Global Rational Approximations of Functions With Factorially Divergent Asymptotic Series

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Contents

Sales Pitch: Some new results from our paper

- The exponential integral $\mathrm{Ei}^+(x) = \mathrm{e}^x \int_0^{\infty e^{i0-}} \frac{\mathrm{e}^{-px}}{1-p} dp$ can be written as a uniformly convergent series of Lerch Φ functions:

$$\mathrm{e}^{-x}\mathrm{Ei}^+(x) = -\Phi(-1, 1, \tfrac{x}{i\pi}) + \sum_{k=1}^{\infty} \Phi(-\mathrm{e}^{\pi i/2^k}, 1, 2^k \tfrac{x}{i\pi}) \quad \text{for } x \in \mathbb{C}\backslash i(-\infty, 0] \tag{1}$$

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- The function $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ satisfies the identity

$$\Psi(x+1) = \ln x - \frac{1}{2}\sum_{k=0}^{\infty}\left[\Psi\left(2^k x + 1\right) - \Psi\left(2^k x + \tfrac{1}{2}\right)\right] \tag{2}$$

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- Given a self-adjoint (bounded or unbounded) operator A defined on a Hilbert space:

$$(A - i\lambda)^{-1} = i\sum_{j=0}^{\infty} \mathrm{e}^{-j\lambda} U_j - i \lim_{\ell\to\infty} \sum_{k=1}^{\ell}\sum_{j=0}^{\infty} (-1)^j 2^{-k} \mathrm{e}^{-j\lambda/2^k} U_{j2^{-k}} \quad , \text{ for } \lambda > 0 \tag{3}$$

where convergence holds in the strong operator topology.

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Divergent Series

- When solving a physical problem by say perturbation theory, one is often left with a series whose radius of convergence is equal to 0.
- Do these coefficients still carry physical meaning?
- Yes!
- In fact in 1952, F. Dyson gave a physical argument that any perturbative expansion obtained in quantum electrodynamics is necessarily divergent [15] .
- What has been realized is that such series do not sum in the ordinary sense of the word. But rather, sum in the Borel sense and more generally the Borel-Écalle sense of summation.

A Bit About Borel Summation

- A key ingredient in Borel and Borel-Écalle summation is the Laplace transform $\mathcal{L}$ and in fact a whole family of operators $\mathcal{L}_\theta$ where $\theta \in [0, 2\pi]$ is the direction of the contour of integration.

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- Assume a function F is analytic in a neighborhood of 0.
- Recall the definition of the Borel transform $\mathcal{B}$ which (for us) acts on the space of formal power series in $1/x$ without constant terms i.e. $x^{-1}\mathbb{C}[\![x^{-1}]\!]$ by the rule

$$\mathcal{B}x^{-n} = \frac{p^{n-1}}{(n-1)!}, \quad n \in \mathbb{N}$$

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- $\mathcal{B}$ is related to the formal inverse Laplace transform $\mathcal{L}^{-1}$. However, $\mathcal{L}^{-1}x^{-n}$ produces an actual function whereas $\mathcal{B}x^{-n}$ is still a formal object.
- Furthermore, assume the germ of F around 0 was obtained by summing **(now in the usual sense of the word)** the Borel transform $\mathcal{B}$ of a formal (possibly factorially divergent) series $\widetilde{f}$ i.e. $F(p) = (S \circ \mathcal{B})\,\widetilde{f}$

Note

The complex p-plane is commonly referred to as the Borel plane

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- We call $f(x)$ the *Borel sum* of F along $\mathbb{R}^+$ and by Watson's lemma is related to the original formal series $\tilde{f}$ by $f(x) \sim \tilde{f}$ along rays as $|x| \to \infty$ $\Re x > 0$.

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$$\forall N \in \mathbb{N}, \quad f(x) - \sum_{k=0}^{N} \frac{c_k}{x^{k+1}} = o(x^{-N-1}) \quad \text{as} \quad |x| \to \infty, \quad \Re x > 0 \tag{5}$$

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- Reaclling $f(x) = \mathcal{L}_0 F(x)$
- f will a priori be holomorphic in a half plane of the form $\{x \in \mathbb{C} : \Re x > \nu\}$.
- We may deform the contour of integration from $\gamma_0 = \mathbb{R}^+$ to $\gamma_\theta = \mathrm{e}^{i\theta}\mathbb{R}^+$ provided F is *Borel summable* along γ_φ for all $\varphi \in (0, \theta]$. This amounts to analytic continuation of $\mathcal{L}_0 F$ to $\mathcal{L}_\theta F$.

A Bit About Borel Summation

- By Watson's lemma [31] or IBP, the asymptotic series of f for large $\Re x$ is related to the Maclaurin series of F :

$$f(x) = \frac{1}{x}F(0) + \frac{1}{x^2}F'(0) + \cdots + \frac{1}{x^n}F^{(n-1)}(0) + \frac{1}{x^n}\int_0^\infty F^{(n)}(p)\mathrm{e}^{-px}\,dp \quad (6)$$

- By Cauchy's theorem, the growing powers of $\frac{d}{dp}$ lead to factorial divergence of the asymptotic series of f, unless F is entire (rarely the case in applications).

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- In fact, the location and type of singularities in the Borel plane carry a lot of information about the problem, their existence implies the presence of Stokes phenomena.
- Each singular point of F corresponds to a Stokes direction of f.

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- Integration by parts gives the *factorial* expansion

$$f(x) = \varphi(1)\,\frac{1}{x} - \varphi'(1)\,\frac{1}{(x)_2} + \cdots + \frac{(-1)^{n-1}}{(x)_n}\,\varphi^{(n-1)}(1) + \frac{(-1)^n}{(x)_n}\int_0^1 s^{x+n-1}\varphi^{(n)}(s)ds \tag{8}$$

where $(x)_k := x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}$, $(x)_k$ are the Pochhammer symbols.

Classical Factorial Expansions

- Without remainder, we have the factorial series, (a formal series, for now)

$$\tilde{\varphi}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi^{(k)}(1)}{(x)_{k+1}} \tag{9}$$

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Note

Since F is analytic at zero, φ is analytic at one. Using Stirling's formula in (9), we see that, for large k, the $(k+1)$'st term of the expansion (9) behaves like

$$(-1)^k \, \Gamma(x) \, \frac{\varphi^{(k)}(1)}{k!} \, k^{-x} \tag{10}$$

Due to the $1/k!$ factor in (10) the series $\tilde{\varphi}(x)$ can converge even if the asymptotic power series obtained from (6) is factorially divergent.

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  2. Limited domain of convergence (usually unrelated to the function one is trying to represent): a half plane.
- The boundary of this half plane is separated by a positive angular distance from the important antistokes rays
- Our new method of generating factorial expansions developed in [9] remedies these two issues giving rise to expansions which converge geometrically in a cut plane.

What is necessary for geometric convergence?

- Recall that we have:

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- For geometric convergence, it is necessary for φ to be analytic in a domain containing the closed disk $\overline{\mathbb{D}_1(1)}$.
- In applications φ is often singular at $s = 0$.

Building geometric convergence into factorial expansions

- We draw on a classical function of three complex variables; the Lerch transcendent.

$$\Phi(z,s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{p^{s-1}e^{-xp}}{1-ze^{-p}}dp, \qquad \Re s > 0, \quad \Re x > 0, \quad z \in \mathbb{C}\backslash[1,\infty) \tag{12}$$

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- For our purposes we are interested in fixing the second parameter $s = 1$ and use the change of variables $u = \mathrm{e}^{-p}$ to obtain

$$\Phi(z,1,x) = \int_0^1 \frac{u^{x-1}}{1-zu} du \tag{13}$$

Building geometric convergence into factorial expansions

- We use the notation $\mathbb{Z}_-$ to denote the non-positive integers.

Lemma

For $|z| < 1$ *and* $x \in \mathbb{C} \backslash \mathbb{Z}_-$ *we have*

$$\Phi\left(\frac{z}{z-1}, 1, x\right) = (1-z) \sum_{j \geqslant 0} z^j \frac{j!}{(x)_{j+1}} \tag{14}$$

Moreover, the series converges absolutely and geometrically in x.

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- It is (14) which provides a prototype for geometrically convergent factorial series.
- The Lerch transcendent will be the building block for our factorial expansions; its presence almost appears native to these problems.

Dyadic Decomposition

Lemma (*Dyadic identity*)

The following identity holds in $\mathbb{C}$:

$$\frac{1}{p} - \left(\frac{1}{1 - \mathrm{e}^{-p}} - \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + \mathrm{e}^{-p/2^k}} \right) = 0 \tag{15}$$

as the left hand side in (15) has only removable singularities.
We have

$$\frac{1}{p} = \frac{1}{1 - \mathrm{e}^{-p}} - \sum_{k=1}^{n} \frac{2^{-k}}{1 + \mathrm{e}^{-p/2^k}} + \rho_{n+1}(p) \tag{16}$$

where

$$\rho_{n+1}(p) = \frac{1}{2^n} \left(\frac{1}{p/2^n} - \frac{1}{1 - \mathrm{e}^{-p/2^n}} \right) \tag{17}$$

as an equality of meromorphic functions.

Sketch of the dyadic decomposition proof

Note

The set of poles within the summation seen in (15) *is equal to* $2\pi i\mathbb{Z}\backslash\{0\}$ *which corresponds to the set of poles of* $(1-\mathrm{e}^{-p})^{-1}$ *with the exception of* $p=0$. *The pole at the origin is removed by the* $1/p$ *term.*

The proof is elementary:

$$\frac{1}{1-x}=\frac{2}{1-x^2}-\frac{1}{x+1}=\frac{4}{1-x^4}-\frac{2}{x^2+1}-\frac{1}{x+1}=\ldots=\frac{2^n}{1-x^{2^n}}-\sum_{j=0}^{n-1}\frac{2^j}{1+x^{2^j}} \tag{18}$$

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which implies, with $x=\mathrm{e}^{-p/2^n}$,

$$\frac{2^{-n}}{1-\mathrm{e}^{-\frac{p}{2^n}}}=\frac{1}{1-\mathrm{e}^{-p}}-\sum_{k=1}^{n}\frac{2^{-k}}{\mathrm{e}^{-\frac{p}{2^k}}+1} \tag{19}$$

which implies (17).

Dyadic Decomposition of the Cauchy Kernel

- Let $\beta \neq 0$. The linear affine transformation $p \to \beta p - \beta s$ gives the following generalization of Lemma 2 for the Cauchy kernel.

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Corollary (*Dyadic decomposition of the Cauchy kernel*)

Assume $\beta \neq 0$, then the following identity holds

$$\frac{1}{s-p} - \left(-\frac{\beta \mathrm{e}^{-\beta s}}{\mathrm{e}^{-\beta s} - \mathrm{e}^{-\beta p}} + \sum_{k=1}^{\infty} \frac{2^{-k} \beta \mathrm{e}^{-\beta s/2^k}}{\mathrm{e}^{-\beta s/2^k} + \mathrm{e}^{-\beta p/2^k}} \right) = 0 \tag{20}$$

as the left hand side in (20) has only removable singularities.
We have

$$\frac{1}{s-p} = -\frac{\beta \mathrm{e}^{-\beta s}}{\mathrm{e}^{-\beta s} - \mathrm{e}^{-\beta p}} + \sum_{k=1}^{n} \frac{2^{-k} \beta \mathrm{e}^{-\beta s/2^k}}{\mathrm{e}^{-\beta s/2^k} + \mathrm{e}^{-\beta p/2^k}} + \rho_{n+1,\beta}(p,s) \tag{21}$$

- Notice that the set poles within the summation for this case is equal to $\frac{2\pi i}{\beta}\mathbb{Z}\backslash\{0\}$; a rotated version of the pole lattice from the expansion of $1/p$.

Exponential Integral Decomposition

- Consider the exponential integral $\mathrm{Ei}^+(x) = \mathrm{e}^x \int_0^{\infty e^{i0-}} \frac{\mathrm{e}^{-px}}{1-p} dp$, Lemma 3 with $\beta = \pi i$ and $s = 1$ we integrate term by term.

Exponential Integral Decomposition

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$$-\int_0^1 \frac{u^{\frac{x}{\pi i}-1}}{1+u} du + \sum_{k=1}^{\infty} \int_0^1 \frac{u^{\frac{x2^k}{\pi i}-1}}{1 + \mathrm{e}^{\pi i/2^k} u} du \qquad (23)$$

- We recognize each of these as Lerch Φ functions

$$-\Phi(-1, 1, \tfrac{x}{i\pi}) + \sum_{k=1}^{\infty} \Phi(-\mathrm{e}^{\pi i/2^k}, 1, 2^k \tfrac{x}{i\pi}) \qquad (24)$$

Exponential Integral Decomposition

- Using the factorial expansion of Φ from Lemma 1 we obtain

$$\mathrm{e}^{-x}\mathrm{Ei}^{+}(x) = -\sum_{m=1}^{\infty}\frac{\Gamma(m)}{2^m}\frac{1}{(y)_m}+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Gamma(m)\mathrm{e}^{-i\pi/2^k}}{(1+\mathrm{e}^{-i\pi/2^k})^m}\frac{1}{(2^k y)_m} \qquad (y=-ix/\pi) \tag{25}$$

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Note

We have expansions similar to (25) for the Airy function in [9] and the method easily extends to Bessel functions J_ν, Y_ν.

Exponential Integral Approximation Errors

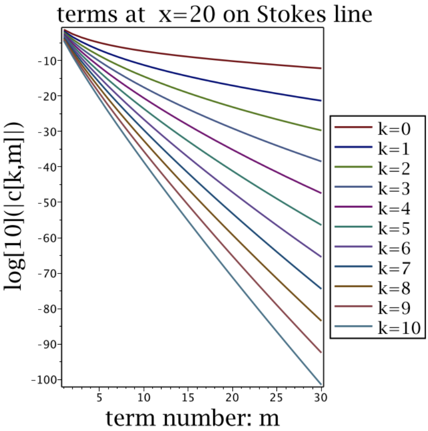

Figure: Size of terms in the successive series on the Stokes ray $\mathbb{R}^+$ with the formula (25). This plot can be used to determine the number of terms to be kept for a given accuracy. To get 10^{-5} accuracy, 10 terms of the first series plus 5 from the second (with $k = 1$).

Stokes Transition of the Exponential Integral Approximation

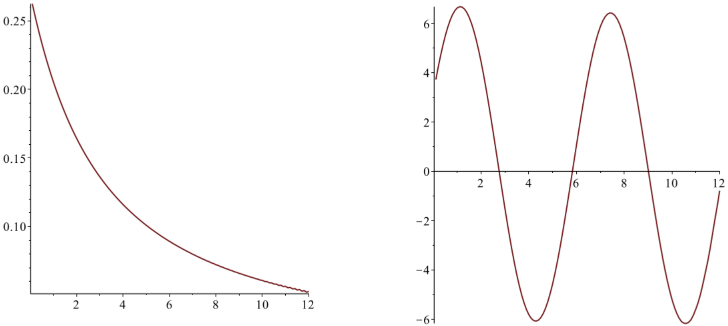

Figure: The classical Stokes transition of Ei^{+} from asymptotically decaying to oscillatory.

- These plots were generated from the rational approximation on either side of the cut $i(-\infty, 0]$. It is quite remarkable that the Stokes phenomenon can be observed through the lens of rational functions!

Dyadic Series For Function Elements

Definition

A function F is called a *function element* if it is analytic at the origin and in a domain of the form $\mathscr{D} = \mathbb{C} \backslash l_\omega$ where l_ω is a half-line originating at $\omega \in \mathbb{C} \backslash \{0\}$ i.e. a cut plane. Furthermore, F is assumed to decay in $\mathscr{D}$ as $|p| \to \infty$.

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Hypotheses of our main theorem:

- Let $\beta \neq 0$ so that $\beta = |\beta| e^{ib}$ with $b \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and θ the angle in the right half plane so that $b + \theta = \pi (\text{mod } 2\pi)$.
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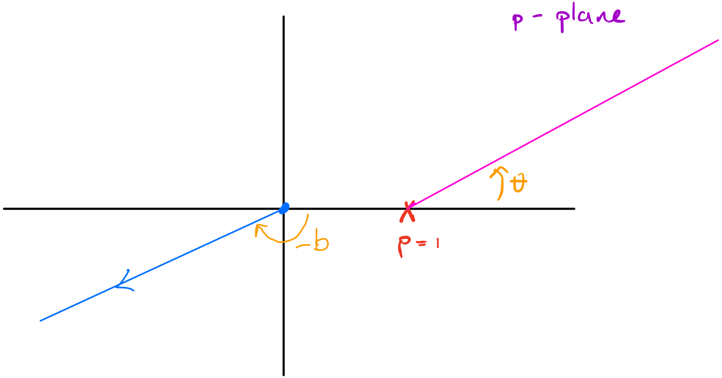
Let f be the Laplace transform of F given by

$$f(x) = \int_0^{\infty e^{-ib}} e^{-xp} F(p)\, dp \qquad \text{for } \arg x \in \left(b - \frac{\pi}{2}, b + \frac{\pi}{2}\right) \tag{26}$$

Borel plane conditions



- : Contour of integration for Laplace transform
- : Branch cut : $1 + e^{i\theta}\mathbb{R}^+$

Dyadic Series For Function Elements

Theorem

- *Then* $f(x)$ *has the dyadic expansion, for all* $x \in \mathbb{C} \backslash e^{ib}(-\infty, 0]$,

$$f(x) = \sum_{m=1}^{n-1} \frac{(m-1)!}{(x/\beta)_m} d_{m,0} + \sum_{k=1}^{N-1} \sum_{m=1}^{\ell} \frac{(m-1)!}{(2^k x/\beta)_m} d_{m,k} + \mathcal{R}_{n,N,\ell}(x) \tag{27}$$

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- *Denoting by* $\Delta F(1 + te^{i\theta})$ *the branch jump of* F, $F(1 + te^{i\theta^+}) - F(1 + te^{i\theta^-})$ *and* $s = 1 + te^{i\theta}$, *the coefficients of the series have the expressions*

$$d_{m,0} = \frac{e^{i\theta}}{2\pi i} \int_0^\infty \Delta F(1 + te^{i\theta}) \frac{e^{\beta s(m-1)}}{(e^{\beta s} - 1)^m} dt \tag{28}$$

$$d_{m,k} = \frac{e^{i\theta}}{2\pi i} \int_0^\infty \Delta F(1 + te^{i\theta}) \frac{e^{\beta s(m-1)/2^k}}{(e^{\beta s/2^k} + 1)^m} dt \tag{29}$$

Dyadic Series For Function Elements

Theorem(continued)

The remainder term has the following closed form:

$$\mathcal{R}_{n,N,\ell}(x) = \frac{e^{i\theta}}{2\pi i} \int_0^\infty \Delta F(1+te^{i\theta}) \left(-\rho_{n,0}(x,t) + \sum_{k=1}^{N-1} \rho_{\ell,k}(x,t) + R_N(x,t) \right) dt \tag{30}$$

The quantities $\rho_{n,0}(x,t)$, $\rho_{\ell,k}(x,t)$ *and* $R_N(x,t)$ *admit geometric decay estimates in various asymptotic regimes of* n, N, ℓ. *(The precise asymptotics can be found in our paper). Moreover, convergence of* (27) *is uniform and geometric in* $\mathbb{C} \backslash e^{ib}(-\infty, 0]$.

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$$(A - i\lambda)^{-1} = i(1 - \mathrm{e}^{-\lambda}U_1)^{-1} - i\sum_{k=1}^{\infty} \frac{1}{2^k}(1 + e^{-\lambda/2^k}U_{2^{-k}})^{-1} \tag{31}$$

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and (3) *follows. Convergence holds in the strong operator topology. For $\lambda < 0$ one simply complex conjugates* (31).

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- In the present work, we use the normalization in which P_I takes the form

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- A reader familiar with resurgence will recognize (33) as the Écalle critical time associated with (32) which ensures the equation takes a form suitable for Borel-Écalle summation.

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- Each Ei has a rational approximation as previously discussed.
- On the following slides we show plots of the pointwise error of the approximation and a plot of the first 100 pole locations for a tritronquée.

Error of Ei Approximation of P_I

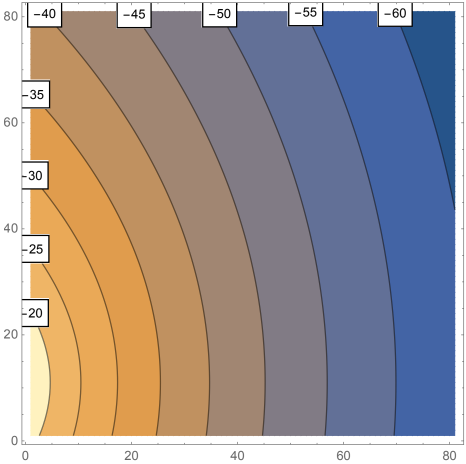

Figure: Modulus of error log-plot generated from a fifty exponential integral approximation. The x, y axes are the number of steps of size $\frac{1}{10}$ in the real and imaginary directions respectively starting at $1 - i$.

Pole locations of a tritronquée solution to P_I.

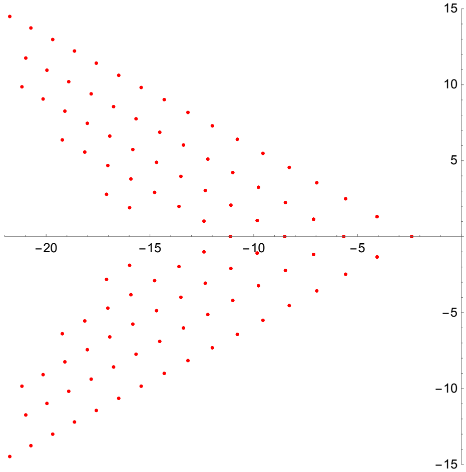

Figure: Pole locations of a tritronquée solution to P_I.

Thank you!

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