Doubly periodic models of the Aztec diamond

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KU Leuven

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Outline of the Talk

1. Background on the Aztec diamond
2. Biased 2×2-periodic Aztec diamond
3. Algebraic properties of arctic curves
4. Matrix-valued orthogonal polynomials

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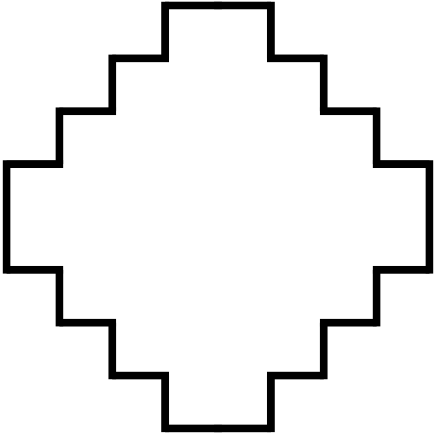
Tilings of the Aztec diamond

Goal: Tile the following region with 2×1 and 1×2 dominos:



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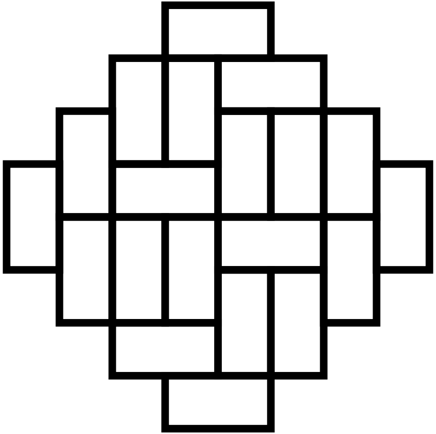
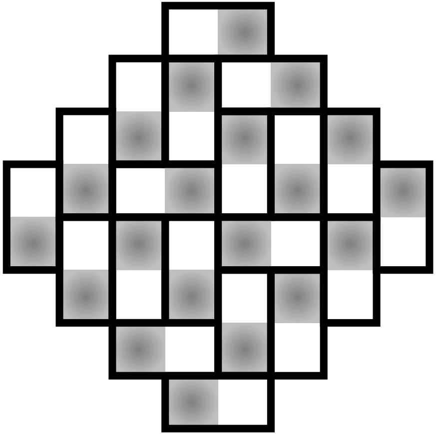
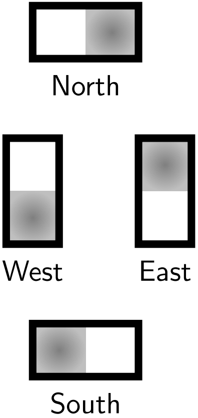
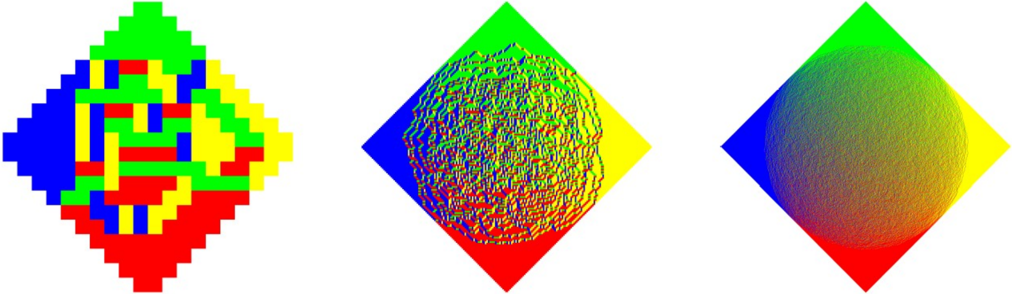


Types of dominos





Uniform tilings of Aztec diamonds



Random equidistributed tilings of size 10, 100, 1000 [taken from Debin, de Kemmeter, Ruelle '23]

[Jockusch, Propp, Shor '98]: **Arctic Circle Theorem**.

Tilings as non-intersecting paths

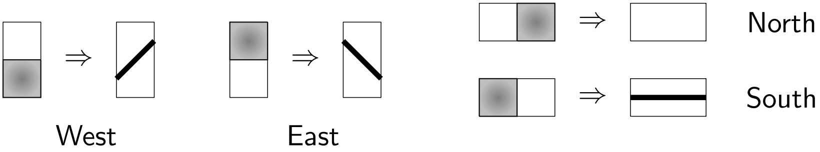


Figure: Line segments on the dominos.

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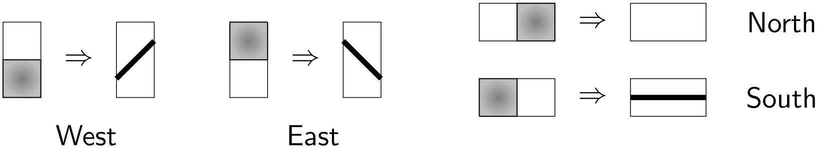

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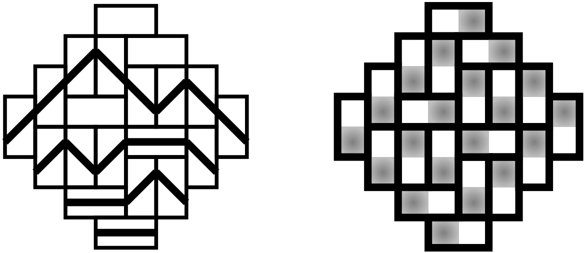

Figure: Non-intersecting paths on a tiled Aztec Diamond.

Fluctuations of the arctic curve

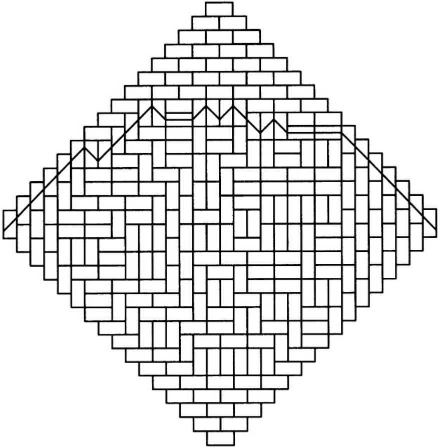

Figure: Upper path separating the frozen from the mixed region, [taken from Johansson '05]

Fluctuations of the arctic curve

Theorem (Johansson '05)

The upper most path, separating the frozen north region from the mixed region, converges to the **Airy process** *in the sense of convergence of finite-dimensional distributions.*

Corollary

The upper most path is distributed according to the **Tracy–Widom distribution** F_2.

Fluctuations of the arctic curve comt.

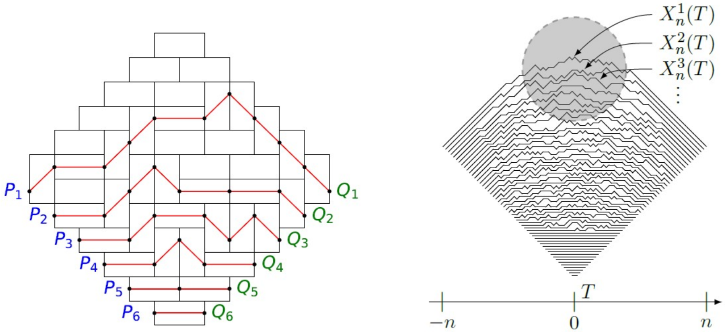

Figure: taken from Debin, de Kemmeter, Ruelle '23

Conjecture

It has been conjectured that the first n **upper most paths** *converge to the* **Airy line ensemble**, *see [Debin, de Kemmeter, Ruelle '23] for numerical evidence.*

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Aztec diamond as a dimer model

Domino tilings of the Aztec diamond are equivalent to **dimer configurations** on part of the square lattice.

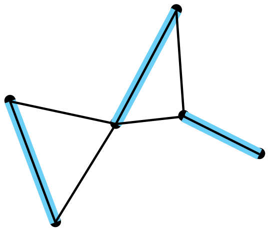
Dimer models

- graph: $G = (V, E)$,
- edge weights: $w\colon E \to \mathbb{R}_+$,
- dimer configurations:

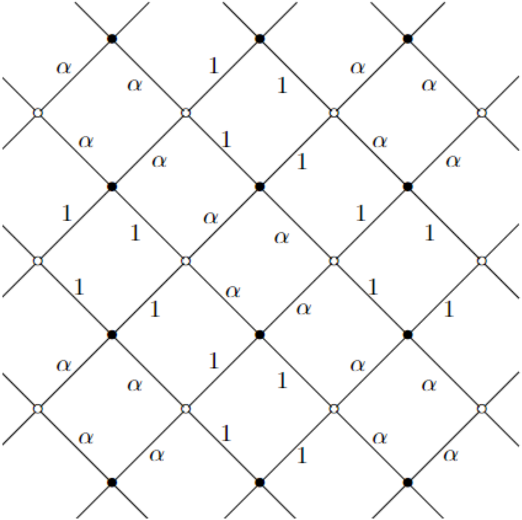
$$\mathcal{M} = \{M\colon M \subseteq E \text{ is a perfect matching (dimer configuration) of } G\}$$

$\Rightarrow$ **Dimer model**:

$$\text{Prob}(M) = \frac{\prod_{e \in M} w(e)}{\sum_{M' \in \mathcal{M}} \prod_{e' \in M'} w(e')}$$



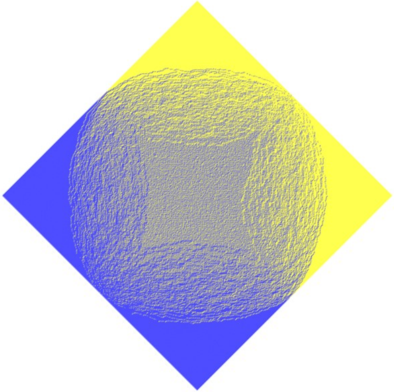
Periodic weightings (unbiased case)



Simplest model with frozen, rough and **smooth** phase,
see [Kenyon, Okounkov, Sheffield '06],

Tiling of a large 2×2-periodic Aztec diamond

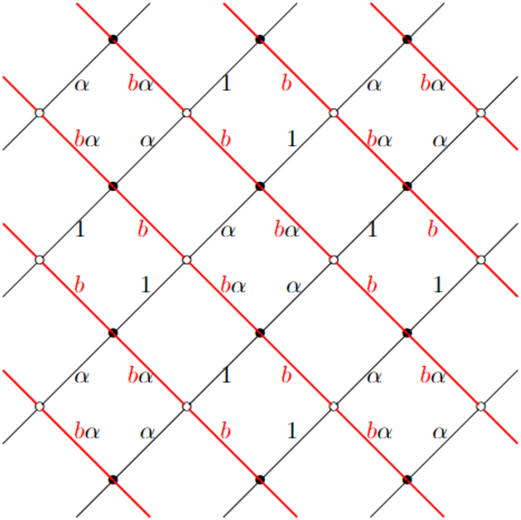
Tilings of large Aztec diamonds under the **unbiased** 2×2-periodic weighting exhibit three phases: **frozen, rough, smooth**.



Taken from Duits, Kuijlaars '21

- **frozen**: dominos are perfectly correlated (no randomness),
- **rough**: domino correlations decay quadratically with distance,
- **smooth**: domino correlations decay exponentially with distance.

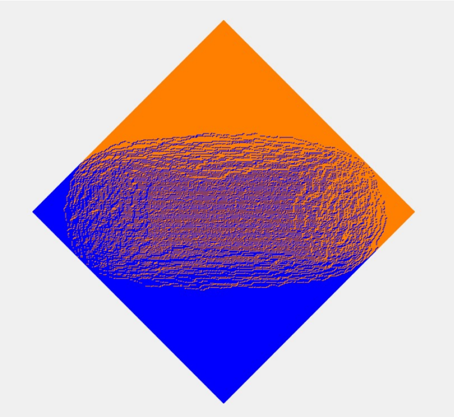
Periodic weightings (biased case)



- model introduced by [Borodin, Duits '23],
- the bias parameter $b > 1$ favors horizontal vs. vertical dominos
- is related to a linear flow on a genus-1 Riemann surface (see [Borodin, Duits '23] and [Chhita, Duits '23])

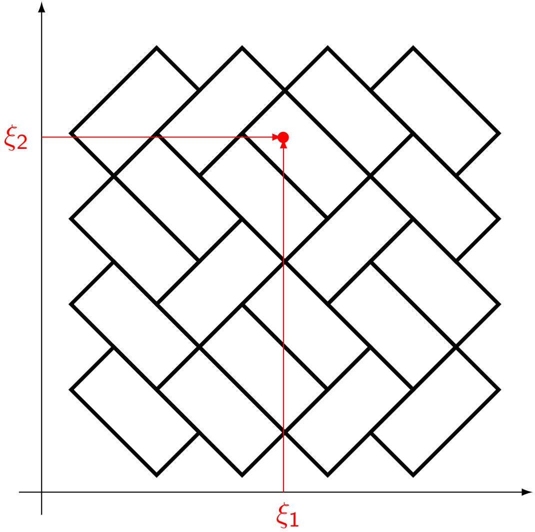
Tiling of a large biased 2×2-periodic Aztec diamond

Tilings of large Aztec diamonds under the **biased** 2×2-periodic weighting exhibit three phases: **frozen, rough, smooth**; but are more "flattened":

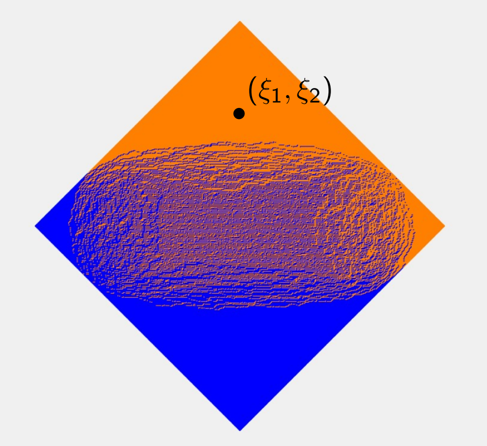


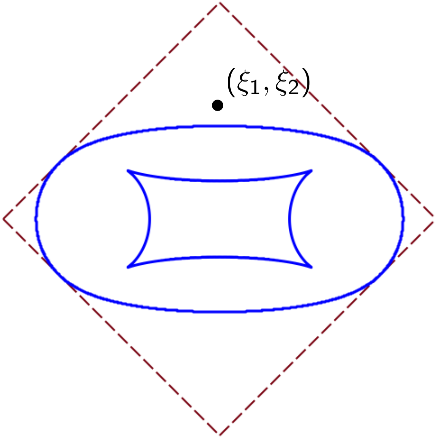
generated through code provided by Christophe Charlier

Global coordinates of the Aztec diamond



Global coordinates of the Aztec diamond cont.





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- What are the domino **fluctuations/correlations** away and near the arctic curve?
  $\Rightarrow$ Dominos give rise to a determinantal point process (see [Kenyon, '97]).
  $\Rightarrow$ What is the **correlation kernel**? (see [Beffara, Chhita, Johansson '18, '22])

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- What about more **general models of the Aztec diamond**?
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In this talk we will focus on **algebraic properties of the arctic curve**.

Techniques

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Towards the end we will mention the **matrix-valued orthogonal polynomials**.

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Aztec diamond and Riemann surfaces

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Aztec diamond and Riemann surfaces

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  $\Rightarrow$ Doubly periodic tiling models are related to **Riemann surfaces** (denoted by $\mathcal{R}$).
- For 2×2-periodic models of the Aztec diamond $\Rightarrow$ $\mathcal{R}$ is of **genus** 1.
- Local domino correlations at $(\xi_1, \xi_2) \in [-1, 1]^2$ are determined by the **location of zeros** of a **meromorphic differential** $d\mathbf{\Phi}_{\xi_1,\xi_2}$ on $\mathcal{R}$.

The meromorphic differential $\mathbf{d\Phi}_{\xi_1,\xi_2}$

Some facts about $\mathbf{d\Phi}_{\xi_1,\xi_2}$:

- $d\mathbf{\Phi}_{\xi_1,\xi_2} = \mathbf{\Phi}'_{\xi_1,\xi_2} dz$ is a meromorphic differential on $\mathcal{R}$.

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- $d\mathbf{\Phi}_{\xi_1,\xi_2}$ has four poles at fixed locations with residues

$$2(1+\xi_1),\ 2(1-\xi_1),\ -2(1+\xi_2),\ -2(1-\xi_2)$$

$\Rightarrow d\mathbf{\Phi}_{\xi_1,\xi_2}$ has **four zeros** on $\mathcal{R}$.

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  $\Rightarrow d\mathbf{\Phi}_{\xi_1,\xi_2}$ has **four zeros** on $\mathcal{R}$.
- location of the **four zeros** as a function of the coordinates $(\xi_1, \xi_2) \in [-1, 1]^2$ determines the **phase of the Aztec diamond**!
  [Duits, Kuijlaars 21'] via matrix-valued orthogonal polynomials;
  [Berggren 21'], [Borodin, Duits 23'], [Berggren, Borodin 23'] via Wiener–Hopf factorizations.

Frozen Region

$$\Phi'_{\xi_1,\xi_2}(s_j)dz = 0$$

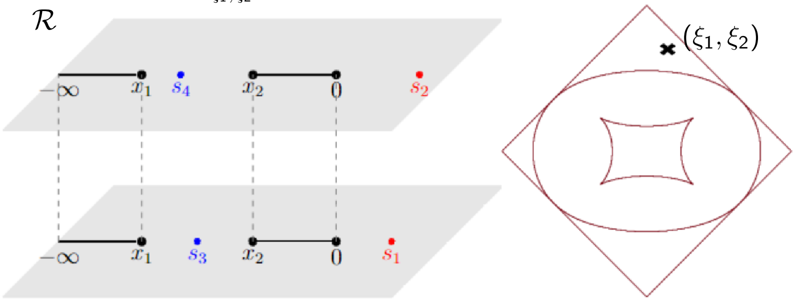

Figure: Location of the zeros for the frozen region

In the **frozen region** domino correlations are deterministic $\Rightarrow$ **No randomness!**

Rough Region

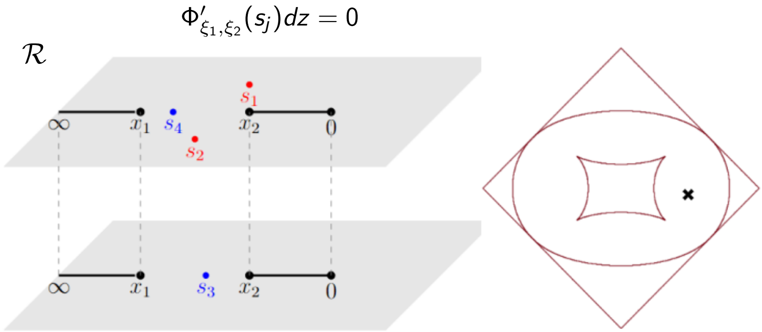

Figure: Location of the zeros for the rough region

In the **rough region** domino correlations decay **quadratically** with the distance.

Smooth Region

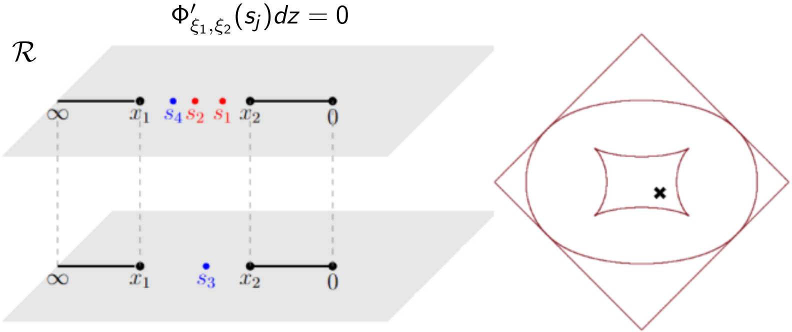

Figure: Location of the zeros for the smooth region

In the **smooth region** domino correlations decay **exponentially**.

Algebraic form of the arctic curve

The **arctic curve** separating the different regions corresponds to $d\boldsymbol{\Phi}_{\xi_1,\xi_2}$ having a **double zero**.

Theorem (Kuijlaars, P.)

The arctic curve of the biased 2×2-periodic Aztec diamond is an algebraic curve of degree 8 and can be written explicitly in terms of **Jacobi theta functions**.

This result should generalize to the $k \times \ell$-**periodic model** recently considered in [Berggren, Borodin 23'].

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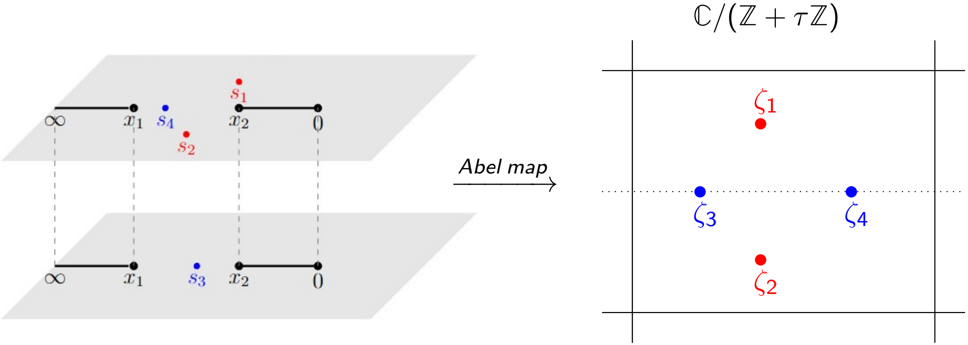
Jacobi theta function:

$$\Theta(z|\tau) := \sum_{k \in \mathbb{Z}} e^{(k^2\tau + 2kz)\pi i}, \quad z \in \mathbb{C}$$

Sum converges absolutely as $\mathrm{Im}(\tau) > 0$. We have:

$$\left.\begin{array}{ll} \text{periodicity} & \Rightarrow \Theta(z+1|\tau) = \Theta(z|\tau) \\ \text{quasi-periodicity} & \Rightarrow \Theta(z+\tau|\tau) = e^{-\pi i\tau - 2\pi i z}\Theta(z|\tau) \end{array}\right\} \textbf{multivalued on } \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$$

The Abel map



$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$
∞
x_1
s_4
s_1
s_2
x_2
0
∞
x_1
s_3
x_2
0
Abel map
ζ_1
ζ_3
ζ_4
ζ_2

Algebraic form of the arctic curve

The degree 8 polynomial $\mathcal{J}$ can be written in terms of Jacobi theta functions of the Riemann surface $\mathcal{R} \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$:

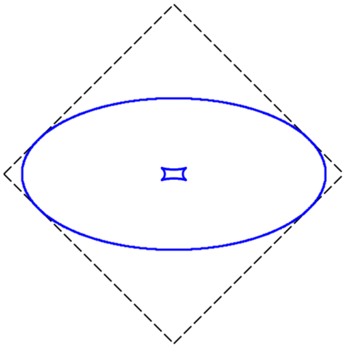
$$\mathcal{J}(\xi_1, \xi_2) = \frac{\prod_{i \neq j} \Theta(\zeta_i - \zeta_j - K \,|\, \tau)}{\left[\prod_{i,\ell} \Theta(\zeta_i - \nu_\ell - K \,|\, \tau)\right]^2} [(1 - \xi_1^2)(1 - \xi_2^2)]^2.$$

Here

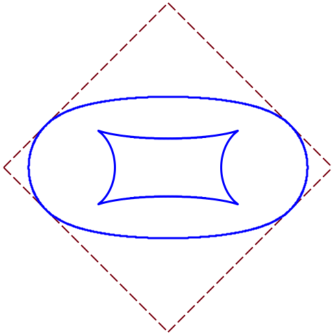
- $\zeta_i = \zeta_i(\xi_1, \xi_2)$ and ν_ℓ are the images of the zeros and poles of the meromorphic differential $d\Phi_{\xi_1,\xi_2}$ under the **Abel map**,
- Θ satisfies $\Theta(-K \,|\, \tau) = 0$, where $K = \frac{1}{2} + \frac{\tau}{2}$ is the **Riemann constant**.

We also have an expression in terms of b and α, but it is too long to fit here.

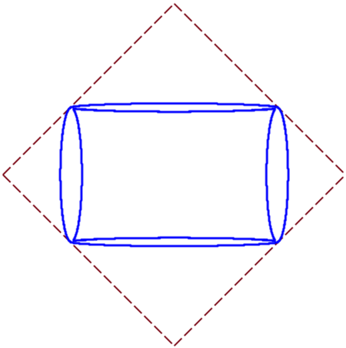
Gallery of arctic curves (increasing periodicity)



bias: $b = 2$; **periodicity**: $\alpha = 1.1$

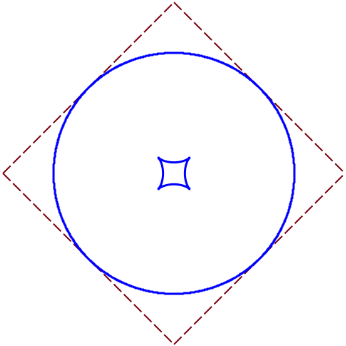


bias: $b = 2$; **periodicity**: $\alpha = 2$

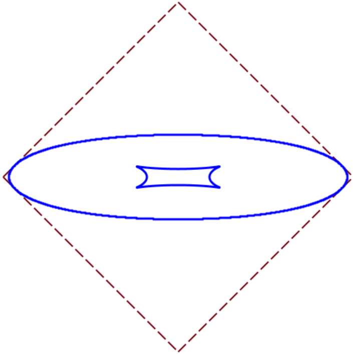


bias: $b = 2$; **periodicity**: $\alpha = 10$

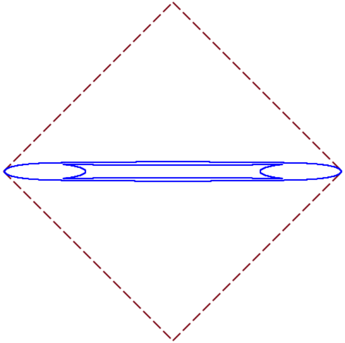
Gallery of arctic curves (increasing bias)



bias: $b = 1$; periodicity: $\alpha = 1.2$



bias: $b = 4$; periodicity: $\alpha = 1.2$



bias: $b = 20$; periodicity: $\alpha = 1.2$

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Matrix-valued orthogonal polynomials

Theorem (Duits, Kuijlaars 21'; informal version)

The domino-domino correlations of periodic dimer models (including the Aztec diamond) can be expressed via the **reproducing kernel** $R_N(w,z)$ *of certain* **matrix-valued orthogonal polynomials**.

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Double integral formula for the correlation kernel in the doubly periodic setting:

$$\left[K_{2N}\big((j,2x+\varepsilon),\,(j',2y+\varepsilon')\big)\right]_{\varepsilon,\varepsilon'\in\{0,1\}} = -\frac{\chi_{j>j'}}{2\pi i}\oint_{\gamma_{0,1}} A_{j',j}(z)z^{y-x}\frac{dz}{z}$$
$$+\frac{1}{(2\pi i)^2}\oint_{\gamma_{0,1}}\oint_{\gamma_{0,1}} A_{j',4N}(w)\mathbf{R_N}(\mathbf{w},\mathbf{z})A_{0,j}(z)\frac{w^y}{z^{x+1}w^N}dzdw.$$

Non-Hermitian orthogonality

[Duits, Kuijlaars '21]: doubly periodic tiling models $\Rightarrow$ contour orthogonality

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Biased 2×2-**periodic Aztec diamond**:

$$W(z) = \frac{1}{(z-1)^2} \begin{pmatrix} (z+b^2)^2 + \alpha^2(b^2+1)^2 z & (b^2+1)(\alpha^2+1)(z+b^2) \\ (b^2+1)(\alpha^{-2}+1)z(z+b^2) & (z+b^2)^2 + \alpha^{-2}(b^2+1)^2 z \end{pmatrix}$$

with the non-Hermitian scalar product between matrix-valued polynomials F, G:

$$\langle F, G \rangle = \frac{1}{2\pi i} \oint_\gamma F(z) W^N(z) G^t(z) dz, \qquad \gamma \text{ a simple curve going around } z = +1.$$

Here N is the **size of the Aztec diamond**.

Non-Hermitian orthogonality

[Duits, Kuijlaars '21]: doubly periodic tiling models $\Rightarrow$ contour orthogonality

Biased 2×2-**periodic Aztec diamond**:

$$W(z) = \frac{1}{(z-1)^2} \begin{pmatrix} (z+b^2)^2 + \alpha^2(b^2+1)^2 z & (b^2+1)(\alpha^2+1)(z+b^2) \\ (b^2+1)(\alpha^{-2}+1)z(z+b^2) & (z+b^2)^2 + \alpha^{-2}(b^2+1)^2 z \end{pmatrix}$$

with the non-Hermitian scalar product between matrix-valued polynomials F, G:

$$\langle F, G \rangle = \frac{1}{2\pi i} \oint_\gamma F(z) W^N(z) G^t(z) dz, \qquad \gamma \text{ a simple curve going around } z = +1.$$

Here N is the **size of the Aztec diamond**.

Disclaimer: As weight matrix is non-Hermitian existence of the MVOP is not guaranteed!

Matrix-valued orthogonal polynomials

We are looking for a polynomial P_N satisfying

- $P_N = z^N I_2 + O(z^{N-1}), \quad \text{as } z \to \infty,$
- $\frac{1}{2\pi i} \oint_\gamma P_N(z) W^N(z) z^k dz = 0, \quad k = 0, \ldots, N-1.$

and a polynomial Q_{N-1} satisfying

- Q_{N-1} is of degree $\leq N-1$,
- $\frac{1}{2\pi i} \oint_\gamma Q_{N-1}(z) W^N(z) z^k dz = \begin{cases} 0 & \text{for } k = 0, \ldots, N-2 \\ -I_2 & \text{for } k = N-1 \end{cases}.$

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Existence and Uniqueness (Duits, Kuijlaars '21) ✓

Relation to the meromorphic differential $d\mathbf{\Phi}_{\xi_1,\xi_2}$

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- Integral formula for $K_N(\cdot\,,\,\cdot) \xrightarrow{\text{steepest descent}}$ mermomorphic differential $d\mathbf{\Phi}_{\xi_1,\xi_2}$

We analyzed the orthogonal polynomials can be analyzed via the **Riemann–Hilbert problem** of Fokas–Its–Kitaev.

Riemann–Hilbert problem for MVOP

Fokas–Its–Kitaev R-H problem for MVOP

Find a 4×4 matrix-valued function $Y = Y^{(N)} \colon \mathbb{C} \setminus \gamma \to \mathbb{C}^{4\times 4}$ satisfying the following properties:

(i) **Analyticity**: $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$,

(ii) **Jump condition**: Y satisfies

$$Y_+(s) = Y_-(s) \begin{pmatrix} I_2 & W^N(s) \\ 0 & I_2 \end{pmatrix}, \qquad s \in \gamma,$$

(iii) **Normalization**: Y satisfies

$$Y(z) = (I_4 + O(z^{-1})) \begin{pmatrix} z^N I_2 & 0_2 \\ 0_2 & z^{-N} I_2 \end{pmatrix} \quad \text{as } z \to \infty.$$

Riemann–Hilbert problem for MVOP

Fokas–Its–Kitaev R-H problem for MVOP cont.

...The R-H problem has a unique solution if and only if the MVOPs P_N and Q_{N-1} exist and are unique, in which case the solution can be written as

$$Y(z) = \begin{pmatrix} P_N(z) & \frac{1}{2\pi i}\oint_\gamma \frac{P_N(s)W^N(s)}{s-z}ds \\ Q_{N-1}(z) & \frac{1}{2\pi i}\oint_\gamma \frac{Q_{N-1}(s)W^N(s)}{s-z}ds \end{pmatrix} \qquad z \in \mathbb{C} \setminus \gamma.$$

Remark

The reproducing kernel can be expressed as

$$R_N(w,z) = \frac{1}{w-z}\begin{pmatrix} 0_2 & I_2 \end{pmatrix} Y(w)^{-1} Y(z) \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix}$$

Deift–Zhou analysis

- R-H problems can be solved asymptotically ($N \to \infty$) via the **Deift–Zhou nonlinear steepest descent analysis**
- Our analysis however leads to **exact** expressions for P_N, Q_{N-1} for finite size N in terms of **Jacobi theta functions**, cf. [Duits, Kuijlaars '21]
- For the general $k \times \ell$-periodic P_N, Q_{N-1} can be computed using the **domino shuffle algorithm**, see [Chhita, Duits '23]

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  Corollary $\Rightarrow$ domino shuffle for the biased 2×2-periodic model can be **linearized** via Jacobi theta functions, c.f. [Borodin, Duits '23] & **KdV finite gap solutions**

References

- T. Berggren and A. Borodin, *Geometry of the doubly periodic Aztec dimer model*, arXiv:2306.07482
- A. Borodin and M. Duits, *Biased* 2×2 *periodic Aztec diamond and an elliptic curve*, Probab. Theory Related Fields, (2023).
- S. Chhita and M. Duits, *On the domino shuffle and matrix refactorizations*, Comm. Math. Phys., **401**(2), 1417–1467, (2023).
- M. Duits and A. B. J. Kuijlaars, *The two-periodic Aztec diamond and matrix valued orthogonal polynomials*, J. Eur. Math. Soc. **23**(4), 1075–1131 (2021).
- R. Kenyon, A. Okounkov, and S. Sheffield, *Dimers and amoebae*, Ann. of Math. **163**, 1019–1056 (2006).

References

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Thank You for your Attention!