Whitney's Extension Theorem for curves in the Heisenberg group

Scott Zimmerman

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The Ohio State University
zimmerman.416@osu.edu

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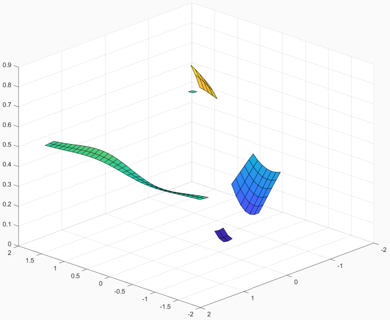

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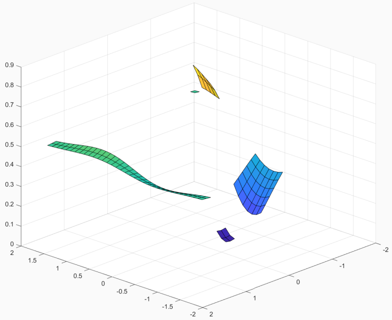

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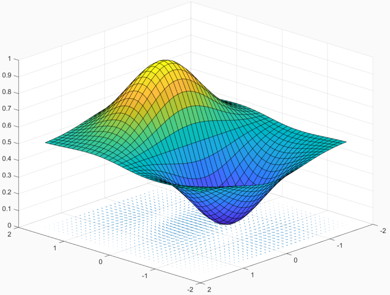

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We write $\|F\|_{C^{m,\omega}}$ to denote the infimum over all possible C.

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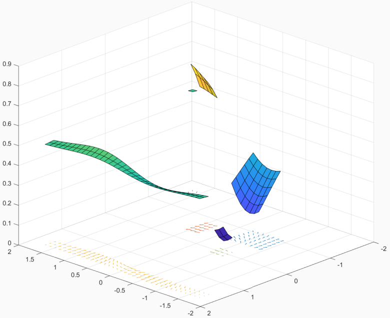

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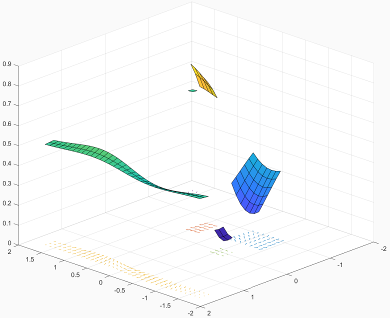

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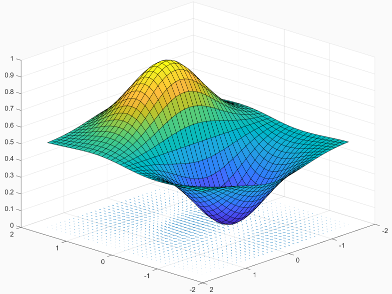

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Theorem (Taylor 1715)

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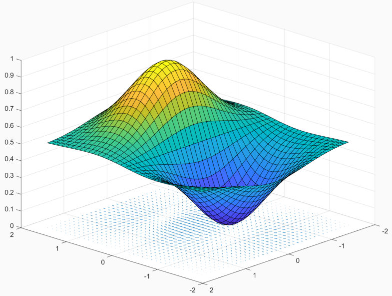

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Theorem (Whitney 1934)

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The classical Whitney Extension Theorem

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}^n$ *is compact and* $\{f_\alpha\}_{|\alpha| \leq m}$ *are continuous real valued functions on* K. *There is some* $F \in C^{m,\omega}(K)$ *such that* $F|_K = f_0$ *and* $\partial^\alpha F|_K = f_\alpha$ *if and only if*

$$\left| f_\alpha(x) - \sum_{|\beta| \leq m - |\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta \right| \leq C|x-y|^{m-|\alpha|}\omega(|x-y|)$$

for any $x, y \in K$, *any* $|\alpha| \leq m$, *and some constant* $C > 0$.

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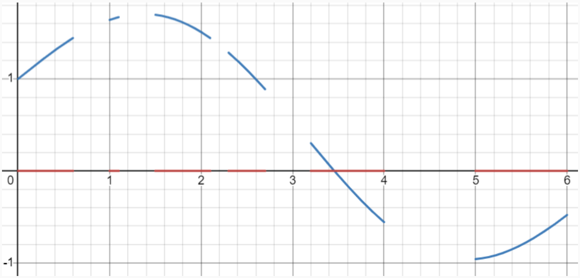

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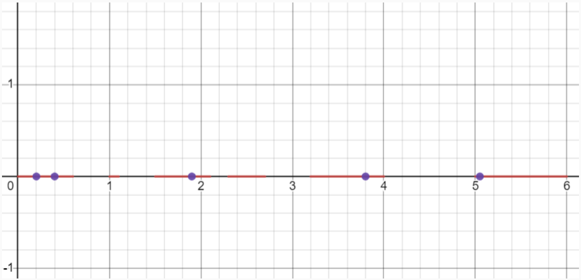

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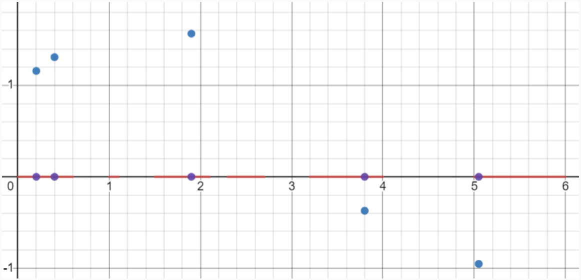

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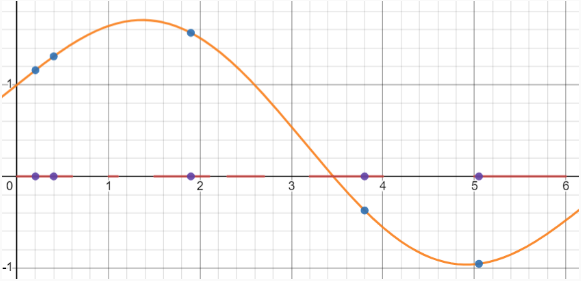

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Fefferman answered it in full (for $C^{m,\omega}$ maps on $\mathbb{R}^n$) in 2009.

A simple application

Thought experiment

Imagine that you are a drone pilot in training.

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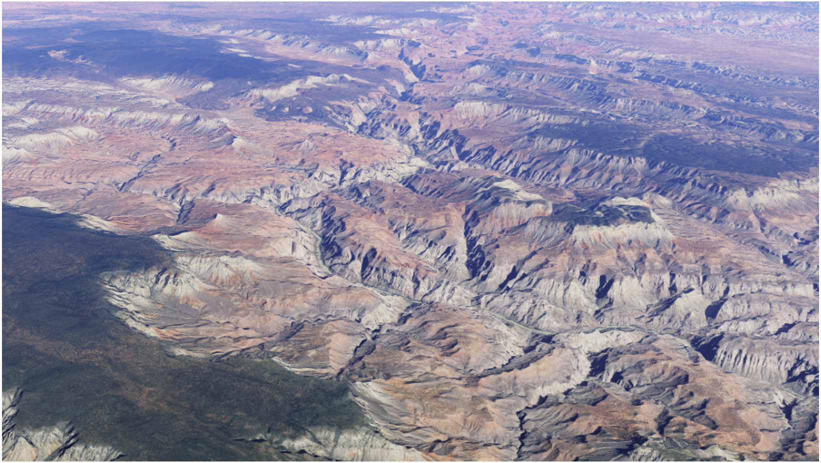

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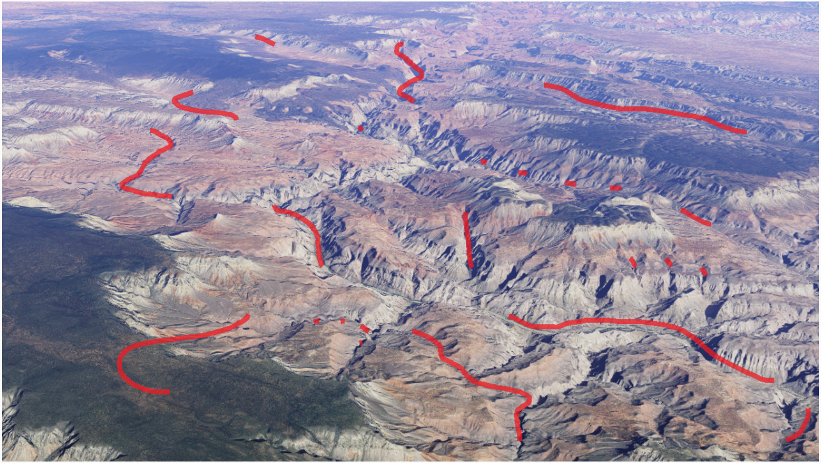

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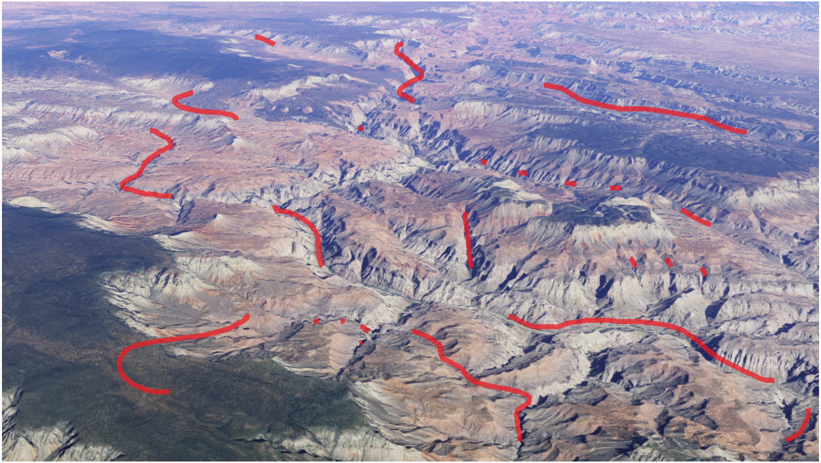

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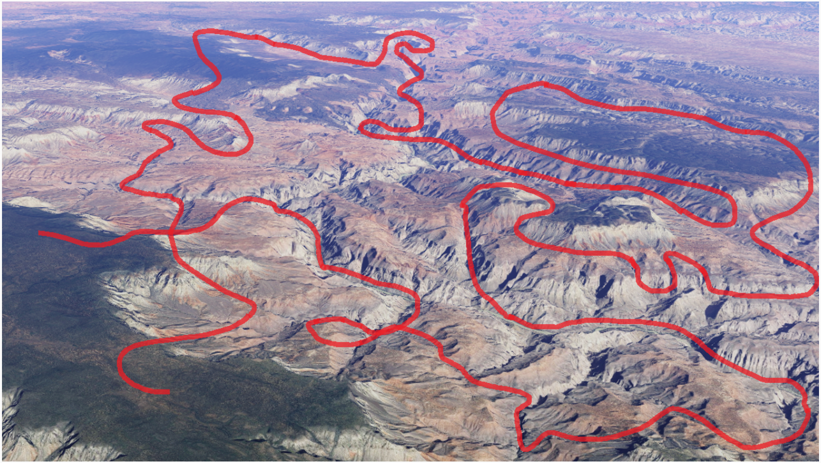

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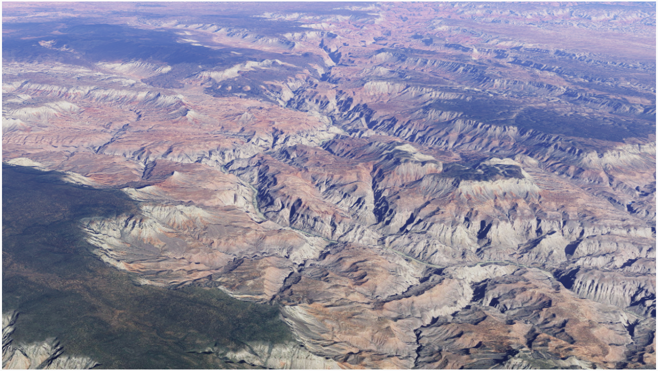
A different scenario

Imagine you are an fixed wing drone pilot in training.



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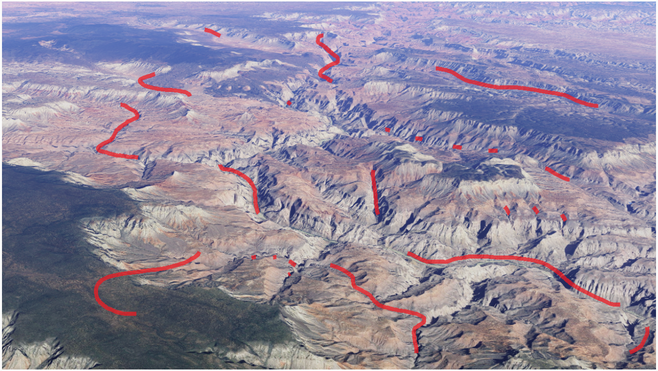

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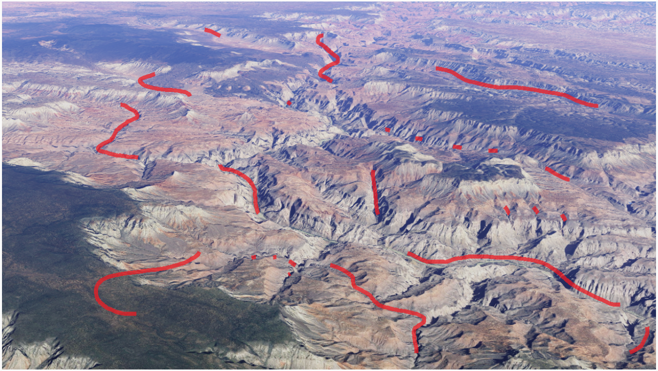

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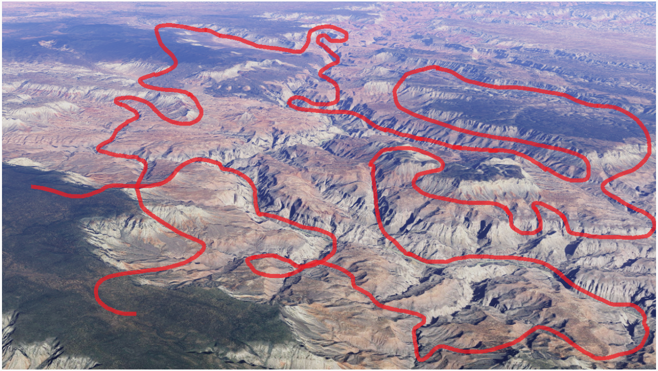

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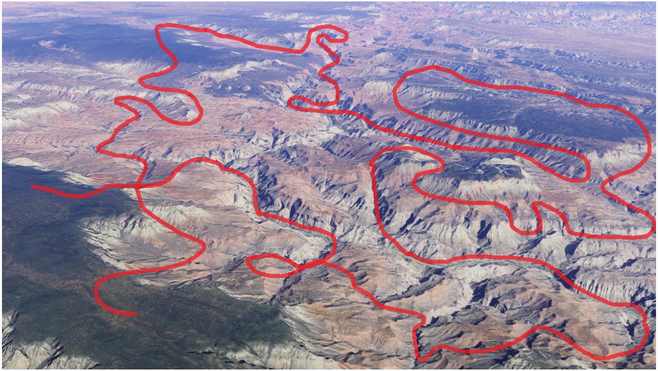

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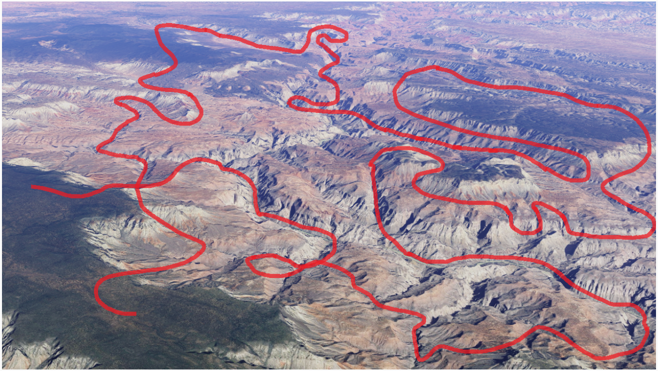

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What is different? Your motion is further restricted by the data.

Sub-Riemannian manifolds

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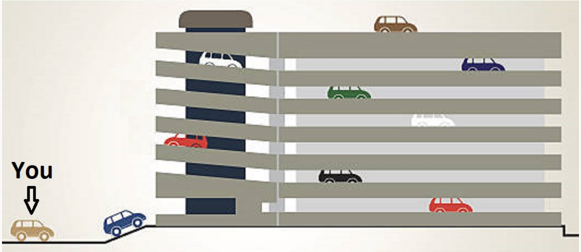

Photo credit: iStock by Getty Images

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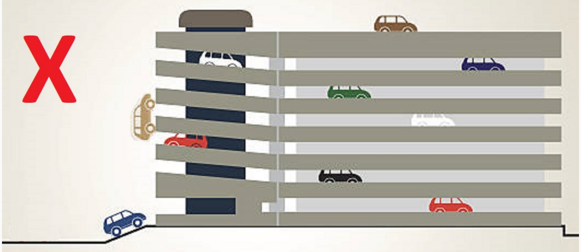

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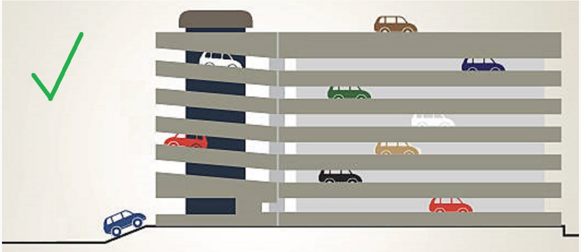

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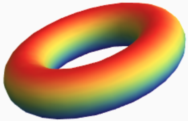

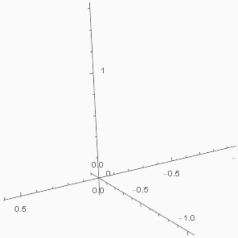

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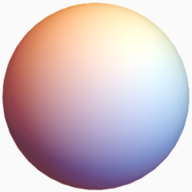

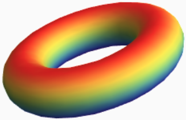

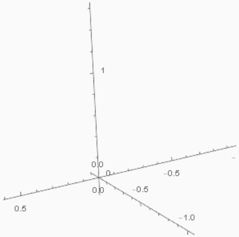

In a **sub-Riemannian manifold**, motion is restricted to a subbundle of the tangent bundle.

The sub-Riemannian Heisenberg group

The Heisenberg group

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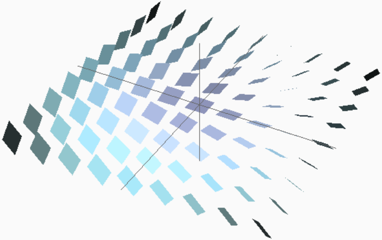

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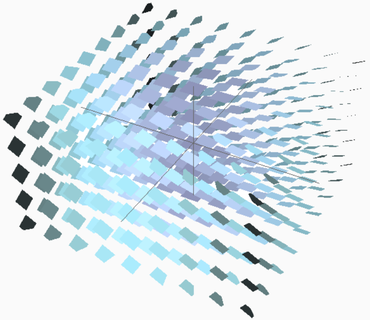

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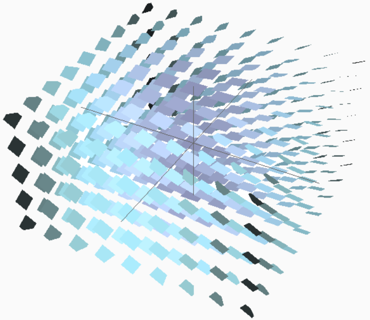

Choose the Riemannian metric which makes X, Y, and $\frac{\partial}{\partial z}$ orthonormal.

The Heisenberg group

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$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + 2(yx' - xy'))$$

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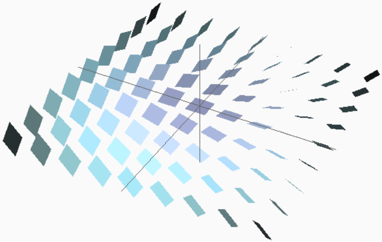

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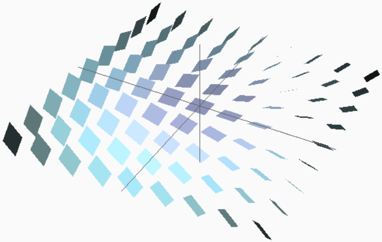

For each $h \neq 0$, the **dilation** $(x, y, z) \mapsto (hx, hy, h^2 z)$ is a group automorphism.

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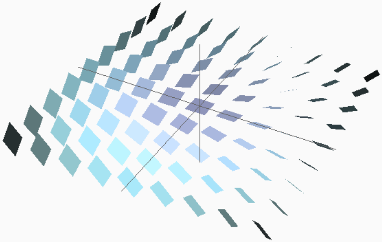

For each $h \neq 0$, the **dilation** $(x, y, z) \mapsto (hx, hy, h^2 z)$ is a group automorphism.

Also, $[X, Y] = -4Z$. All of this implies that $\mathbb{H}$ is a Carnot group.

The Heisenberg group

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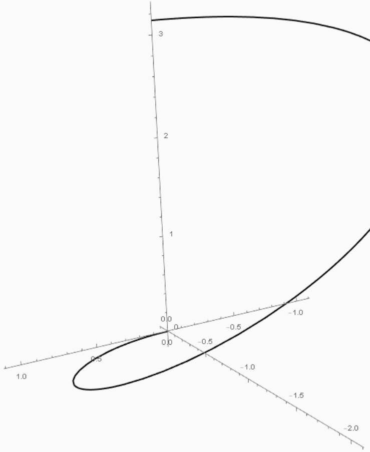

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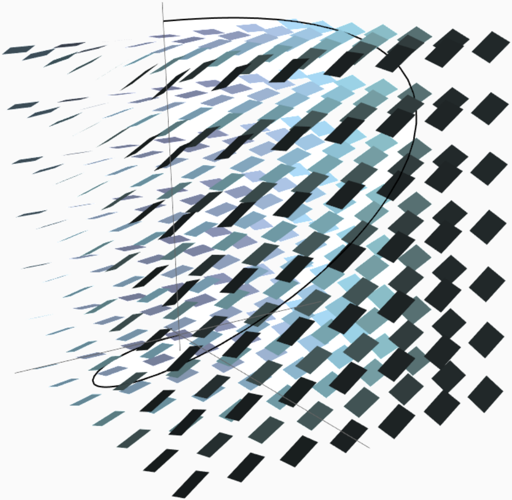

Curves in the Heisenberg group

Horizontal curves

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A curve $\gamma = (x, y, z)$ is horizontal if and only if

$$\begin{aligned}\gamma'(t) &= AX_{\gamma(t)} + BY_{\gamma(t)} \\ &= A\left(\frac{\partial}{\partial x} + 2y(t)\frac{\partial}{\partial z}\right) + B\left(\frac{\partial}{\partial y} - 2x(t)\frac{\partial}{\partial z}\right) \\ &= A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + 2(Ay(t) - Bx(t))\frac{\partial}{\partial z}.\end{aligned}$$

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Takeaway: A curve in the xy-plane can always be lifted to a horizontal curve by integrating this equation.

Understanding the Heisenberg group

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Understanding the Heisenberg group

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We want to find the shortest curve in the xy-plane whose signed area equals the fixed constant α.

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Takeaway:

There is a strong relationship between vertical motion and area in the xy-plane.

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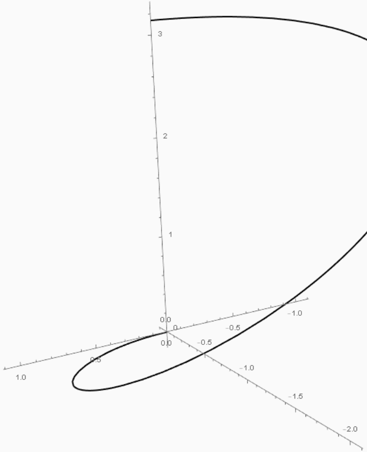
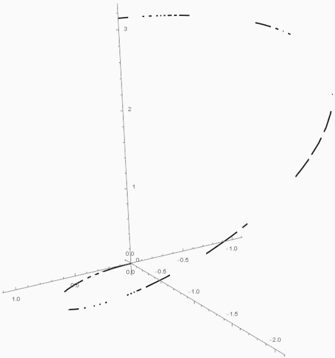


Whitney extensions in the Heisenberg group

Whitney's Extension Theorem into $\mathbb{H}$



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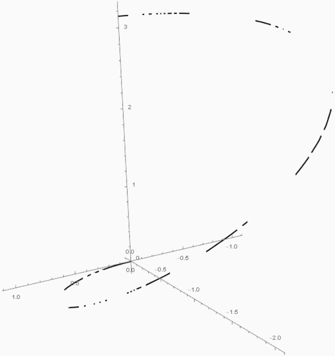

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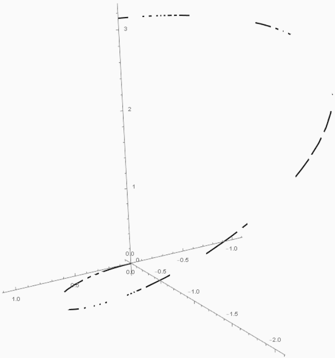

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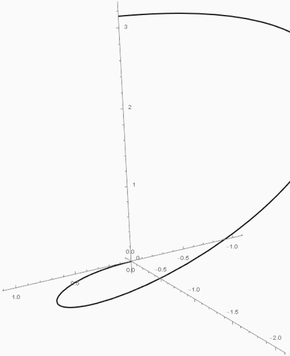

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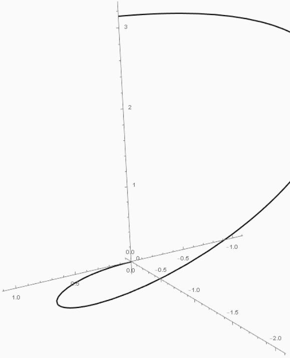

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Question: When does a horizontal $C^{m,\omega}$ extension of γ exist?

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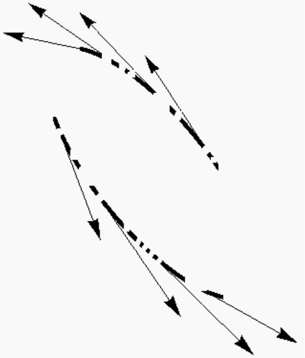

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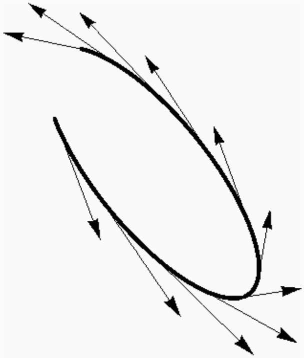

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Theorem (Whitney 1934)

C^1 extension of γ

$$\Updownarrow$$

uniform convergence on K of

$$\frac{\gamma(x+h) - \gamma(x)}{h} \to \gamma_1(x)$$

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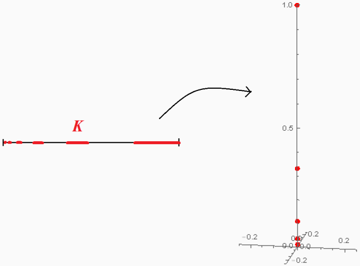

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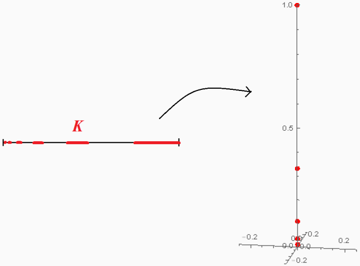

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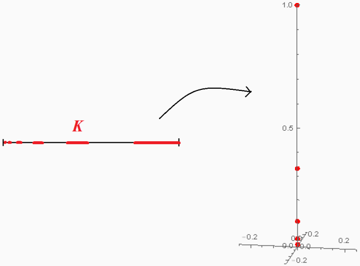

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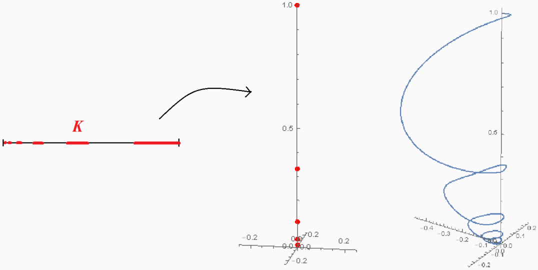

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The **Pansu derivative** at $x \in \mathbb{R}$ of a horizontal curve γ in $\mathbb{H}$ is

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whenever this limit exists.

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Note: If γ is Lipschitz and this limit exists, the limit must have the form $(a, b, 0)$.

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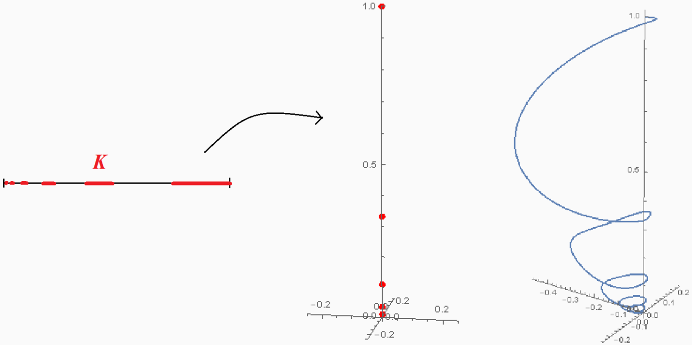

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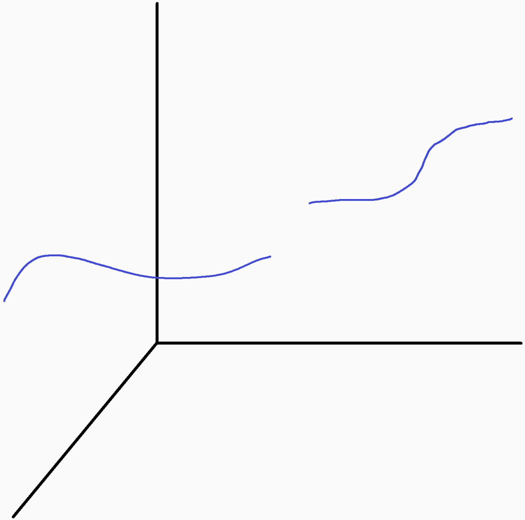
A proof



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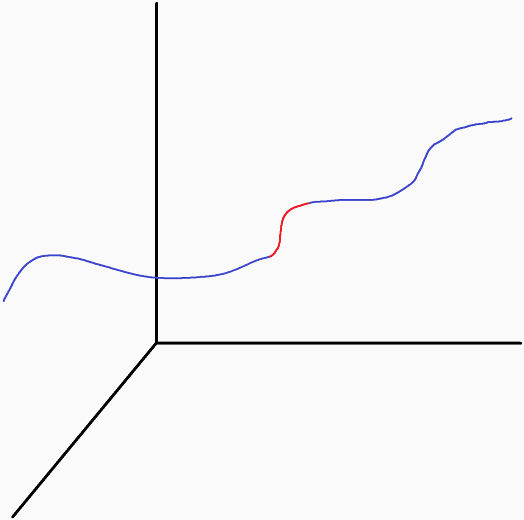

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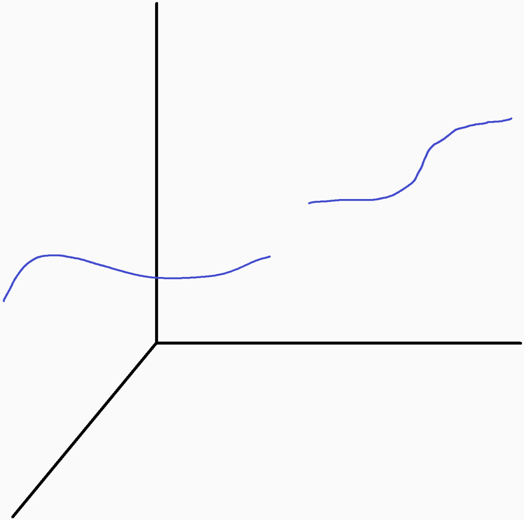

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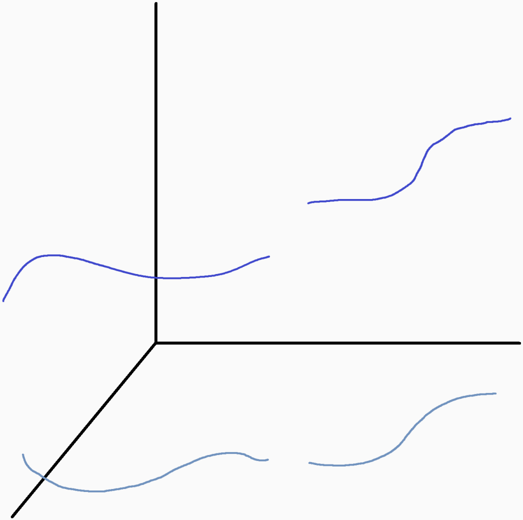

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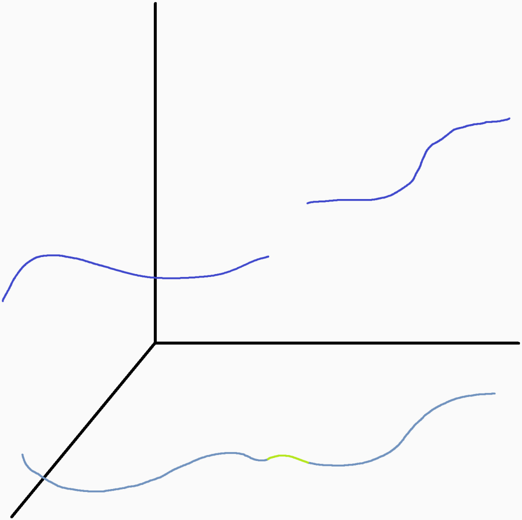

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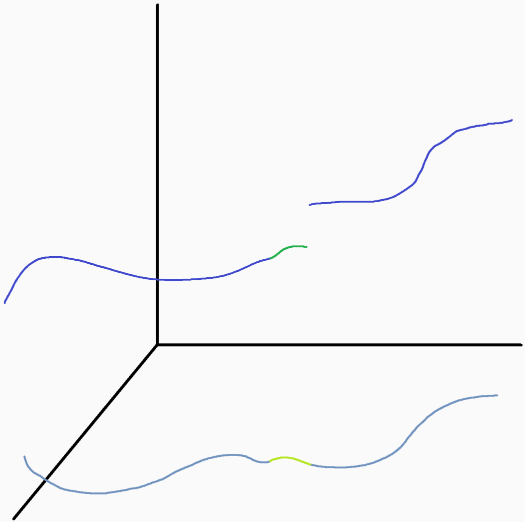

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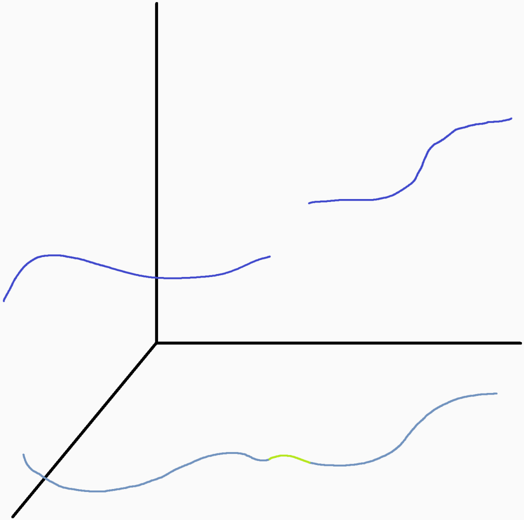

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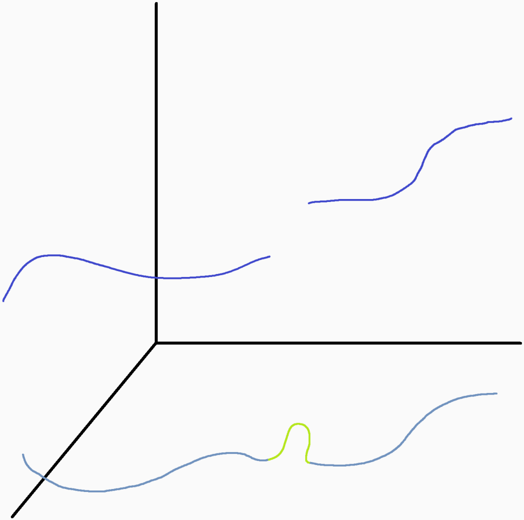

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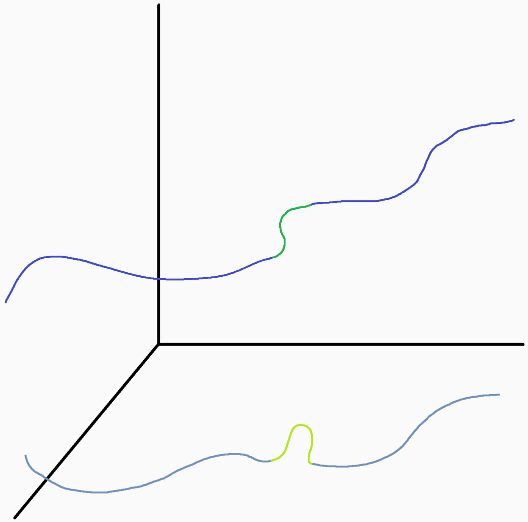

The C^m case

Now consider Whitney's C^m question into $\mathbb{H}$ (with extra data)
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Unfortunately, little is known about higher order Pansu derivatives.

The C^m case

Maybe we can build on the "area / velocity" relationship from before:

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$$A(a,b) = z(b) - z(a) - 2\int_a^b (T_a^m x)' T_a^m y - (T_a^m y)' T_a^m x$$

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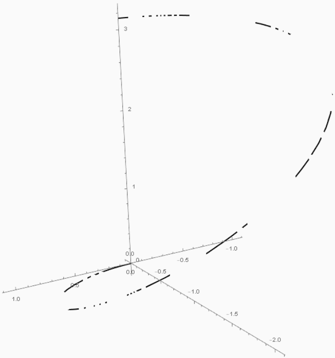
Theorem (Pinamonti, Speight, SZ 2019)

$$C^m \textit{ horizontal extension of } \gamma \iff$$

$$\textit{uniform convergence of } \frac{A(a,b)}{V(a,b)} \to 0 \textit{ on } K$$

Whitney's Extension Theorem into $\mathbb{H}$



Let's go back to Whitney's original question.

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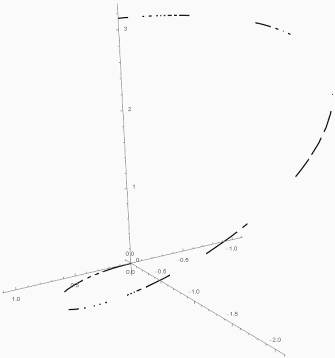

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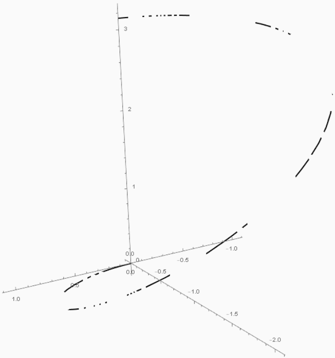
Whitney's Extension Theorem into $\mathbb{H}$



Let's go back to Whitney's original question.

Suppose $K \subset \mathbb{R}$ is compact and $\gamma : K \to \mathbb{H}$ is continuous.

When does a horizontal $C^{m,\omega}$ extension of γ exist?

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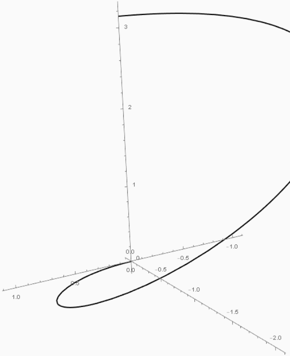

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The finiteness principle

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous.
There is a $C^{m,\omega}$ extension $F : \mathbb{R} \to \mathbb{R}$ of f if and only if,
for every choice X of $m + 2$ points in K, there is a $C^{m,\omega}$ extension
$F_X : \mathbb{R} \to \mathbb{R}$ of $f|_X$ where $\|F\|_{C^{m,\omega}}$ is bounded uniformly.

The finiteness principle

Theorem (Whitney 1934)

Suppose $K \subset \mathbb{R}$ *is compact and* $f : K \to \mathbb{R}$ *is continuous.*
There is a $C^{m,\omega}$ *extension* $F : \mathbb{R} \to \mathbb{R}$ *of* f *if and only if, for every choice* X *of* $m+2$ *points in* K*, there is a* $C^{m,\omega}$ *extension* $F_X : \mathbb{R} \to \mathbb{R}$ *of* $f|_X$ *where* $\|F\|_{C^{m,\omega}}$ *is bounded uniformly.*

Theorem (SZ 2021)

Suppose $K \subset \mathbb{R}$ *is perfect and* $\gamma : K \to \mathbb{H}$ *is continuous.*
Assume that, for every choice X *of* $m+2$ *points in* K*, there is a* $C^{m,\omega}$ *extension* $\Gamma_X : \mathbb{R} \to \mathbb{H}$ *of* $\gamma|_X$ *where* $\|\Gamma_X\|_{C^{m,\omega}}$ *is bounded uniformly and*

$$\frac{A(\Gamma_X; a, b)}{V(\Gamma_X; a, b)} \leq \omega(|b-a|) \quad \forall a, b \in K.$$

Then there is a horizontal $C^{m,\sqrt{\omega}}$ *extension* $\Gamma : \mathbb{R} \to \mathbb{R}$ *of* γ

The finiteness principle

Theorem (Speight, SZ 2023)

$C^{m,\omega}$ *horizontal extension of* $\gamma \iff A(a,b) \leq CV_\omega(a,b)$ *on* K

The finiteness principle

Theorem (Speight, SZ 2023)

$$C^{m,\omega} \text{ horizontal extension of } \gamma \iff A(a,b) \leq CV_\omega(a,b) \text{ on } K$$

Theorem (Pinamonti, Speight, SZ 2023)

Suppose $K \subset \mathbb{R}$ is perfect and $\gamma : K \to \mathbb{H}$ is continuous. There is a horizontal $C^{m,\omega}$ extension $\Gamma : \mathbb{R} \to \mathbb{R}$ of γ if and only if, for every choice X of $m+2$ points in K, there is a $C^{m,\omega}$ extension $\Gamma_X : \mathbb{R} \to \mathbb{H}$ of $\gamma|_X$ where $\|\Gamma_X\|_{C^{m,\omega}}$ is bounded uniformly and

$$A(\Gamma_X; a, b) \leq CV_\omega(\Gamma_X; a, b) \quad \forall a, b \in K.$$

Higher dimensional domains

My white whale

Suppose $K \subset \mathbb{R}^k$ for $k > 1$ and f is a mapping from K into $\mathbb{H}^n$.

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Proposition

C^1 horizontal extension of $f \implies$ unif. convergence of Pansu d.q. on K

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Proposition

C^1 horizontal extension of f $\Longrightarrow$ unif. convergence of Pansu d.q. on K

Conjecture

C^1 horizontal extension of f $\Longleftrightarrow$ unif. convergence of Pansu d.q. on K

Thank you!

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