Lecture 1

1. Ordinary Differential equations

Differential equation — any equation which contains derivatives of a function we are looking for.

Example (Newton's first law)

$$F = m a$$

acceleration

force acting on object

mass of object that is moving

Let $u(t)$ be the position function of object at time t.

Then, velocity at time t denoted by $v(t) = \frac{du}{dt}$

Then, acceleration at time t denoted by $a(t) = \frac{dv(t)}{dt} = \frac{d^2u}{dt^2}$

Therefore, given F, m we get a differential equation on u:

$$(*) \quad F = m\frac{d^2u}{dt^2}$$

- Solve differential equation — find $u(t)$ that satisfies $(*)$.

(2) Classification of differential equations:

<u>Order:</u>

The <u>order</u> of a differential equation – the "largest" derivative of unknown function present in the equation.

<u>Example</u>

$$\underbrace{y''}_{\text{2nd derivative}} - 3\underbrace{y'}_{\text{1st derivative}} + 5 = 0 \quad \rightarrow \text{ order } = 2$$

$$\underbrace{\left(\frac{d^3y}{dt^3}\right)^2}_{\text{3rd derivative}} - e^{ty}\underbrace{\frac{dy}{dt}}_{\text{1st derivative}} = \underbrace{\frac{d^5y}{dt^5}}_{\text{5th derivative}} \quad \rightarrow \text{ order } = 5$$

<u>Notation:</u> y' – 1st derivative, $y^{(k)}$ – kth derivative, but $y^k = \underbrace{y \cdot y \cdots y}_{k\text{-times}}$

<u>Linear vs Non linear:</u>

<u>Def.</u> A <u>linear</u> ordinary diff. equation is any differential equation that can be written in the form

$$(**) \quad a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

where $a_i(t), g(t)$ – given functions of t and $y(t)$ – unknown function

Otherwise, if cannot be written in (**), then <u>non-linear</u> equation.

③ Example

1) $y'' - 3y' + 5 = 0$ $\rightsquigarrow$ linear

$\Rightarrow$ $\underbrace{1}_{a_2(t)} \cdot y'' \underbrace{-3}_{a_1(t)} \cdot y' + \underbrace{0}_{a_0(t)} \cdot y = \underbrace{-5}_{g(t)}$

2) $y''' + 2\sin(t)\cdot e^{3t} \cdot y'' + \cos(t) y = 3e^t$

$\Rightarrow$
$a_3(t) = 1$
$a_2(t) = 2\sin(t) e^{3t}$
$a_1(t) = 0$
$a_0(t) = \cos(t)$
$g(t) = 3e^t$

$\rightsquigarrow$ $1 \cdot y''' + (2\sin(t) e^{3t}) y'' + 0 \cdot y' + \cos(t) y = 3e^t$ $\rightsquigarrow$ linear

3) $\left(\frac{d^3y}{dt^3}\right)^2 + y = 0$ $\rightsquigarrow$ non-linear as have $(y''')^{2}$

4) $\frac{d^3y}{dt^3} - e^{ty}\frac{dy}{dt} = \frac{d^5y}{dt^5}$ $\rightsquigarrow$

$\rightsquigarrow$ $y^{(5)} - y^{(3)} + e^{ty} y' = 0$ $\rightsquigarrow$ non-linear

depends on t and $\underline{y}$

Homogeneous vs Nonhomogeneous

Def. A linear differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

is homogeneous if $g(t) = 0$ for any t, i.e. equations of form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = 0$$

Otherwise, the linear equation is nonhomogeneous

Example:

1) $y'' + y' + y = t$ – linear but nonhomogeneous because $g(t) = t$.

2) $\underbrace{\sin(t)}_{a_2(t)} y'' + \underbrace{(1-t^2)}_{a_1(t)} y' + \underbrace{\cos(t)}_{a_0(t)} y = \underbrace{0}_{g(t)}$ – linear and homogeneous as $g(t) \equiv 0$ for every t.

Def. A solution to a differential equation is any function that satisfies it, i.e., if you plug in that function into equation, then you get true equality.

(5) Example

Consider a diff. eq.

$$(y')^2 - 5ty = 5t^2+1.$$

(a) Is $y(t) = -t$ a solution of the given equation?

$$y(t) = -t \to y'(t) = -1$$

when $y(t)=-t$

$$(-1)^2 - 5\cdot t\cdot(-t) = 1+5t^2 = 5t^2+1 \quad \text{true}$$

$\to$ $y(t) = -t$ is a solution.

(b) Is $y(t) = t^2$ a solution of the given equation?

$$y(t) = t^2 \to y'(t) = 2t$$

$$(2t)^2 - 5\cdot t\cdot t^2 = 4t^2 - 5t^3 \neq 5t^2+1 \rightsquigarrow$$

$\rightsquigarrow$ $y(t) = t^2$ is not a solution.

Lecture 2

① Second-order differential equations

Example Consider $y''-y=0$.

(1) Is $y(t)=e^t$ a solution of $y''-y=0$

$y(t)=e^t \to y'(t)=e^t \to y''(t)=e^t$

$e^t-e^t=0 \rightsquigarrow y(t)=e^t$ is a solution of $y''-y=0$

(2) Is $y(t)=2e^t$ a solution?

$y(t)=y'(t)=y''(t)=2e^t$

$2e^t-2e^t=0 \rightsquigarrow y(t)=2e^t$ is a solution of $y''-y=0$

(3) $y(t)=c_1e^t$, where c_1 is a constant is a solution

because $y''(t)=c_1e^t \rightsquigarrow \underbrace{c_1e^t}_{y''(t)}-\underbrace{c_1e^t}_{y}=0$ ✓

(4) Similarly $y(t)=c_2e^{-t}$, where c_2 is a constant is a solution because

$y'(t)=-c_2e^{-t} \to y''(t)=-c_2\cdot(-1)e^{-t}=c_2e^{-t}$

$\underbrace{c_2e^{-t}}_{y''(t)}-\underbrace{c_2e^{-t}}_{y(t)}=0$ ✓

②

(5) $y(t) = c_1 e^t + c_2 e^{-t}$, where c_1, c_2 - constant

is a solution

$$y''(t) = c_1 e^t + c_2 e^{-t} = y(t)$$

<u>Thm</u> If $y_1(t), y_2(t)$ are solutions of the linear <u>homogeneous</u> equation

$$y'' + p(t)y' + q(t)y = 0,$$

then so is $c_1 y_1(t) + c_2 y_2(t)$ for arbitrary constants c_1, c_2.

<u>Thm</u> If $y_1(t), y_2(t)$ are solutions of the linear <u>homogeneous</u> equation $y'' + p(t)y' + q(t)y = 0$ (v)

and $y_1(t) \neq \text{const} \cdot y_2(t)$ and $y_2(t) \neq \text{const} \cdot y_1(t)$,

$y_1(t)$ and $y_2(t)$ are linearly independent

Then $y(t) = c_1 y_1(t) + c_2 y_2(t)$, where c_1, c_2 - any constants, gives <u>all solutions</u> of (v)

(gives <u>general solution</u> of (v)

<u>Example</u> We showed that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions of $y'' - y = 0$ (linear homogeneous)

$e^t \neq \text{const} \cdot e^{-t}$ and $e^{-t} \neq \text{const} \cdot e^t$

Therefore, $y(t) = c_1 e^t + c_2 e^{-t}$ is a <u>general solution</u> of $y'' - y = 0$.

(3)

Linear Homogeneous equations with constant coefficients

(*) $ay'' + by' + cy = 0$, where

a, b, c – given constants

<u>Def.</u> The characteristic equation of (*) is
$ar^2 + br + c = 0$ ← quadratic equation on r.

Gives us two roots r_1, r_2 where

$$r_1 = \frac{-b + \sqrt{D}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{D}}{2a},$$

where $D = b^2 - 4ac$.

<u>Possible cases:</u>

1. $D > 0 \rightarrow$ two distinct real roots r_1, r_2
   $r_1 \neq r_2$

2. $D = 0 \rightarrow$ repeated real root
   $\left(r_1 = r_2 = r = -\frac{b}{2a}\right)$

3. $D < 0 \rightarrow$ two complex conjugate roots
   (explain later)

Case of two distinct real roots r_1, r_2

General solution of (*) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$ where c_1, c_2 – any constants.

Example

Find general solution of

$$y'' - 4y = 0.$$

Solution: 1. The characteristic equation is

$$r^2 - 4 = 0.$$

solve equation: $r^2 = 4 \ (= 2^2)$

$r_1 = 2$ and $r_2 = -2$

distinct real roots

2 General solution is

$$(**) \quad y(t) = c_1 e^{2t} + c_2 e^{-2t}$$

, where c_1, c_2 – any constants.

What does it mean to solve initial value problem

$$y'' - 4y = 0, \quad y(0) = 4, \quad y'(0) = 4 \ ?$$

Answer: Find solution of $y'' - 4y = 0$ such that $y(0) = 4$, $y'(0) = 4$; i.e., find c_1, c_2 ~~such that~~ in $(**)$ such that $y(0) = 4$, $y'(0) = 4$.

(5) Find c_1, c_2:

We have general solution of

$y'' - 4y = 0$ is $y(t) = c_1 e^{2t} + c_2 e^{-2t}$

$\rightsquigarrow$ $y(0) = c_1 + c_2 = 4$

$y'(t) = 2c_1 e^{2t} - 2c_2 e^{-2t} \rightsquigarrow y'(0) = 2c_1 - 2c_2 = 4$

Find c_1, c_2 s.t. $\begin{cases} c_1 + c_2 = 4 \\ 2c_1 - 2c_2 = 4 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & -2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -4 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right] \to \begin{cases} c_1 = 3 \\ c_2 = 1 \end{cases}$$

$\rightsquigarrow$ The solution of IVP $\begin{cases} y'' - 4y = 0 \\ y(0) = 4 \\ y'(0) = 4 \end{cases}$

is $y(t) = 3e^{2t} + e^{-2t}$

6

Example

(a) Solve the IVP:

$$y'' - y' - 6y = 0$$
$$y(0) = \alpha$$
$$y'(0) = 4 \quad , \text{ where } \alpha - \text{some number}$$

Solution: 1. Find general solution of
$$y'' - y' - 6y = 0.$$

The characteristic equation is:
$$r^2 - r - 6 = 0$$
$$D = (-1)^2 - 4 \cdot 1 \cdot (-6) = 1 + 24 = 25 = 5^2$$
$$r_1 = \frac{-(-1)+5}{2 \cdot 1} = 3 \quad \text{and} \quad r_2 = \frac{-(-1)-5}{2 \cdot 1} = -2$$

General solution of the equation.
$$y(t) = c_1 e^{3t} + c_2 e^{-2t}, \quad \text{where } c_1, c_2 - \text{any constant}$$

2. Find c_1, c_2 s.t. $y(0) = \alpha$ and $y'(0) = 4$

$$y(0) = c_1 + c_2 = \alpha$$
$$y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t} \to y'(0) = 3c_1 - 2c_2 = 4$$

$$\begin{cases} c_1 + c_2 = \alpha \\ 3c_1 - 2c_2 = 4 \end{cases} \to \left[\begin{array}{cc|c} 1 & 1 & \alpha \\ 3 & -2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & \alpha \\ 0 & -5 & 4-3\alpha \end{array}\right] \to$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & \alpha - \frac{3\alpha-4}{5} \\ 0 & 1 & \frac{3\alpha-4}{5} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2\alpha+4}{5} \\ 0 & 1 & \frac{3\alpha-4}{5} \end{array}\right] \to \quad c_1 = \frac{2\alpha+4}{5}, \quad c_2 = \frac{3\alpha-4}{5}$$

(7)

Therefore, $y(t) = \frac{2\alpha+4}{5} e^{3t} + \frac{3\alpha-4}{5} e^{-2t}$

(b) Find α such that $\lim_{t\to+\infty} y(t) = 0$ where $y(t)$ is the solution of the IVP in (a):

Notice that $\lim_{t\to+\infty} e^{3t} = +\infty$

$\lim_{t\to+\infty} e^{-2t} = 0$

We don't want to see e^{3t} to have $\lim_{t\to\infty} y(t) = 0$. Therefore, need $\frac{2\alpha+4}{5} = 0$

i.e. $\boxed{\alpha = -2}$

Lecture 3

Characteristic

$$ay'' + by' + cy = 0 \rightsquigarrow ar^2 + br + c = 0$$

① Repeated real roots:

$$r_1 = r_2 = r$$

General solution is

$$y(t) = c_1 e^{rt} + c_2 t e^{rt},$$ where c_1, c_2 - any constants

Example What is the general solution of

$$9y'' + 6y' + y = 0 \, ?$$

Solution: Characteristic equation:

$$9r^2 + 6r + 1 = 0$$
$$(3r+1)^2 = 0$$
$$3r + 1 = 0$$
$$r = -\frac{1}{3}$$

$$D = 6^2 - 4 \cdot 9 \cdot 1 = 36 - 36 = 0$$
$$r_1 = r_2 = \frac{-6}{2 \cdot 9} = -\frac{1}{3}$$

General solution is: $y(t) = c_1 e^{-\frac{1}{3}t} + c_2 t e^{-\frac{1}{3}t}$

(b) Find solution s.t. $y(0) = 2$ and $y'(0) = 0$:

$$y(0) = \boxed{c_1 = 2}$$

$$y'(t) = -\frac{1}{3} c_1 e^{-\frac{1}{3}t} + c_2 e^{-\frac{1}{3}t} - \frac{1}{3} c_2 t e^{-\frac{1}{3}t}$$

$$y'(0) = -\frac{1}{3} c_1 + c_2 = 0 \rightsquigarrow \boxed{c_2 = \frac{1}{3} c_1 = \frac{1}{3} \cdot 2 = \frac{2}{3}}$$

$$y(t) = 2e^{-\frac{1}{3}t} + \frac{2}{3} t e^{-\frac{1}{3}t}$$

(2)

Complex conjugate roots

Example Find roots of characteristic equation for $y''+2y'+3y=0$.

Solution: Characteristic equation:

$$r^2+2r+3=0$$

$$D = 2^2-4\cdot1\cdot3 = 4-12 = -8$$

what is $\sqrt{-8}$?

Denote by i a complex number such that $i^2=-1$. (imaginary unit)

Then $-8 = i^2\cdot 8$ and $\sqrt{-8} = \sqrt{i^2}\cdot\sqrt{8} = i\sqrt{8} =$

$= i\cdot 2\sqrt{2}$

Then, $r_1 = \dfrac{-2+i\cdot2\sqrt{2}}{2} = -1+i\sqrt{2}$

$r_2 = \dfrac{-2-i\cdot2\sqrt{2}}{2} = -1-i\sqrt{2}$

Numbers $\lambda+i\mu$ and $\lambda-i\mu$, where λ,μ — real are called complex conjugate numbers.

For example, $(-1)+i\sqrt{2}$ and $(-1)-i\sqrt{2}$ are complex conjugate. (Here $\lambda=-1$ and $\mu=\sqrt{2}$)

What is general solution in case of complex conjugate roots $\lambda \pm i\mu$?

Answer: $y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$, where c_1, c_2 – any constants

Example Find general solution of

$$y'' + 2y' + 3y = 0$$

Solution: We found that the roots of characteristic equation are

$$r_{1,2} = -1 \pm i\sqrt{2}$$

Therefore, the general solution is

$$y(t) = c_1 e^{-t}\cos(\sqrt{2}t) + c_2 e^{-t}\sin(\sqrt{2}t).$$

Example (a) Solve IVP

$$y'' + 4y' + 13y = 0$$

$$y(0) = 3$$

$$y'(0) = -9$$

Solution: 1) Characteristic equation

$$r^2 + 4r + 13 = 0$$

$$D = 4^2 - 4\cdot 1\cdot 13 = 4\cdot(4-13) = -4\cdot 9 = -6^2$$

(4)

$$r_{1,2} = \frac{-4 \pm i\cdot 6}{2\cdot 1} = \underset{\lambda}{\underset{\shortparallel}{-2}} \pm \underset{\mu}{\underset{\shortparallel}{3}}\, i$$

2) General solution

$$y(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t),$$

where c_1, c_2 - constants.

3) Find c_1, c_2 s.t. $y(0)=3$ and $y'(0)=-9$

$$y(0) = c_1 e^{-2\cdot 0}\overset{1}{\overset{\shortparallel}{\cos(3\cdot 0)}} + c_2 e^{-2\cdot 0}\overset{0}{\overset{\shortparallel}{\sin(3\cdot 0)}} =$$

$$= \boxed{c_1 = 3}$$

$$y'(t) = -2c_1 e^{-2t}\cos(3t) - 3c_1 e^{-2t}\sin(3t) -$$

$$-2c_2 e^{-2t}\sin(3t) + 3c_2 e^{-2t}\cos(3t) =$$

$$= (3c_2 - 2c_1)e^{-2t}\cos(3t) - (2c_2 + 3c_1)e^{-2t}\sin(3t)$$

$$y'(0) = 3c_2 - 2c_1 = -9$$

$$c_1 = 3 \Rightarrow 3c_2 = -9 + 2\cdot 3 = -3 \Rightarrow$$

$$\Rightarrow c_2 = -1$$

Solution of IVP is $y(t) = 3e^{-2t}\cos(3t) - e^{-2t}\sin(3t)$.

(b) What is $\lim_{t\to+\infty} y(t)$, where $y(t)$ is the solution of IVP in (a)?

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty}\left(3e^{-2t}\cos(3t) - e^{-2t}\sin(3t)\right) = 0$$ as $e^{-2t} \xrightarrow[t\to+\infty]{} 0$ and $-1 \le \sin(3t) \le 1$, $-1 \le \cos(3t) \le 1$