The KdV equation with steplike initial data and connections with finite-gap solutions

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Outline of the Talk

1. Background

2. Riemann–Hilbert problems

3. Applications: KdV with steplike initial data
Outline of the Talk

1. Background

2. Riemann–Hilbert problems

3. Applications: KdV with steplike initial data
The KdV equation

The KdV (Korteweg–de-Vries) equation is a nonlinear wave equation given by

\[
\frac{\partial}{\partial t} q(x, t) = 6 \left( \frac{\partial}{\partial x} q(x, t) \right) q(x, t) - \frac{\partial^3}{\partial x^3} q(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+
\]

- Introduced by Boussinesque in 1877 and later studied by Korteweg and de-Vries in 1895.
- Models shallow water waves and admits soliton solutions, see Scott Russell 1834: "wave of translation".
- First example of an integrable PDE (linearizable via the scattering transform), see Gardner, Greene, Kruskal, Miura 1968/ Lax 1968.
- Admits finite-gap solutions deeply related to compact Riemann surfaces, see Its, Matveev 1975.

Are special solutions of the KdV equation generic? ⇒ Riemann-Hilbert approach (later)
KdV solitons

One-soliton solution: observed by Scott Russell in 1834 in a water canal

\[ q_{1 \text{ soliton}}(x, t) = -\frac{c}{\text{sech}^2 \left[ \frac{\sqrt{c}}{2} (x - ct) \right]} \]

Multi-soliton solutions: observed by Zabusky and Kruskal 1965 (see also Fermi–Pasta–Ulam–Tsingou experiment)

Figure: A two-soliton solution at time \( t = -1, 0, 1 \) (taken from Dunajski 2012)
Lax pairs

**Lax 1968**: Define the Schrödinger operator

\[
L = L(q) = -\frac{\partial^2}{\partial x^2} + q(x, t)
\]

and

\[
P = P(q) = -4\frac{\partial^3}{\partial x^3} + 6q(x, t)\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x} q(x, t)
\]

The following equivalence holds:

\[
q(x, t) \text{ solves the KdV Eq.} \iff \begin{cases}
L\psi(z, x, t) = z^2\psi(z, x, t) \\
P\psi(z, x, t) = \frac{\partial}{\partial t}\psi(z, x, t)
\end{cases}
\]

**Proof**: Both conditions are equivalent to the **Lax pair equation** \( \frac{\partial}{\partial t} L = [P, L] \).
Periodic KdV solutions

$q(x, t)$ solves the KdV equation $\implies$ spectrum $\sigma(L)$ is **conserved** in time:

$$\sigma\left(L(q(x, t))\right) = \sigma\left(L(q(x, 0))\right)$$
q(x, t) solves the KdV equation \( \implies \) spectrum \( \sigma(L) \) is **conserved** in time:

\[
\sigma \left( L(q(x, t)) \right) = \sigma \left( L(q(x, 0)) \right)
\]

If \( q(x, t) \) is **periodic** in \( x \), then

\[
\sigma(L(q)) = \bigcup_{i=0}^{\infty} [E_{2i}, E_{2i+1}]
\] (1)

\( \implies \) **Bandstructure**!
Finite gap potentials

$q$ is called a **finite-gap potential** if

$$\sigma(L(q)) = \bigcup_{i=0}^{g}[E_{2i}, E_{2i+1}], \quad \text{with} \quad E_{2g+1} = +\infty,$$

(2)

i.e. $E_k = +\infty$ for $k \geq 2g + 1$.

$\Rightarrow$ As the KdV flow is **isospectral** finite-gap initial data remains finite-gap for all time.
Related to a finite gap spectrum $\bigcup_{i=0}^{g}[E_{2i}, E_{2i+1}]$ define a Riemann surface by gluing two copies of $\mathbb{C}$ along $[E_{2i}, E_{2i+1}]$. 
Finite gap KdV solutions

Theorem (Akhiezer, Dubrovin, Its, Matveev)

All reflectionless periodic finite gap solutions of the KdV equation with spectrum

$$\sigma(L(q)) = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, \infty]$$

can be described explicitly in terms of the Jacobi theta function related to the hyperelliptic Riemann surface with two sheets $\mathbb{C} \setminus \bigcup_{i=0}^{g}[E_{2i}, E_{2i+1}]$ ($E_{2g+1} = \infty$) glued along the spectrum, and related quantities:

$$q(x, t) = -2 \frac{\partial^2}{\partial x^2} \log \Theta(Ux + Wt + D) - 2h$$

These solutions can be characterized by a Riemann–Hilbert problem.

Jacobi Theta functions

The genus 1 Jacobi Theta function (here $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \cong \mathcal{R}$):

$$\Theta(z|\tau) := \sum_{k \in \mathbb{Z}} e^{(k^2\tau + 2kz)i\pi}, \quad z \in \mathbb{C}$$

Sum converges absolutely as $\text{Im}(\tau) > 0$. We have:

- **periodicity** $\Rightarrow \Theta(z + 1|\tau) = \Theta(z|\tau)$

- **quasi-periodicity** $\Rightarrow \Theta(z + \tau|\tau) = e^{-\pi i \tau - 2\pi iz} \Theta(z|\tau)$

**Multivalued** holomorphic function on $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \cong \mathcal{R}$.

Applications (see Olver et al. NIST):

- Number Theory: Riemann Zeta function, sum of squares...
- Physics: string theory, statistical mechanics
- **Integrable wave equations and Riemann-Hilbert theory**
Jacobi Theta functions

Figure: Jacobi Theta function (source Wikipedia)
1. Background

2. Riemann–Hilbert problems

3. Applications: KdV with steplike initial data
What is a Riemann–Hilbert problem?

\[ \Sigma \ldots \text{finite union of smooth oriented arcs} \]

\[ m(z) \ldots \text{holomorphic vector-valued function on } \mathbb{C} \setminus \Sigma \]
What is a Riemann–Hilbert problem?

\[ m_{\pm}(k) := \lim_{z \to k_{\pm}} m(z) \]
**Definition of Riemann–Hilbert problem**

Given $\Sigma$, and a **jump matrix** $v(k)$, $k \in \Sigma$, find a holomorphic vector-valued function $m(z)$ on $\mathbb{C} \setminus \Sigma$, such that

$$m_+(k) = m_-(k)v(k), \quad k \in \Sigma.$$ 

and

$$\lim_{z \to \infty} m(z) = m_\infty$$

**Remark:** $m(z)$ is a row vector $\Rightarrow$ matrix multiplication from the right.
Example: scalar R-H problem

Find a scalar-valued function $\gamma : \mathbb{C}/([-ic, -ia] \cup [ia, ic]) \to \mathbb{C}$ s.t.:

\[
\begin{align*}
  \gamma(z) & = z^2 + a^2 z^2 + c^2 \\
  \lim_{z \to \infty} \gamma(z) & = 1,
\end{align*}
\]
Example: scalar R-H problem

Find a scalar-valued function $\gamma: \mathbb{C}/([-ic, -ia] \cup [ia, ic]) \rightarrow \mathbb{C}$ s.t.:

- $\gamma_+(k) = i\gamma_-(k), \quad k \in [ia, ic]$
Example: scalar R-H problem

Find a scalar-valued function \( \gamma : \mathbb{C}/([-ic, -ia] \cup [ia, ic]) \to \mathbb{C} \) s.t.:

- \( \gamma_+(k) = i\gamma_-(k), \quad k \in [ia, ic] \)
- \( \gamma_+(k) = -i\gamma_-(k), \quad k \in [-ia, -ic] \)

\[ R \quad \]

\[ ic \]

\[ \text{lim}_{z \to \infty} \gamma(z) = 1, \]

\( \gamma(z) \) has at most fourth root singularities at the endpoints \( \pm ia, \pm ic \).

\( \Rightarrow \) Unique solution \( \gamma(z) = \frac{z^2}{4} + \frac{a^2}{4}z^2 + \frac{c^2}{4} \)
The task is to find a scalar-valued function $\gamma : \mathbb{C}/([-ic, -ia] \cup [ia, ic]) \rightarrow \mathbb{C}$ s.t.:

- $\gamma_{+}(k) = i \gamma_{-}(k)$, $k \in [ia, ic]$
- $\gamma_{+}(k) = -i \gamma_{-}(k)$, $k \in [-ia, -ic]$
- $\lim_{z \rightarrow \infty} \gamma(z) = 1,$
Example: scalar R-H problem

Find a scalar-valued function $\gamma : \mathbb{C}/([-ic, -ia] \cup [ia, ic]) \to \mathbb{C}$ s.t.:

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- $\lim_{z \to \infty} \gamma(z) = 1,$
- $\gamma(z)$ has at most fourth root singularities at the endpoints $\pm ia, \pm ic.$

$\Rightarrow$ Unique solution $\gamma(z) = \left(\frac{z^2+a^2}{z^2+c^2}\right)^{1/4}$
Let \( q_{gap}(x, t) \) be a periodic 1-gap KdV solution with \( \sigma(L(q)) = [-c^2, -a^2] \cup [0, \infty) \).
Let $q_{\text{gap}}(x, t)$ be a periodic 1-gap KdV solution with $\sigma(L(q)) = [-c^2, -a^2] \cup [0, \infty)$.

$\Rightarrow q_{\text{gap}}(x, t)$ can be characterized by a **R-H problem**:
R-H problem for $q_{\text{gap}}$

Find a vector-valued function, holomorphic in $\mathbb{C} \setminus [-ic, ic]$

$$\psi(z, x, t) = (\psi_1(z, x, t), \psi_2(z, x, t))$$

satisfying the

- **jump condition** $\psi^+(k, x, t) = \psi^-(k, x, t)v(k, x, t), \quad k \in [-ic, ic]$

$$v(k, x, t) = \begin{cases} 
\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in [ic, ia], \\
\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, & k \in [-ia, -ic], \\
\begin{pmatrix} e^{-i\Omega} & 0 \\ 0 & e^{i\Omega} \end{pmatrix}, & k \in [ia, -ia],
\end{cases}$$

with $\Omega = Ux + Wt + D$ and $\sigma(L(q)) = [-c^2, -a^2] \cup [0, \infty)$. 
1 gap R-H problem cont.

- the **symmetry condition**,

\[
\psi(-z, x, t) = \psi(z, x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

- and the **normalization condition**,

\[
\lim_{z \to \infty} \psi(z, x, t) = (1 \ 1).
\]
1 gap R-H problem cont.

- the symmetry condition,

\[ \psi(-z, x, t) = \psi(z, x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

- and the normalization condition,

\[ \lim_{z \to \infty} \psi(z, x, t) = (1 \ 1). \]

**Question**: How can we obtain a 1 gap solution \( q_{gap}(x, t) \) from \( \psi(z, x, t) \)?
1 gap R-H problem cont.

- the **symmetry condition**, 

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\psi(-z, x, t) = \psi(z, x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

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\lim_{z \to \infty} \psi(z, x, t) = (1 \ 1).
\]

**Question**: How can we obtain a 1 gap solution \( q_{\text{gap}}(x, t) \) from \( \psi(z, x, t) \)?

**Answer**: If \( \psi(z, x, t) = (1 \ 1) + \frac{Q_{\text{gap}}(x, t)}{2z i} (-1 \ 1) + O\left(\frac{1}{z^2}\right) \) then

\[
q_{\text{gap}}(x, t) = \frac{\partial}{\partial x} Q_{\text{gap}}(x, t) - 2h - a^2 - c^2
\]

is a 1 gap solution of the KdV equation.
Sketch of proof

Note: Uniqueness is equivalent to

\[ \psi_0 \text{ satisfies R-H problem with normalization } \lim_{z \to \infty} \psi_0(z) = (0 \ 0) \implies \psi_0(z) \equiv (0 \ 0) \]

(assume two solutions \( \psi, \tilde{\psi} \), define \( \psi_0 = \psi - \tilde{\psi} \) ...)

Step 1: Show that a unique R-H solution \( \psi \) exists.

Step 2: Define Lax pair \( L, P \) with potential \( q_{\text{gap}} = \frac{\partial}{\partial x} Q_{\text{gap}}(x, t) - 2h - a^2 - c^2 \).

Step 3: Show that \( \psi_0 = L \psi - z^2 \psi \) (or \( \psi_0 = P \psi - \frac{\partial}{\partial t} \psi \)) solve modified R-H problem and vanish at infinity.

Step 4: By uniqueness of \( \psi \) conclude \( L \psi - z^2 \psi = 0 \) \( P \psi - \frac{\partial}{\partial t} \psi = 0 \)

Lax pair equation \implies The potential \( q_{\text{gap}} \) solves the KdV equation (for details see P., Teschl '21).
Sketch of proof

**Note:** Uniqueness is equivalent to

$$
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$$

(assume two solutions $$\psi, \tilde{\psi}$$, define $$\psi_0 = \psi - \tilde{\psi}$$ ...)

**Step 1:** Show that a unique R-H solution $$\psi$$ exists.
Note: Uniqueness is equivalent to

$$\psi_0 \text{ satisfies R-H problem with normalization } \lim_{z \to \infty} \psi_0(z) = (0 \ 0) \implies \psi_0(z) \equiv (0 \ 0)$$

(assume two solutions $\psi$, $\tilde{\psi}$, define $\psi_0 = \psi - \tilde{\psi}$ ...)

**Step 1**: Show that a unique R-H solution $\psi$ exists.

**Step 2**: Define Lax pair $L, P$ with potential $q_{gap} = \frac{\partial}{\partial x} Q_{gap}(x, t) - 2h - a^2 - c^2$. 
Sketch of proof

**Note:** Uniqueness is equivalent to

\[
\psi_0 \text{ satisfies R-H problem with normalization } \lim_{z \to \infty} \psi_0(z) = (0 \ 0) \implies \psi_0(z) \equiv (0 \ 0)
\]

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**Step 1:** Show that a unique R-H solution \( \psi \) exists.

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**Step 3:** Show that \( \psi_0 = L\psi - z^2 \psi \) (or \( \psi_0 = P\psi - \frac{\partial}{\partial t} \psi \)) solve modified R-H problem and vanish at infinity.
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**Step 1**: Show that a unique R-H solution \( \psi \) exists.

**Step 2**: Define Lax pair \( L, P \) with potential \( q_{\text{gap}} = \frac{\partial}{\partial x} Q_{\text{gap}}(x, t) - 2h - a^2 - c^2 \).

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**Step 4**: By uniqueness of \( \psi \) conclude

\[
L \psi - z^2 \psi = 0 \\
P \psi - \frac{\partial}{\partial t} \psi = 0
\]

**Lax pair equation \( \Rightarrow \)** The potential \( q_{\text{gap}} \) solves the KdV equation

(for details see P., Teschl '21).
Solution of R-H problem

The explicit solution $\psi = (\psi_1, \psi_2)$ is given by (here $A$ is the Abel map):

$$
\psi_1(z) = \left( \frac{z^2 + a^2}{z^2 + c^2} \right)^{1/4} \frac{\Theta \left( A(z) - i\pi - \frac{i\Omega}{2} \right) \Theta \left( A(z) - \frac{i\Omega}{2} \right) \Theta^2 \left( \frac{\pi i}{2} \right)}{\Theta \left( A(z) - i\pi \right) \Theta \left( A(z) \right) \Theta \left( \frac{\pi i}{2} - \frac{i\Omega}{2} \right) \Theta \left( \frac{\pi i}{2} + \frac{i\Omega}{2} \right)},
$$

$$
\psi_2(z) = \psi_1(-z).
$$

For the explicit derivation via a scalar R-H problem on the torus see P., Teschl '21.

From $\psi$ we obtain the 1-gap Its–Matveev KdV solution:

$$
q_{gap}(x, t) = -2 \frac{\partial^2}{\partial x^2} \log \Theta(Ux + Wt + D) - 2h
$$
Outline of the Talk

1. Background

2. Riemann–Hilbert problems

3. Applications: KdV with steplike initial data
The steplike KdV Cauchy problem

Consider the KdV initial value problem

\[ \frac{\partial}{\partial t} q(x, t) = 6 \left( \frac{\partial}{\partial x} q(x, t) \right) q(x, t) - \frac{\partial^3}{\partial x^3} q(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+ \]

with \textbf{steplike} initial data \( q(x, 0) = q_0(x) \) \( (c > 0) \):

\[ \begin{align*}
q_0(x) & \to 0, \quad \text{as } x \to +\infty, \\
q_0(x) & \to -c^2, \quad \text{as } x \to -\infty,
\end{align*} \]

Technical details:

- \( \int_0^{+\infty} e^{C_0 x} (|q_0(x)| + |q_0(-x) + c^2|) dx < \infty, \quad C_0 > c > 0, \)
- \( \int_{\mathbb{R}} (x^6 + 1)|q_0^{(i)}(x)| dx < \infty, \quad i = 1, \ldots, 11 \)
Theorem (Egorova, Grunert, Teschl ’09)

This Cauchy problem has a unique global solution \( q(\cdot, t) \in C^3(\mathbb{R}) \) satisfying

\[
\int_{0}^{+\infty} |x|(|q(x, t)| + |q(-x, t) + c^2|)dx < \infty, \quad t \in \mathbb{R}_+.
\]
A numerical solution

Figure: Numerically computed solution $q(x, t)$ of the KdV equation at time $t = 10$, with initial condition $q(x, 0) = \frac{1}{2}(\text{erf}(x) - 1) - 5\text{sech}(x - 1)$ [taken from Egorova, Gladka, Kotlyarov, Teschl '13]
Asymptotic behaviour

We observe the following behaviour:

- $x < -6c^2t$: decaying dispersive tail
- $-6c^2t < x < 4c^2t$: elliptic wave
- $4c^2t < x$: finitely many solitons

The **elliptic wave region** is related to the 1 gap solutions from the previous slides!
Figure: Modulated 1 gap solution
Question: How to get a quantitative and rigorous result?

Answer: Riemann–Hilbert method!
The **direct scattering transform** (in the absence of solitons):

\[ q(x, t) \mapsto S(t) = \{ R(k, t), k \in \mathbb{R}; \chi(k, t), k \in [-ic, ic] \} \]

**Theorem (cf. Gardner, Greene, Kruskal, Miura ’68/Lax ’68)**

\[
\begin{align*}
q(x, t) \text{ satisfies KdV Eq. } & \iff R(k, t) = R(k, 0)e^{8ik^3t}, \\
& \chi(k, t) = \chi(k, 0)e^{8ik^3t},
\end{align*}
\]

This effectively **linearizes** the KdV equation.
(Inverse) Scattering Transform

Key Insight:

The **inverse scattering transform** (IST) can be formulated as a **Riemann–Hilbert problem**.
$M(z) = M(z, x, t)$ is uniquely characterized by the following Riemann–Hilbert problem:
Find a vector-valued function $M(z) = M(z, x, t)$ which is holomorphic away from $\mathbb{R} \cup [-ic, ic]$ and satisfies:

- The **jump condition** $M_+(k) = M_-(k)V(k)$
  \[
  V(k) = \begin{cases} 
  \begin{pmatrix}
  1 - |R(k)|^2 & -\overline{R(k)}e^{-\Phi(k)} \\
  R(k)e^{\Phi(k)} & 1
  \end{pmatrix}, & k \in \mathbb{R}, \\
  \begin{pmatrix}
  1 & 0 \\
  \chi(k)e^{\Phi(k)} & 1
  \end{pmatrix}, & k \in (0, ic], \\
  \begin{pmatrix}
  1 & \chi(k)e^{-\Phi(k)} \\
  0 & 1
  \end{pmatrix}, & k \in [-ic, 0),
  \end{cases}
  \]

Where the phase function $\Phi(k) = \Phi(k, x, t)$ is given by $\Phi(k) = 8ik^3t + 2ikx$.

Here $R(k), \chi(k)$ is the scattering data of the initial data $q_0(x)$. 
Steplike KdV Riemann–Hilbert problem cont.

- the **symmetry condition**: \( M(-z) = M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \),

- and the **normalization condition**: \( \lim_{z \to \infty} M(z) = (1 \ 1) \).
Steplike KdV Riemann–Hilbert problem cont.

- the **symmetry condition**: \( M(-z) = M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \),

- and the **normalization condition**: \( \lim_{z \to \infty} M(z) = (1 \ 1) \).

**Question**: How can we obtain the steplike solution \( q(x, t) \) from \( M(z, x, t) \)?
Steplike KdV Riemann–Hilbert problem cont.

- the **symmetry condition**: \( M(-z) = M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \),

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**Question**: How can we obtain the steplike solution \( q(x, t) \) from \( M(z, x, t) \)?

**Answer**: If \( M(z, x, t) = (1 \ 1) + \frac{Q(x,t)}{2zi} (-1 \ 1) + O\left(\frac{1}{z^2}\right) \) then

\[
q(x, t) = \frac{\partial}{\partial x} Q(x, t)
\]

is the solution of the steplike KdV Cauchy problem with \( q(x, 0) = q_0(x) \).
Deift–Zhou nonlinear steepest descent method

General idea of the Deift–Zhou nonlinear steepest descent method for R-H problems:

Step 1: Start with a R-H problem (e.g. the steplike KdV problem for $M$)

Step 2: Perform a series of

- jump matrix factorizations
- matrix conjugations
- contour deformations

to arrive at a R-H problem which is a perturbation of an explicitly solvable R-H problem.

Step 3: Solve this simple R-H problem, and bound the error.
The jump contour for the initial R-H problem for the steplike KdV problem:
Deformation and conjugation steps

After a few conjugation and deformation steps we obtain an equivalent Riemann–Hilbert problem with jump contour (see Egorova et al. ’13):

- **dashed contour**: jump matrices converge exponentially to the identity matrix,
- **interval** $[-ic, ic]$: jump matrices equal to the 1 gap R-H problem from before,
- **points** $\pm ia$: need a local parametrix solution (exponential convergence nonuniform).
Main result

**Theorem (Egorova, P., Teschl '23 / P. '23)**

In the transition region, $-6c^2 + \varepsilon < x/t < 4c^2 - \varepsilon$ with $\varepsilon > 0$, the solution $q(x, t)$ with steplike initial data $q_0(x)$ satisfies:

$$q(x, t) = q_{\text{gap}}(x, t) + O(t^{-1}),$$

where

$$q_{\text{gap}}(x, t) = -2 \frac{\partial^2}{\partial x^2} \log \Theta(Ux + Wt + D \tau) - 2h$$

is a 1 gap periodic solution of the KdV equation, with $h, U, W, D, \tau$ depending only on the slowly varying parameter $\xi = \frac{x}{t}$. 
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A scalar Riemann–Hilbert problem on the torus: Applications to the KdV equation
Thank you!