int_a^b |(T_a^m x)'| + |(T_a^m y)'|$$

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Theorem (Pinamonti, Speight, SZ 2019)

C^m horizontal extension of ( )

uniform convergence of $\dfrac{A(a,b)}{V(a,b)} \to 0$ *on* K

Whitney's Extension Theorem into $\mathbb{H}$

Let's go back to Whitney's original question.

Whitney's Extension Theorem into H

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Suppose K R is compact
and : K ! H is continuous.

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Suppose K R is compact and : K ! H is continuous.

When does a horizontal $C^{m;!}$ extension of exist?

Whitney's Extension Theorem into $\mathbb{H}$

Let's go back to Whitney's original question.

Suppose K $\mathbb{R}$ is compact and $: K \to \mathbb{H}$ is continuous.

When does a horizontal $C^{m,\omega}$ extension of exist?

The finiteness principle

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous.
There is a $C^{m,\omega}$ extension $F : \mathbb{R} \to \mathbb{R}$ of f if and only if,
for every choice X of $m+2$ points in K, there is a $C^{m,\omega}$ extension
$F_X : \mathbb{R} \to \mathbb{R}$ of $f|_X$ where $\|F\|_{C^{m,\omega}}$ is bounded uniformly.

The finiteness principle

Theorem (Whitney 1934)

Suppose K R is compact and f : K ! R is continuous.
There is a $C^{m;!}$ extension F : R ! R of f if and only if,
for every choice X of m + 2 points in K, there is a $C^{m;!}$ extension
F_X : R ! R of $f j_X$ where $kFk_{C^{m;!}}$ is bounded uniformly.

Theorem (SZ 2021)

Suppose K R is perfect and : K ! H is continuous.
Assume that, for every choice X of m + 2 points in K, there is a $C^{m;!}$
extension $_X$: R ! H of j_X where $k \; {}_X k_{C^{m;!}}$ is bounded uniformly and

$$\frac{A(\; _X; a; b)}{V(\; _X; a; b)} \quad ! (jb \quad aj) \quad 8a; b \; 2 \; K:$$

Then there is a horizontal $C^{m;^p\overline{\ }}$ extension : R ! R of

The finiteness principle

Theorem (Speight, SZ 2023)

$C^{m;!}$ horizontal extension of ( ) $A(a;b)$ $CV_{!}(a;b)$ on K

The finiteness principle

Theorem (Speight, SZ 2023)

$$C^{m;!} \text{ horizontal extension of } \quad (\,) \quad A(a;b) \quad CV_{!}(a;b) \text{ on } K$$

Theorem (Pinamonti, Speight, SZ 2023)

Suppose K R is perfect and $: K \; ! \; \mathrm{H}$ is continuous.
There is a horizontal $C^{m;!}$ extension $: \mathrm{R} \; ! \; \mathrm{R}$ of if and only if, for every choice X of $m+2$ points in K, there is a $C^{m;!}$ extension $_X : \mathrm{R} \; ! \; \mathrm{H}$ of j_X where $k \; _X k_{C^{m;!}}$ is bounded uniformly and

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Higher dimensional domains

My white whale

Suppose $K \quad R^k$ for $k > 1$ and f is a mapping from K into H^n.

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When can f be extended to a horizontal C^1 map on R^k?

Suppose K $\mathbb{R}^k$ for $k > 1$ and f is a mapping from K into $\mathbb{H}^n$.

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Proposition

C^1 horizontal extension of f $\implies$ unif. convergence of Pansu d.q. on K

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Suppose K $\mathbb{R}^k$ for $k > 1$ and f is a mapping from K into $\mathbb{H}^n$.

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Conjecture

C^1 horizontal extension of f $(\ )$ unif. convergence of Pansu d.q. on K

Thank you!

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