On domain properties of Bessel-type operators

Jonathan Stanfill (The Ohio State University)

Based on joint work with

Fritz Gesztesy, Michael M. H. Pang

Analysis and Operator Theory Seminar

Bessel-type operators studied

In a nutshell, our aim was to derive domain properties of Bessel-type operators associated with the singular second-order differential expressions of the form

$$\tau_{s_a,s_b} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2} + \frac{s_b^2 - (1/4)}{(x-b)^2} + q(x), \quad x \in (a,b),$$
$$s_a, s_b \in [0,\infty),\ q \in L^\infty((a,b);dx),\ q \text{ real-valued a.e. on } (a,b),$$

where $(a,b) \subset \mathbb{R}$ is a bounded interval.

Bessel-type operators studied

In a nutshell, our aim was to derive domain properties of Bessel-type operators associated with the singular second-order differential expressions of the form

$$\tau_{s_a,s_b} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2} + \frac{s_b^2 - (1/4)}{(x-b)^2} + q(x), \quad x \in (a,b),$$
$$s_a, s_b \in [0,\infty),\ q \in L^\infty((a,b);dx),\ q \text{ real-valued a.e. on } (a,b),$$

where $(a,b) \subset \mathbb{R}$ is a bounded interval.

The classical Bessel operator is associated with the expression having a singularity at only one endpoint and $q = 0$:

$$\tau_{s_a,1/2,q=0} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2}, \quad s_a \in [0,\infty),\ x \in (a,b).$$

Some motivation/background

Our interest in $L^2((a,b);dx)$-realizations of τ_{s_a,s_b} has its origins in the linear operators (more precisely, the Friedrichs extensions of $\tau_{0,0}\big|_{C_0^\infty((a,b))}$) underlying the following Hardy-type inequalities:

$$\int_a^b |f'(x)|^2\,dx \geqslant \frac{1}{4}\int_a^b \frac{|f(x)|^2}{x^2}\,dx, \quad f \in H_0^1((a,b)),$$

where $H_0^1((a,b))$ is the standard Sobolev space on (a,b) obtained upon completion of $C_0^\infty((a,b))$ in the norm of $H^1((a,b))$, that is,

$$H_0^1((a,b)) = \big\{g \in L^2((a,b);dx) \,\big|\, g \in AC([a,b]);\ g(a) = 0 = g(b);\\ g' \in L^2((a,b);dx)\big\}.$$

Some motivation/background

Our interest in $L^2((a,b);dx)$-realizations of τ_{s_a,s_b} has its origins in the linear operators (more precisely, the Friedrichs extensions of $\tau_{0,0}\big|_{C_0^\infty((a,b))}$) underlying the following Hardy-type inequalities:

$$\int_a^b |f'(x)|^2\,dx \geqslant \frac{1}{4}\int_a^b \frac{|f(x)|^2}{x^2}\,dx, \quad f \in H_0^1((a,b)),$$

$$\int_a^b |f'(x)|^2\,dx \geqslant \frac{1}{4}\int_a^b \frac{|f(x)|^2}{d_{(a,b)}(x)^2}\,dx, \quad f \in H_0^1((a,b)),$$

where $H_0^1((a,b))$ is the standard Sobolev space on (a,b) obtained upon completion of $C_0^\infty((a,b))$ in the norm of $H^1((a,b))$, that is,

$$H_0^1((a,b)) = \big\{g \in L^2((a,b);dx)\,\big|\, g \in AC([a,b]);\ g(a) = 0 = g(b); \\ g' \in L^2((a,b);dx)\big\}.$$

Some motivation/background

Our interest in $L^2((a,b);dx)$-realizations of τ_{s_a,s_b} has its origins in the linear operators (more precisely, the Friedrichs extensions of $\tau_{0,0}\big|_{C_0^\infty((a,b))}$) underlying the following Hardy-type inequalities:

$$\int_a^b |f'(x)|^2\,dx \geqslant \frac{1}{4}\int_a^b \frac{|f(x)|^2}{x^2}\,dx, \quad f\in H_0^1((a,b)),$$

$$\int_a^b |f'(x)|^2\,dx \geqslant \frac{1}{4}\int_a^b \frac{|f(x)|^2}{d_{(a,b)}(x)^2}\,dx, \quad f\in H_0^1((a,b)),$$

$$\begin{aligned}\int_a^b |f'(x)|^2\,dx \geqslant{}& \frac{\pi^2}{4(b-a)^2}\int_a^b \frac{|f(x)|^2}{\sin^2(\pi(x-a)/(b-a))}\,dx\\ &+\frac{\pi^2}{4(b-a)^2}\int_a^b |f(x)|^2\,dx, \quad f\in H_0^1((a,b)),\end{aligned}$$

where $H_0^1((a,b))$ is the standard Sobolev space on (a,b) obtained upon completion of $C_0^\infty((a,b))$ in the norm of $H^1((a,b))$, that is,

$$\begin{aligned}H_0^1((a,b)) = \big\{g\in L^2((a,b);dx)\,\big|\, g\in AC([a,b]);\ g(a)=0=g(b);\\ g'\in L^2((a,b);dx)\big\}.\end{aligned}$$

Proof idea from a related Bessel-type operator

One first proof of the refinement (taking $a = 0$, $b = \pi$ for simplicity)

$$\int_0^\pi dx\, |f'(x)|^2 \geqslant \frac{1}{4} \int_0^\pi \frac{|f(x)|^2}{\sin^2(x)}\, dx + \frac{1}{4} \int_0^\pi |f(x)|^2\, dx, \quad f \in H_0^1((0,\pi)),$$

and the optimality of both constants rested on the exact solvability of the one-dimensional Schrödinger equation with potential q_s, $s \in [0, \infty)$, given by

$$q_s(x) = \frac{s^2 - (1/4)}{\sin^2(x)}, \quad x \in (0, \pi),$$

which was first studied by Rosen and Morse in 1932, Pöschl and Teller in 1933, and Lotmar in 1935. A discussion of the underlying singular periodic problem on $\mathbb{R}$ was later presented by Scarf in 1958.

In particular, considering the differential expression

$$\tau_s - \frac{d^2}{dx^2} + \frac{s^2 - (1/4)}{\sin^2(x)}, \quad s \in [0, \infty),\ x \in (0, \pi),$$

one can show that two linearly independent solutions of $\tau_s u = 0$ are given by

In particular, considering the differential expression

$$\tau_s - \frac{d^2}{dx^2} + \frac{s^2 - (1/4)}{\sin^2(x)}, \quad s \in [0, \infty),\ x \in (0, \pi),$$

one can show that two linearly independent solutions of $\tau_s u = 0$ are given by

$$u_{0,s}(0,x) = [\sin(x)]^{(1+2s)/2} {}_2F_1\big((1/4) + (s/2), (1/4) + (s/2); 1 + s; \sin^2(x)\big), \qquad s \in [0,1),$$

Here ${}_2F_1(\,\cdot\,,\,\cdot\,;\,\cdot\,;\,\cdot\,)$ denotes the hypergeometric function.

In particular, considering the differential expression

$$\tau_s - \frac{d^2}{dx^2} + \frac{s^2 - (1/4)}{\sin^2(x)}, \quad s \in [0, \infty),\ x \in (0, \pi),$$

one can show that two linearly independent solutions of $\tau_s u = 0$ are given by

$$u_{0,s}(0,x) = [\sin(x)]^{(1+2s)/2} {}_2F_1\big((1/4) + (s/2), (1/4) + (s/2); 1 + s; \sin^2(x)\big), \\ s \in [0,1),$$

$$\widehat{u}_{0,s}(0,x) = \begin{cases} (2s)^{-1}[\sin(x)]^{(1-2s)/2} \\ \quad \times {}_2F_1\big((1/4) - (s/2), (1/4) - (s/2); 1 - s; \sin^2(x)\big), & s \in (0,1), \\ [\sin(x)]^{1/2} {}_2F_1\big(1/4, 1/4; 1; \sin^2(x)\big) \\ \quad \times \int_x^c [\sin(x')]^{-1} \big[{}_2F_1\big(1/4, 1/4; 1; \sin^2(x')\big)\big]^{-2}\, dx', & s = 0. \end{cases}$$

Here ${}_2F_1(\,\cdot\,, \cdot\,; \cdot\,; \cdot\,)$ denotes the hypergeometric function.

One can then use extension theory and spectral theory to show that

$$\left(-\frac{d^2}{dx^2}+\frac{s^2-(1/4)}{\sin^2(x)}-[(1/2)+s]^2 I\right)\bigg|_{C_0^\infty((0,\pi))}\geqslant 0, \quad s\in[0,\infty).$$

Thus, setting $s=0$ and integration by parts yields

$$\int_0^\pi |f'(x)|^2\,dx \geqslant \frac{1}{4}\int_0^\pi \frac{|f(x)|^2}{\sin^2(x)}\,dx+\frac{1}{4}\int_0^\pi |f(x)|^2\,dx, \quad f\in C_0^\infty((0,\pi)).$$

One then extends using the typical Fatou's lemma argument since $C_0^\infty((0,\pi))$ is dense in $H_0^1((0,\pi))$, while the optimality of the constants can be shown using oscillation theory.

Alternate proof

If one is primarily interested in the refined Hardy inequality, the following appears to be the quickest derivation:

Alternate proof

If one is primarily interested in the refined Hardy inequality, the following appears to be the quickest derivation: Introduce the factorization of $\tau_s - (s + (1/2))^2$,

$$\delta_s^+ \delta_s = \tau_s - (s + (1/2))^2 = -\frac{d^2}{dx^2} + \frac{s^2 - (1/4)}{\sin^2(x)} - (s + (1/2))^2, \ s \in [0, \infty), \ x \in (0, \pi),$$

where the differential expressions δ_s, δ_s^+ are given by

$$\delta_s = \frac{d}{dx} - [s + (1/2)] \cot(x), \ \delta_s^+ = -\frac{d}{dx} - [s + (1/2)] \cot(x).$$

Thus, $\delta_s^+ \delta_s \big|_{C_0^\infty((0,\pi))} \geqslant 0$, and taking $s = 0$, yields the inequality for $f \in C_0^\infty((0, \pi))$.

Bessel-type operators

We now return to

$$\tau_{s_a,s_b} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2} + \frac{s_b^2 - (1/4)}{(x-b)^2} + q(x), \quad x \in (a,b),$$
$$s_a, s_b \in [0,\infty),\ q \in L^\infty((a,b);dx),\ q \text{ real-valued a.e. on } (a,b),$$

where $(a,b) \subset \mathbb{R}$ is a bounded interval.

Bessel-type operators

We now return to

$$\tau_{s_a,s_b} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2} + \frac{s_b^2 - (1/4)}{(x-b)^2} + q(x), \quad x \in (a,b),$$
$$s_a, s_b \in [0,\infty),\ q \in L^\infty((a,b);dx),\ q \text{ real-valued a.e. on } (a,b),$$

where $(a,b) \subset \mathbb{R}$ is a bounded interval. The associated **maximal** and **preminimal** operators in $L^2((a,b);dx)$ are then defined by

$$\begin{aligned}
&T_{s_a,s_b,max} f = \tau_{s_a,s_b} f,\\
&f \in \operatorname{dom}(T_{s_a,s_b,max}) = \big\{g \in L^2((a,b);dx) \,\big|\, g, g' \in AC_{loc}((a,b));\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tau_{s_a,s_b} g \in L^2((a,b);dx)\big\},\\
&T_{s_a,s_b,min,0} f = \tau_{s_a,s_b} f,\\
&f \in \operatorname{dom}(T_{s_a,s_b,min,0}) = \big\{g \in \operatorname{dom}(T_{s_a,s_b,max}) \,\big|\, \operatorname{supp}(g) \subset (a,b) \text{ is compact}\big\}.
\end{aligned}$$

Limit circle/limit point

We recall that the celebrated Weyl alternative can be stated as follows:

Theorem (Weyl's Alternative)

Assume Hypothesis 1. Then the following alternative holds:

(i) **Limit circle:** For every $z \in \mathbb{C}$, all solutions u of $(\tau - z)u = 0$ are in $L^2((a, b); dx)$ near b (resp., near a).

(ii) **Limit point:** For every $z \in \mathbb{C}$, there exists at least one solution u of $(\tau - z)u = 0$ which is not in $L^2((a, b); dx)$ near b (resp., near a). In this case, for each $z \in \mathbb{C}\backslash\mathbb{R}$, there exists precisely one solution u_b (resp., u_a) of $(\tau - z)u = 0$ (up to constant multiples) which lies in $L^2((a, b); dx)$ near b (resp., near a).

Thus, boundary conditions are only needed at limit circle endpoints.

Limit circle/limit point for our example

One can show that τ_{s_a,s_b} is in the limit circle case at a (resp., b) if and only if $s_a^2 \in [0,1)$ (resp., $s_b^2 \in [0,1)$) and in the limit point case at a (resp., b) if and only if $s_a^2 \in [1,\infty)$ (resp., $s_b^2 \in [1,\infty)$).
For simplicity, we will always assume $s_a, s_b \in [0,\infty)$.

Generalized boundary values

Viewing q as a bounded perturbation (which does not influence operator domains) one can focus on the case $q = 0$ and hence obtains (for $c, d \in (a, b)$ with c sufficiently close to a and d sufficiently close to b) the solutions to $\tau_{s_a,s_b} u = 0$,

$$u_{a,s_a,1/2}(0, x; q = 0) = (x - a)^{(1/2)+s_a}, \ s_a \in [0, 1), \ x \in (a, c),$$

$$\widehat{u}_{a,s_a,1/2}(0, x; q = 0) = \begin{cases} (2s_a)^{-1}(x - a)^{(1/2)-s_a}, & s_a \in (0, 1), \\ (x - a)^{1/2}\ln(1/(x - a)), & s_a = 0, \end{cases} \quad x \in (a, c),$$

The solutions u are called the **principal** (or small) solutions while the solutions $\widehat{u}$ are called the **nonprincipal** (or large) solutions near each endpoint.

Generalized boundary values

Viewing q as a bounded perturbation (which does not influence operator domains) one can focus on the case $q = 0$ and hence obtains (for $c, d \in (a, b)$ with c sufficiently close to a and d sufficiently close to b) the solutions to $\tau_{s_a,s_b} u = 0$,

$$u_{a,s_a,1/2}(0, x; q = 0) = (x - a)^{(1/2)+s_a}, \ s_a \in [0, 1), \ x \in (a, c),$$

$$\widehat{u}_{a,s_a,1/2}(0, x; q = 0) = \begin{cases} (2s_a)^{-1}(x - a)^{(1/2)-s_a}, & s_a \in (0, 1), \\ (x - a)^{1/2}\ln(1/(x - a)), & s_a = 0, \end{cases} \quad x \in (a, c),$$

$$u_{b,1/2,s_b}(0, x; q = 0) = -(b - x)^{(1/2)+s_b}, \ s_b \in [0, 1), \ x \in (d, b),$$

$$\widehat{u}_{b,1/2,s_b}(0, x; q = 0) = \begin{cases} (2s_b)^{-1}(b - x)^{(1/2)-s_b}, & s_b \in (0, 1), \\ (b - x)^{1/2}\ln(1/(b - x)), & s_b = 0, \end{cases} \quad x \in (d, b).$$

The solutions u are called the **principal** (or small) solutions while the solutions $\widehat{u}$ are called the **nonprincipal** (or large) solutions near each endpoint.

Making the transition from $q = 0$ to $q \in L^\infty((a,b);dx)$, the principal and nonprincipal solutions will have the same leading behavior near $x = a, b$, with easy control over the remainder terms, so that for $g \in \operatorname{dom}(T_{s_a,s_b,max})$, one can define generalized boundary values in the limit circle cases

$$\begin{aligned}
\widetilde{g}(a) &= -W(u_{a,s_a,1/2}(0,\,\cdot\,),g)(a) \\
&= \begin{cases} \lim_{x\downarrow a} g(x)\big/\big[(2s_a)^{-1}(x-a)^{(1/2)-s_a}\big], & s_a \in (0,1), \\ \lim_{x\downarrow a} g(x)\big/\big[(x-a)^{1/2}\ln(1/(x-a))\big], & s_a = 0, \end{cases} \\
\widetilde{g}\,'(a) &= W(\widehat{u}_{a,s_a,1/2}(0,\,\cdot\,),g)(a) \\
&= \begin{cases} \lim_{x\downarrow a} \big[g(x) - \widetilde{g}(a)(2s_a)^{-1}(x-a)^{(1/2)-s_a}\big]\big/(x-a)^{(1/2)+s_a}, & s_a \in (0,1), \\ \lim_{x\downarrow a} \big[g(x) - \widetilde{g}(a)(x-a)^{1/2}\ln(1/(x-a))\big]\big/(x-a)^{1/2}, & s_a = 0. \end{cases}
\end{aligned}$$

The minimal operator can now be explicitly characterized by

$$\begin{aligned}&T_{s_a,s_b,min} f = \tau f,\\ &f \in \operatorname{dom}(T_{s_a,s_b,min}) = \big\{g \in \operatorname{dom}(T_{s_a,s_b,max}) \,\big|\, \widetilde{g}(a) = \widetilde{g}\,'(a) = \widetilde{g}(b) = \widetilde{g}\,'(b) = 0\big\},\end{aligned}$$

while the Friedrichs extension $T_{s_a,s_b,F}$ of $T_{s_a,s_b,min}$ now permits the particularly simple characterization in terms of the generalized boundary values,

$$T_{s_a,s_b,F} f = \tau f, \quad f \in \operatorname{dom}(T_{s_a,s_b,F}) = \big\{g \in \operatorname{dom}(T_{s_a,s_b,max}) \,\big|\, \widetilde{g}(a) = \widetilde{g}(b) = 0\big\}.$$

(If $x = a$ or $x = b$ is in the limit point case, the boundary conditions at that endpoint are dropped.)

First-order singular operators

Motivated by the second proof of the Hardy inequality refinement mentioned earlier, we introduce the differential expressions α_{s_a}, $\alpha_{s_a}^{+}$ by

$$\begin{aligned}
\alpha_{s_a} &= \frac{d}{dx} - \frac{s_a + (1/2)}{x-a} = (x-a)^{s_a+(1/2)} \frac{d}{dx} (x-a)^{-s_a-(1/2)}, \\
\alpha_{s_a}^{+} &= -\frac{d}{dx} - \frac{s_a + (1/2)}{x-a} = -\alpha_{-s_a-1}, \quad s_a \in \mathbb{R},\ x \in (a,b),
\end{aligned}$$

and one can confirm that

$$\tau_{s_a,1/2,q=0} = \alpha_{s_a}^{+} \alpha_{s_a} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2}, \quad s_a \in \mathbb{R},\ x \in (a,b).$$

One can then define maximal, preminimal, and minimal operators associated with α_{s_a} similarly to before (removing the derivative being absolutely continuous).

Theorem

Let $s_a \in \mathbb{R}$.

(i) For all $s_a \in \mathbb{R}\backslash\{0\}$,

$$\mathrm{dom}(A_{s_a,min}) = H_0^1((a,b)).$$

Theorem

Let $s_a \in \mathbb{R}$.

(i) For all $s_a \in \mathbb{R}\backslash\{0\}$,

$$\mathrm{dom}(A_{s_a,min}) = H_0^1((a,b)).$$

(ii) For $s_a = 0$, $H_0^1((a,b)) \subsetneqq \mathrm{dom}(A_{0,min})$.

Theorem

Let $s_a \in \mathbb{R}$.

(i) For all $s_a \in \mathbb{R}\backslash\{0\}$,

$$\mathrm{dom}(A_{s_a,min}) = H_0^1((a,b)).$$

(ii) For $s_a = 0$, $H_0^1((a,b)) \subsetneqq \mathrm{dom}(A_{0,min})$. Moreover,

$$\begin{aligned}\mathrm{dom}(A_{0,min}) = \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ f(a) = 0 = f(b);\\ \alpha_0 f \in L^2((a,b);dx)\big\},\end{aligned}$$

$$\begin{aligned}\mathrm{dom}(A_{0,max}) &= \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ \alpha_0 f \in L^2((a,b);dx)\big\}\\ &= \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ f(a) = 0;\\ &\qquad \alpha_0 f \in L^2((a,b);dx)\big\}.\end{aligned}$$

In fact, the boundary conditions can be replaced by

$$\lim_{x\downarrow a} \frac{f(x)}{[(x-a)\ln(1/(x-a))]^{1/2}} = 0, \quad \lim_{x\uparrow b} \frac{f(x)}{(b-x)^{1/2}} = 0.$$

Theorem continued

(iii) For $s_a \in (-\infty, -1] \cup (0, \infty)$ one has,

$$\begin{aligned}\operatorname{dom}(A_{s_a,max}) &= \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ \alpha_{s_a} f \in L^2((a,b);dx)\big\} \\ &= \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ f(a) = 0; \\ &\qquad \alpha_{s_a} f \in L^2((a,b);dx)\big\}.\end{aligned}$$

The boundary condition $f(a) = 0$ can be replaced by

$$\lim_{x \downarrow a} \frac{f(x)}{(x-a)^{1/2}} = 0.$$

Theorem continued

(iii) For $s_a \in (-\infty, -1] \cup (0, \infty)$ one has,

$$\begin{aligned}\mathrm{dom}(A_{s_a,max}) &= \big\{ f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ \alpha_{s_a} f \in L^2((a,b);dx) \big\} \\ &= \big\{ f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ f(a) = 0; \\ &\qquad\qquad \alpha_{s_a} f \in L^2((a,b);dx) \big\}.\end{aligned}$$

The boundary condition $f(a) = 0$ can be replaced by

$$\lim_{x \downarrow a} \frac{f(x)}{(x-a)^{1/2}} = 0.$$

(iv) For $s_a \in (-1, 0)$,

$$H_0^1((a,b)) = \mathrm{dom}(A_{s_a,min}) \subsetneq \mathrm{dom}(A_{s_a,max}).$$

Theorem continued

(iii) For $s_a \in (-\infty, -1] \cup (0, \infty)$ one has,

$$\begin{aligned}\mathrm{dom}(A_{s_a,max}) &= \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ \alpha_{s_a} f \in L^2((a,b);dx)\big\} \\ &= \big\{f \in L^2((a,b);dx) \,\big|\, f \in AC_{loc}((a,b));\ f(a) = 0; \\ &\qquad\qquad \alpha_{s_a} f \in L^2((a,b);dx)\big\}.\end{aligned}$$

The boundary condition $f(a) = 0$ can be replaced by

$$\lim_{x \downarrow a} \frac{f(x)}{(x-a)^{1/2}} = 0.$$

(iv) For $s_a \in (-1, 0)$,

$$H_0^1((a,b)) = \mathrm{dom}(A_{s_a,min}) \subsetneq \mathrm{dom}(A_{s_a,max}).$$

Remark about the proof: Our proof of these results relies on using Hardy-type inequalities to help with estimates, but different proof techniques are used in other papers to obtain similar results.

Question

By how much does $\mathrm{dom}(A_{0,min})$ miss out on coinciding with $H_0^1((a,b))$?

Question

By how much does $\mathrm{dom}(A_{0,min})$ miss out on coinciding with $H_0^1((a,b))$?

In fact, not by much as the following elementary consideration shows:

Question

By how much does $\mathrm{dom}(A_{0,min})$ miss out on coinciding with $H_0^1((a,b))$?

In fact, not by much as the following elementary consideration shows:

Suppose $0 \neq f \in \mathrm{dom}(A_{0,max})$ and denote $h = A_{0,max} f$. Then the first-order differential equation $(\alpha_0 f)(x) = h(x)$ for $x \in (a, a+1)$, say (i.e., assuming $b > a+1$ without loss of generality), implies

$$f(x) = C(x-a)^{1/2} - (x-a)^{1/2} \int_x^{a+1} (t-a)^{-1/2} h(t)\, dt, \quad x \in (a, a+1),$$

for some $C \in \mathbb{C}$. Thus a Cauchy estimate yields

$$|f(x)| \leqslant |C|(x-a)^{1/2} + (x-a)^{1/2}[\ln(1/(x-a))]^{1/2} \|h\|_{L^2((a,a+1);dt)},$$

hence $f(a) = 0$ and $f \in L^2((a, a+1); dx)$.

In addition,

$$f'(x) = 2^{-1}C(x-a)^{-1/2} - 2^{-1}(x-a)^{-1/2}\int_x^{a+1}(t-a)^{-1/2}h(t)\,dt + h(x),$$
$$x \in (a, a+1),$$

and the same Cauchy estimate implies

$$|f'(x)| \leqslant |C/2|(x-a)^{-1/2} + (x-a)^{-1/2}[\ln(1/(x-a))]^{1/2}\|h\|_{L^2((a,a+1);dt)}$$
$$+ |h(x)|, \quad x \in (a, a+1).$$

In addition,

$$f'(x) = 2^{-1}C(x-a)^{-1/2} - 2^{-1}(x-a)^{-1/2}\int_x^{a+1}(t-a)^{-1/2}h(t)\,dt + h(x),$$
$$x \in (a, a+1),$$

and the same Cauchy estimate implies

$$|f'(x)| \leqslant |C/2|(x-a)^{-1/2} + (x-a)^{-1/2}[\ln(1/(x-a))]^{1/2}\|h\|_{L^2((a,a+1);dt)}$$
$$+ |h(x)|, \quad x \in (a, a+1).$$

In particular, the possible failure of f' being L^2 near $x = a$ happens only on a logarithmic scale!

In addition,

$$f'(x) = 2^{-1}C(x-a)^{-1/2} - 2^{-1}(x-a)^{-1/2}\int_x^{a+1}(t-a)^{-1/2}h(t)\,dt + h(x),$$
$$x \in (a, a+1),$$

and the same Cauchy estimate implies

$$|f'(x)| \leqslant |C/2|(x-a)^{-1/2} + (x-a)^{-1/2}[\ln(1/(x-a))]^{1/2}\|h\|_{L^2((a,a+1);dt)}$$
$$+ |h(x)|, \quad x \in (a, a+1).$$

In particular, the possible failure of f' being L^2 near $x = a$ happens only on a logarithmic scale!

This line of ideas along with appropriate two-weight Hardy-type inequalities provide an alternate proof of some of the statements of the previous theorem.

Domain properties of Bessel-type operators

We now return to Bessel-type operators once more. Recall the differential expression

$$\tau_{s_a,s_b} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2} + \frac{s_b^2 - (1/4)}{(x-b)^2} + q(x), \quad x \in (a,b),$$
$$s_a, s_b \in [0,\infty),\ q \in L^\infty((a,b);dx),\ q \text{ real-valued a.e. on } (a,b),$$

where $(a,b) \subset \mathbb{R}$ is a bounded interval.

We would like to apply the previous theorem to better understand some domains for operators associated with the τ_{s_a,s_b}. We begin with the simpler case of the classic Bessel operator, that is, $s_b = 1/2$ and $q = 0$.

The previous theorem leads to a somewhat bewilderingly variety of seemingly different, yet obviously equivalent, characterizations of $T_{s_a,1/2,F,q=0}$ and $\operatorname{dom}\big(T^{1/2}_{s_a,1/2,F,q=0}\big)$ as follows (we note $T_{s_a,1/2,F,q=0} \geqslant \varepsilon I$ for some $\varepsilon > 0$):

The previous theorem leads to a somewhat bewilderingly variety of seemingly different, yet obviously equivalent, characterizations of $T_{s_a,1/2,F,q=0}$ and $\operatorname{dom}\left(T_{s_a,1/2,F,q=0}^{1/2}\right)$ as follows (we note $T_{s_a,1/2,F,q=0} \geqslant \varepsilon I$ for some $\varepsilon > 0$):

$$
\begin{aligned}
T_{s_a,1/2,F,q=0} &= A_{s_a,min}^* A_{s_a,min} = -A_{-s_a-1,max} A_{s_a,min} \\
&= T_{s_a,1/2,max,q=0}\big|_{\operatorname{dom}\left(T_{s_a,1/2,F,q=0}^{1/2}\right)} = T_{s_a,1/2,max,q=0}\big|_{\operatorname{dom}(A_{s_a,min})} \\
&= \begin{cases} T_{0,1/2,max,q=0}\big|_{\operatorname{dom}(A_{0,min})}, & s_a = 0, \\ T_{s_a,1/2,max,q=0}\big|_{H_0^1((a,b))}, & s_a \in (0,\infty), \\ T_{s_a,1/2,max,q=0}\big|_{\{f \in \operatorname{dom}(T_{s_a,1/2,max,q=0}) \,|\, \widetilde{f}(a)=0=f(b)\}}, & s_a \in [0,1), \\ T_{s_a,1/2,max,q=0}\big|_{\{f \in \operatorname{dom}(T_{s_a,1/2,max,q=0}) \,|\, f(b)=0\}}, & s_a \in [1,\infty), \end{cases}
\end{aligned}
$$

$$
Q_{T_{s_a,1/2,F,q=0}}(f,g) = (\alpha_{s_a} f, \alpha_{s_a} g)_{L^2((a,b);dx)},
$$

$$
f,g \in \operatorname{dom}(Q_{T_{s_a,1/2,F,q=0}}) = \operatorname{dom}(A_{s_a,min}), \quad s_a \in [0,\infty),
$$

$$
\begin{aligned}
\operatorname{dom}\left(T_{s_a,1/2,F,q=0}^{1/2}\right) &= \operatorname{dom}(Q_{T_{s_a,1/2,F,q=0}}) = \operatorname{dom}(A_{s_a,min}) \\
&= \begin{cases} \operatorname{dom}(A_{0,min}), & s_a = 0, \\ H_0^1((a,b)), & s_a \in (0,\infty). \end{cases}
\end{aligned}
$$

Moreover, the analog of the previous theorem (introducing an appropriate new factorization taking into account singularities at both endpoints) then leads to the following characterizations of $T_{s_a,s_b,F}$ and $\operatorname{dom}\big(|T_{s_a,s_b,F}|^{1/2}\big)$.

Theorem

Let $s_a, s_b \in [0,\infty)$, then,

$$\begin{aligned}
T_{s_a,s_b,F} &= A^*_{s_a,s_b,min}A_{s_a,s_b,min} + q - \widetilde{q} = A_{-s_a-1,-s_b-1,min}A_{s_a,s_b,min} + q - \widetilde{q} \\
&= T_{s_a,s_b,max}\big|_{\operatorname{dom}(|T_{s_a,s_b,F}|^{1/2})} = T_{s_a,s_b,max}\big|_{\operatorname{dom}(A_{s_a,s_b,min})} \\
&= \begin{cases} T_{0,0,max}\big|_{\operatorname{dom}(A_{0,0,min})}, & s_a = s_b = 0, \\ T_{0,s_b,max}\big|_{\operatorname{dom}(A_{0,s_b,min})}, & s_a = 0,\ s_b \in (0,\infty), \\ T_{s_a,0,max}\big|_{\operatorname{dom}(A_{s_a,0,min})}, & s_a \in (0,\infty),\ s_b = 0, \\ T_{s_a,s_b,max}\big|_{H_0^1((a,b))}, & s_a, s_b \in (0,\infty), \\ T_{s_a,s_b,max}, & s_a, s_b \in [1,\infty), \end{cases}
\end{aligned}$$

Theorem continued

and

$$Q_{T_{s_a,s_b,F}}(f,g) = (\alpha_{s_a,s_b} f, \alpha_{s_a,s_b} g)_{L^2((a,b);dx)} + \big(f, \big[q - \widetilde{q}\big] g\big)_{L^2((a,b);dx)},$$

$$f,g \in \operatorname{dom}(Q_{T_{s_a,s_b,F}}) = \operatorname{dom}(A_{s_a,s_b,min}), \quad s_a, s_b \in [0,\infty),$$

$$\operatorname{dom}\left(|T_{s_a,s_b,F}|^{1/2}\right) = \operatorname{dom}(Q_{T_{s_a,s_b,F}}) = \operatorname{dom}(A_{s_a,s_b,min})$$

$$= \begin{cases} \operatorname{dom}(A_{0,s_b,min}), & s_a = 0,\ s_b \in (0,\infty), \\ \operatorname{dom}(A_{s_a,0,min}), & s_a \in (0,\infty),\ s_b = 0, \\ H_0^1((a,b)), & s_a, s_b \in (0,\infty). \end{cases}$$

Connection to the confluent Heun differential equation

We start by relating $\tau_{s_a,s_b,q=0}u = zu$ to the confluent Heun differential equation,

$$w''(\xi) + \left(\frac{\gamma}{\xi} + \frac{\delta}{\xi - 1} + \varepsilon\right) w'(\xi) + \frac{\nu\xi - \mu}{\xi(\xi - 1)} w(\xi) = 0, \quad \gamma, \delta, \varepsilon, \mu, \nu, \in \mathbb{C}, \ \xi \in (0,1).$$

One observes that the confluent Heun differential equation has regular singularities at $\xi = 0, 1$ and an irregular singularity of rank 1 at $\xi = \infty$.

Connection to the confluent Heun differential equation

We start by relating $\tau_{s_a,s_b,q=0}u = zu$ to the confluent Heun differential equation,

$$w''(\xi) + \left(\frac{\gamma}{\xi} + \frac{\delta}{\xi - 1} + \varepsilon\right) w'(\xi) + \frac{\nu\xi - \mu}{\xi(\xi - 1)} w(\xi) = 0, \quad \gamma, \delta, \varepsilon, \mu, \nu, \in \mathbb{C}, \ \xi \in (0, 1).$$

One observes that the confluent Heun differential equation has regular singularities at $\xi = 0, 1$ and an irregular singularity of rank 1 at $\xi = \infty$. Eliminating the first-order term w' in the standard manner by introducing the change of dependent variable

$$w \mapsto v, \quad v(\xi) = e^{\varepsilon\xi/2}\xi^{\gamma/2}(\xi - 1)^{\delta/2} w(\xi), \ \xi \in (0, 1),$$

transforms the equation into the normal form

$$v''(\xi) + \left(A + \frac{B}{\xi} + \frac{C}{\xi - 1} + \frac{D}{\xi^2} + \frac{E}{(\xi - 1)^2}\right) v(\xi) = 0,$$

where

$$A = -\varepsilon^2/4, \quad B = [2\mu + (\delta - \varepsilon)]/2, \quad C = [2\nu - (\varepsilon + \gamma)\delta - 2\mu]/2,$$
$$D = (2 - \gamma)\gamma/4, \quad E = (2 - \delta)\delta/4.$$

The equivalence to $\tau_{s_a,s_b,q=0}u = zu$ is now easily seen by setting

$$A = z, \quad B = C = 0, \quad D = (1/4) - s_a^2, \quad E = (1/4) - s_b^2,$$

combined with the variable changes

$$(0,1) \ni \xi \mapsto (x-a)/(b-a), \quad v(\xi) = u(x), \quad x \in (a,b).$$

This shows that the "two-point" Bessel-type differential equation $\tau_{s_a,s_b,q=0}u = zu$ is a special case of the confluent Heun differential equation. (These considerations extend to a constant potential term $q(x) = q_0 \in \mathbb{R}$.)

The equivalence to $\tau_{s_a,s_b,q=0}u = zu$ is now easily seen by setting

$$A = z, \quad B = C = 0, \quad D = (1/4) - s_a^2, \quad E = (1/4) - s_b^2,$$

combined with the variable changes

$$(0,1) \ni \xi \mapsto (x-a)/(b-a), \quad v(\xi) = u(x), \quad x \in (a,b).$$

This shows that the "two-point" Bessel-type differential equation $\tau_{s_a,s_b,q=0}u = zu$ is a special case of the confluent Heun differential equation. (These considerations extend to a constant potential term $q(x) = q_0 \in \mathbb{R}$.)

Remark: From the original equation, the substitutions needed are

$$\gamma = 1 + 2s_a, \quad \delta = 1 - 2s_b, \quad \varepsilon = 2i(a-b)z^{1/2},$$
$$\mu = (1/2)(1+2s_a)\left[2i(a-b)z^{1/2} + 2s_b - 1\right], \quad \nu = 2i(a-b)z^{1/2}(1 + s_a - s_b),$$

which is clearly not an isospectral transformation!

Returning to the confluent Heun differential equation in the special case $\varepsilon = 0$, that is, $A = z = 0$, the resulting choice

$$\gamma = 1 + 2s_a, \quad \delta = 1 - 2s_b, \quad \varepsilon = 0,$$
$$\mu = (1/2)(1 + 2s_a)(2s_b - 1), \quad \nu = 0,$$

reduces the equation to the hypergeometric differential equation

$$w''(\xi) + \left(\frac{1+2s_a}{\xi} + \frac{1-2s_b}{\xi - 1}\right) w'(\xi) + \frac{(1+2s_a)(1-2s_b)}{2\xi(\xi-1)} w(\xi) = 0, \quad \xi \in (0,1).$$

Thus, the **principal** and **nonprincipal** solutions of $\tau_{s_a,s_b,q=0} u = 0$ can be explicitly expressed in terms of appropriate hypergeometric functions (not just powers!).

Final comments and questions

The Bessel potential has many applications in physics, and the Bessel-type potential at both endpoints provides a confining potential of this same type. So it would be natural to study the spectral theory of such operators in more detail.

What physical models are these operators related to?

What about studying Weyl m-functions and spectral functions (such as the spectral zeta function) associated with these operators?

We would then need to better understand certain properties of the underlying solutions to $\tau y = zy$, such as large and small asymptotics in the spectral parameter z.

Thanks!