The KdV equation with steplike initial data and connections with finite-gap solutions

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Outline of the Talk

1. Background
2. Riemann–Hilbert problems
3. Applications: KdV with steplike initial data

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1. Background

2. Riemann–Hilbert problems

3. Applications: KdV with steplike initial data

The KdV equation

The KdV (Korteweg–de-Vries) equation is a nonlinear wave equation given by

$$\frac{\partial}{\partial t} q(x,t) = 6\left(\frac{\partial}{\partial x} q(x,t)\right) q(x,t) - \frac{\partial^3}{\partial x^3} q(x,t), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

- Introduced by Boussinesque in 1877 and later studied by Korteweg and de-Vries in 1895.
- Models shallow water waves and admits **soliton** solutions, see Scott Russell 1834: "wave of translation".
- First example of an **integrable PDE** (linearizable via the scattering transform), see Gardner, Greene, Kruskal, Miura 1968/ Lax 1968.
- Admits **finite-gap solutions** deeply related to compact Riemann surfaces, see Its, Matveev 1975.

Are special solutions of the KdV equation generic? $\Rightarrow$ **Riemann-Hilbert approach** (later)

KdV solitons

One-soliton solution: observed by Scott Russell in 1834 in a water canal

$$q_{1\ soliton}(x,t) = -\frac{c}{2}\text{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]$$

Multi-soliton solutions: observed by Zabusky and Kruskal 1965 (see also Fermi–Pasta–Ulam–Tsingou experiment)

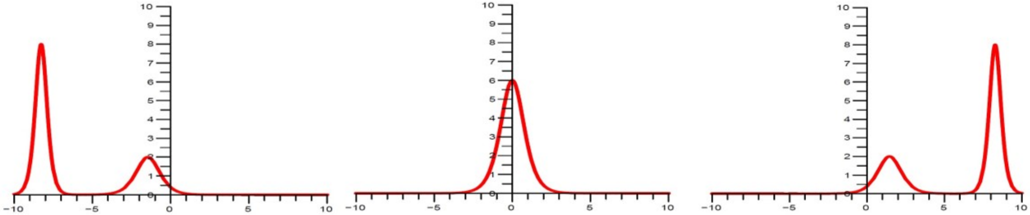

Figure: A two-soliton solution at time $t = -1, 0, 1$ (taken from Dunajski 2012)

Lax pairs

Lax 1968: Define the Schrödinger operator

$$L = L(q) = -\frac{\partial^2}{\partial x^2} + q(x,t)$$

and

$$P = P(q) = -4\frac{\partial^3}{\partial x^3} + 6q(x,t)\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}q(x,t)$$

The following equivalence holds:

$$q(x,t) \text{ solves the KdV Eq.} \iff \begin{cases} L\psi(z,x,t) = z^2\psi(z,x,t) \\ P\psi(z,x,t) = \frac{\partial}{\partial t}\psi(z,x,t) \end{cases} \quad \text{is solvable}$$

Proof: Both conditions are equivalent to the **Lax pair equation** $\frac{\partial}{\partial t}L = [P, L]$.

Periodic KdV solutions

$q(x,t)$ solves the KdV equation $\implies$ spectrum $\sigma(L)$ is **conserved** in time:

$$\sigma\Big(L(q(x,t))\Big) = \sigma\Big(L(q(x,0))\Big)$$

Periodic KdV solutions

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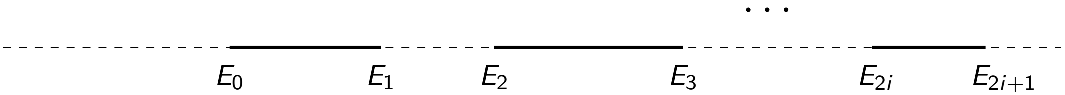
If $q(x,t)$ is **periodic** in x, then

$$\sigma(L(q)) = \cup_{i=0}^{\infty}[E_{2i}, E_{2i+1}] \tag{1}$$

$\Rightarrow$ **Bandstructure**!

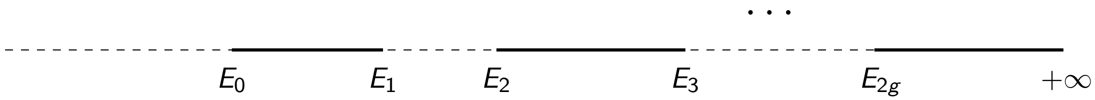


Finite gap potentials

q is called a **finite-gap potential** if

$$\sigma(L(q)) = \cup_{i=0}^{g}[E_{2i}, E_{2i+1}], \qquad \text{with} \quad E_{2g+1} = +\infty, \tag{2}$$

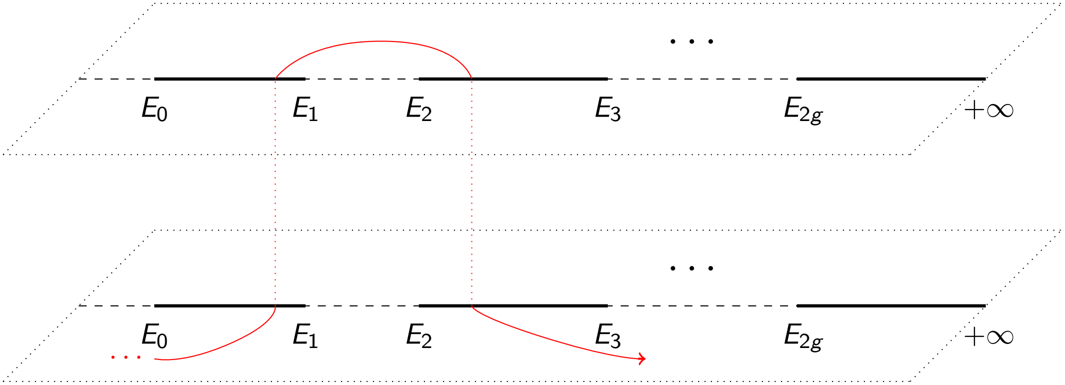
i.e. $E_k = +\infty$ for $k \geq 2g+1$.



$\Rightarrow$ As the KdV flow is **isospectral** finite-gap initial data remains finite-gap for all time.

Riemann surface

Related to a finite gap spectrum $\cup_{i=0}^{g}[E_{2i}, E_{2i+1}]$ define a Riemann surface by gluing two copies of $\mathbb{C}$ along $[E_{2i}, E_{2i+1}]$.



Finite gap KdV solutions

Theorem (Akhiezer, Dubrovin, Its, Matveev)

All reflectionless periodic finite gap solutions of the KdV equation with spectrum

$$\sigma(L(q)) = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, \infty]$$

can be described explicitly in terms of the Jacobi theta function related to the hyperelliptic Riemann surface with two sheets $\mathbb{C} \setminus \cup_{i=0}^{g} [E_{2i}, E_{2i+1}]$ *(*$E_{2g+1} = \infty$*) glued along the spectrum, and related quantities:*

$$q(x,t) = -2 \frac{\partial^2}{\partial x^2} \log \Theta(Ux + Wt + D) - 2h$$

These solutions can be characterized by a **Riemann–Hilbert problem**.

Belokolos, A. Bobenko, V. Enol'skii, A. Its and V. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer Series in Nonlinear Dynamics, Berlin, 1994.

Jacobi Theta functions

The genus 1 **Jacobi Theta function** (here $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \cong \mathcal{R}$):

$$\Theta(z|\tau) := \sum_{k\in\mathbb{Z}} e^{(k^2\tau+2kz)\pi \mathrm{i}}, \quad z \in \mathbb{C}$$

Sum converges absolutely as $\mathrm{Im}(\tau) > 0$. We have:

- periodicity $\Rightarrow \Theta(z+1|\tau) = \Theta(z|\tau)$
- quasi-periodicity $\Rightarrow \Theta(z+\tau|\tau) = e^{-\pi i\tau - 2\pi i z}\Theta(z|\tau)$

multivalued holomorphic function on $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \cong \mathcal{R}$.

Applications (see Olver et al. NIST):

- Number Theory: Riemann Zeta function, sum of squares...
- Physics: string theory, statistical mechanics
- **Integrable wave equations and Riemann-Hilbert theory**

Jacobi Theta functions

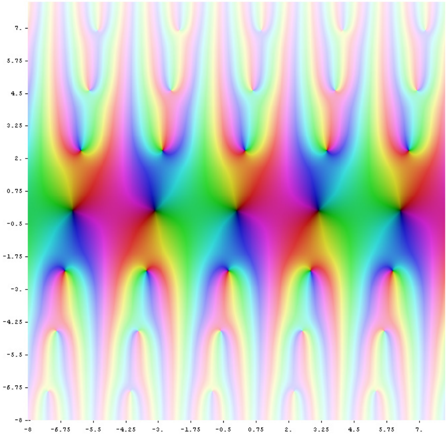


Figure: Jacobi Theta function (*source Wikipedia*)

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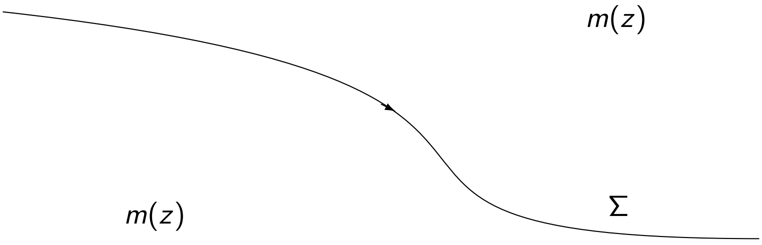
2. Riemann–Hilbert problems

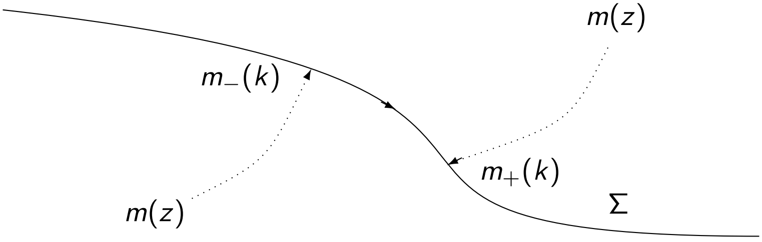
3. Applications: KdV with steplike initial data

What is a Riemann–Hilbert problem?



Σ ... finite union of smooth oriented arcs
$m(z)$ ... holomorphic vector-valued function on $\mathbb{C} \setminus \Sigma$

What is a Riemann–Hilbert problem?



$$m_\pm(k) := \lim_{z \to k\pm} m(z)$$

Definition of Riemann–Hilbert problem

Riemann-Hilbert problem

Given Σ, and a **jump matrix** $v(k)$, $k \in \Sigma$, find a holomorphic vector-valued function $m(z)$ on $\mathbb{C} \setminus \Sigma$, such that

$$m_+(k) = m_-(k)v(k), \quad k \in \Sigma.$$

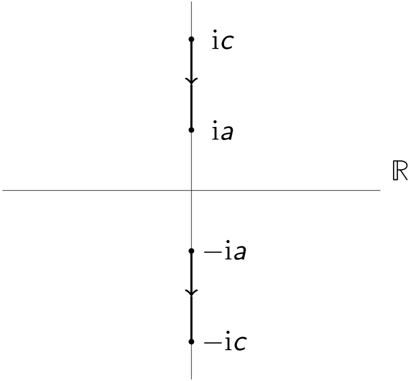
and

$$\lim_{z \to \infty} m(z) = m_\infty$$

Remark: $m(z)$ is a row vector $\Rightarrow$ matrix multiplication from the right.

Example: scalar R-H problem

Find a scalar-valued function $\gamma\colon \mathbb{C}/([-\mathrm{i}c, -\mathrm{i}a] \cup [\mathrm{i}a, \mathrm{i}c]) \to \mathbb{C}$ s.t.:



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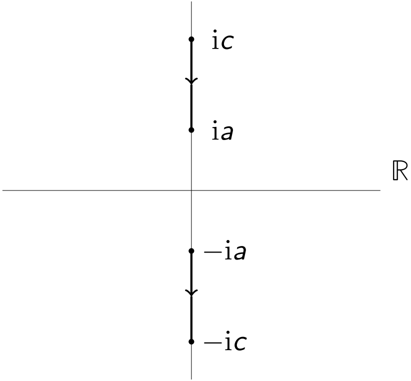

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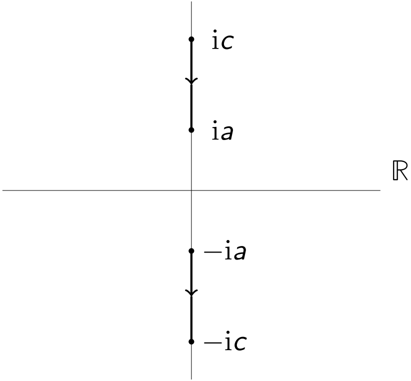

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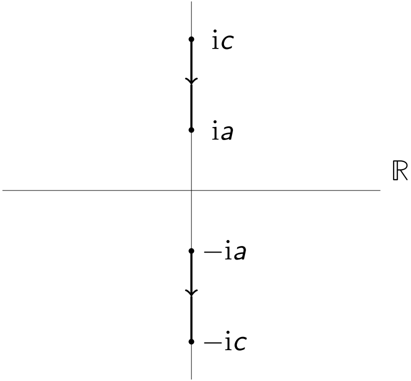

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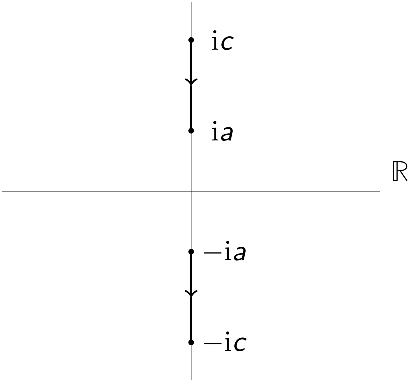
- $\lim_{z\to\infty} \gamma(z) = 1$,
- $\gamma(z)$ has at most fourth root singularities at the endpoints $\pm\mathrm{i}a$, $\pm\mathrm{i}c$.



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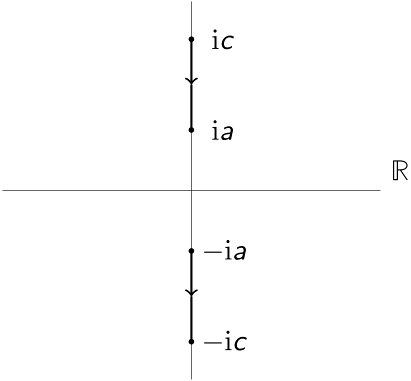

$\Rightarrow$ Unique solution $\gamma(z) = \left(\frac{z^2+a^2}{z^2+c^2}\right)^{1/4}$

1 gap KdV solution

Let $q_{gap}(x, t)$ be a periodic 1-gap KdV solution with $\sigma(L(q)) = [-c^2, -a^2] \cup [0, \infty)$.

1 gap KdV solution

Let $q_{gap}(x,t)$ be a periodic 1-gap KdV solution with $\sigma(L(q)) = [-c^2, -a^2] \cup [0, \infty)$.

$\Rightarrow$ $q_{gap}(x,t)$ can be characterized by a **R-H problem**:

R-H problem for q_{gap}

Find a vector-valued function, holomorphic in $\mathbb{C} \setminus [-\mathrm{i}c, \mathrm{i}c]$

$$\psi(z,x,t) = (\psi_1(z,x,t),\ \psi_2(z,x,t))$$

satisfying the

- **jump condition** $\psi_+(k,x,t) = \psi_-(k,x,t)v(k,x,t), \quad k \in [-\mathrm{i}c, \mathrm{i}c]$

$$v(k,x,t) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \\ \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c], \\ \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Omega} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\Omega} \end{pmatrix}, & k \in [\mathrm{i}a, -\mathrm{i}a], \end{cases}$$

with $\Omega = Ux + Wt + D$ and $\sigma(L(q)) = [-c^2, -a^2] \cup [0, \infty)$.

1 gap R-H problem cont.

- the **symmetry condition**,

$$\psi(-z, x, t) = \psi(z, x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

- and the **normalization condition**,

$$\lim_{z \to \infty} \psi(z, x, t) = (1 \;\; 1).$$

1 gap R-H problem cont.

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Question: How can we obtain a 1 gap solution $q_{gap}(x,t)$ from $\psi(z,x,t)$?

1 gap R-H problem cont.

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Question: How can we obtain a 1 gap solution $q_{gap}(x, t)$ from $\psi(z, x, t)$?

Answer: If $\psi(z, x, t) = \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{Q_{gap}(x,t)}{2z\mathrm{i}} \begin{pmatrix} -1 & 1 \end{pmatrix} + O\left(\frac{1}{z^2}\right)$ then

$$q_{gap}(x, t) = \frac{\partial}{\partial x} Q_{gap}(x, t) - 2h - a^2 - c^2$$

is a 1 gap solution of the KdV equation.

Sketch of proof

Note: Uniqueness is equivalent to

$$\psi_0 \text{ satisfies R-H problem with normalization } \lim_{z\to\infty} \psi_0(z) = (0 \quad 0) \Longrightarrow \psi_0(z) \equiv (0 \quad 0)$$

(assume two solutions ψ, $\widetilde{\psi}$, define $\psi_0 = \psi - \widetilde{\psi}$ ...)

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Step 4: By uniqueness of ψ conclude

$$L\psi - z^2\psi = 0$$
$$P\psi - \frac{\partial}{\partial t}\psi = 0$$

Lax pair equation $\Rightarrow$ The potential q_{gap} solves the KdV equation
(for details see P., Teschl '21).

Solution of R-H problem

The explicit solution $\psi = (\psi_1, \psi_2)$ is given by (here A is the Abel map):

$$\psi_1(z) = \left(\frac{z^2+a^2}{z^2+c^2}\right)^{1/4} \frac{\Theta\left(A(z) - \mathrm{i}\pi - \frac{\mathrm{i}\Omega}{2}\right)\Theta\left(A(z) - \frac{\mathrm{i}\Omega}{2}\right)\Theta^2\left(\frac{\pi \mathrm{i}}{2}\right)}{\Theta\left(A(z)-\mathrm{i}\pi\right)\Theta\left(A(z)\right)\Theta\left(\frac{\pi\mathrm{i}}{2} - \frac{\mathrm{i}\Omega}{2}\right)\Theta\left(\frac{\pi\mathrm{i}}{2} + \frac{\mathrm{i}\Omega}{2}\right)},$$
$$\psi_2(z) = \psi_1(-z).$$

For the explicit derivation via a scalar R-H problem on the torus see P., Teschl '21.

From ψ we obtain the 1-gap Its–Matveev KdV solution:

$$q_{gap}(x,t) = -2\frac{\partial^2}{\partial x^2}\log\Theta(Ux + Wt + D) - 2h$$

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3. Applications: KdV with steplike initial data

The steplike KdV Cauchy problem

Consider the KdV initial value problem

$$\frac{\partial}{\partial t}q(x,t) = 6\Big(\frac{\partial}{\partial x}q(x,t)\Big)q(x,t) - \frac{\partial^3}{\partial x^3}q(x,t), \qquad (x,t)\in\mathbb{R}\times\mathbb{R}_+$$

with **steplike** initial data $q(x,0) = q_0(x)$ $(c > 0)$:

$$\begin{cases} q_0(x) \to 0, & \text{as} \quad x \to +\infty, \\ q_0(x) \to -c^2, & \text{as} \quad x \to -\infty, \end{cases}$$

Technical details:

- $\int_0^{+\infty} \mathrm{e}^{C_0 x}(|q_0(x)| + |q_0(-x) + c^2|)dx < \infty, \quad C_0 > c > 0,$
- $\int_{\mathbb{R}}(x^6+1)|q_0^{(i)}(x)|dx < \infty, \quad i = 1, \ldots, 11$

Existence result

Theorem (Egorova, Grunert, Teschl '09)

This Cauchy problem has a unique global solution $q(\cdot, t) \in C^3(\mathbb{R})$ satisfying

$$\int_0^{+\infty} |x|(|q(x,t)| + |q(-x,t) + c^2|)dx < \infty, \qquad t \in \mathbb{R}_+.$$

A numerical solution

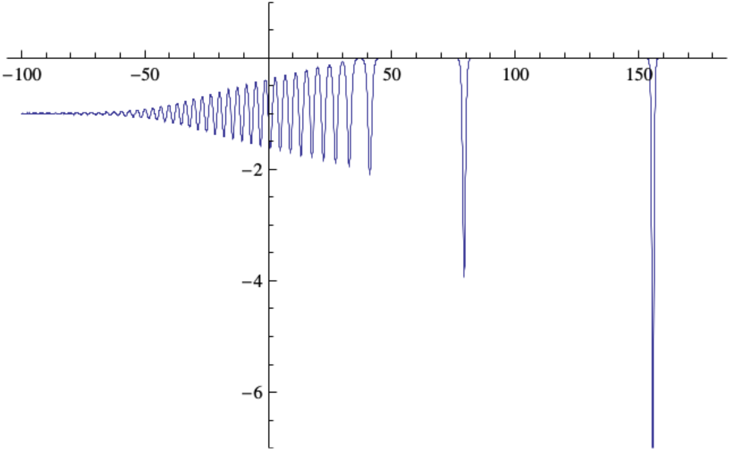
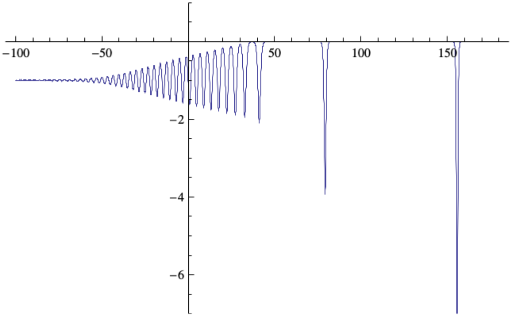


Figure: Numerically computed solution $q(x,t)$ of the KdV equation at time $t = 10$, with initial condition $q(x,0) = \frac{1}{2}(\mathrm{erf}(x) - 1) - 5\mathrm{sech}(x-1)$ [taken from Egorova, Gladka, Kotlyarov, Teschl '13]

Asymptotic behaviour

We observe the following behaviour:

- $x < -6c^2t$: decaying dispersive tail
- $-6c^2t < x < 4c^2t$: elliptic wave
- $4c^2t < x$: finitely many solitons



The **elliptic wave region** is related to the 1 gap solutions from the previous slides!

Modulated 1 gap solution

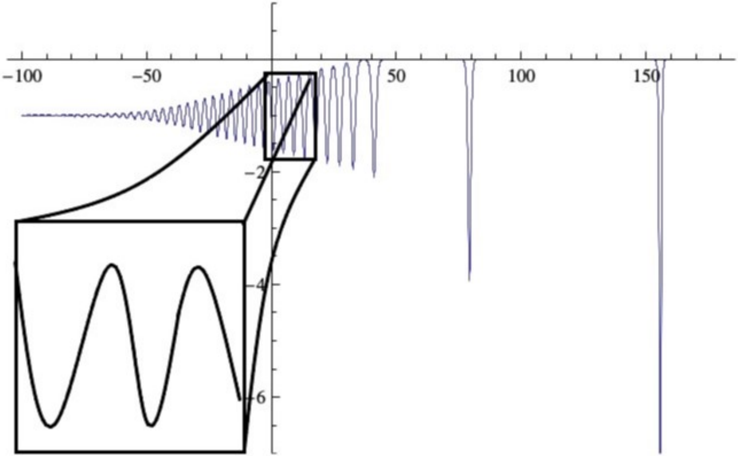


Figure: Modulated 1 gap solution

Rigorous analysis

Question: How to get a quantitative and rigorous result?

Answer: **Riemann–Hilbert method**!

Gardner, Greene, Kruskal, Miura method

The **direct scattering transform** (in the absence of solitons):

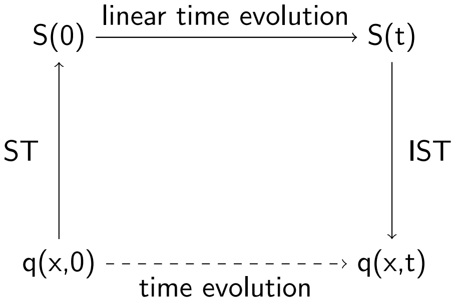
$$q(x,t) \mapsto S(t) = \{R(k,t),\ k \in \mathbb{R};\ \chi(k,t),\ k \in [-ic, ic]\}$$

Theorem (cf. Gardner, Greene, Kruskal, Miura '68/Lax '68)

$$q(x,t) \textit{ satisfies KdV Eq.} \iff \begin{array}{l} R(k,t) = R(k,0)\mathrm{e}^{8\mathrm{i}k^3 t}, \\ \chi(k,t) = \chi(k,0)\mathrm{e}^{8\mathrm{i}k^3 t}, \end{array}$$

This effectively **linearizes** the KdV equation.

(Inverse) Scattering Transform



Key Insight:

The **inverse scattering transform** (IST) can be formulated as a **Riemann–Hilbert problem**.

Steplike KdV Riemann–Hilbert problem

$M(z) = M(z, x, t)$ is **uniquely** characterized by the following Riemann–Hilbert problem:

Steplike KdV Riemann–Hilbert problem cont.

Find a vector-valued function $M(z) = M(z, x, t)$ which is holomorphic away from $\mathbb{R} \cup [-\mathrm{i}c, \mathrm{i}c]$ and satisfies:

- The **jump condition** $M_+(k) = M_-(k)V(k)$

$$V(k) = \begin{cases} \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-\Phi(k)} \\ R(k)\mathrm{e}^{\Phi(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ \chi(k)\mathrm{e}^{\Phi(k)} & 1 \end{pmatrix}, & k \in (0, \mathrm{i}c], \\ \begin{pmatrix} 1 & \chi(k)\mathrm{e}^{-\Phi(k)} \\ 0 & 1 \end{pmatrix}, & k \in [-\mathrm{i}c, 0), \end{cases}$$

Where the phase function $\Phi(k) = \Phi(k, x, t)$ is given by $\Phi(k) = 8\mathrm{i}k^3 t + 2\mathrm{i}kx$.
Here $R(k)$, $\chi(k)$ is the scattering data of the initial data $q_0(x)$.

Steplike KdV Riemann–Hilbert problem cont.

- the **symmetry condition**: $M(-z) = M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
- and the **normalization condition**: $\lim_{z\to\infty} M(z) = (1 \;\; 1)$.

Steplike KdV Riemann–Hilbert problem cont.

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Question: How can we obtain the steplike solution $q(x,t)$ from $M(z,x,t)$?

Steplike KdV Riemann–Hilbert problem cont.

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Question: How can we obtain the steplike solution $q(x,t)$ from $M(z,x,t)$?

Answer: If $M(z,x,t) = (1 \;\; 1) + \frac{Q(x,t)}{2z\mathrm{i}}(-1 \;\; 1) + O\left(\frac{1}{z^2}\right)$ then

$$q(x,t) = \frac{\partial}{\partial x} Q(x,t)$$

is the solution of the steplike KdV Cauchy problem with $q(x,0) = q_0(x)$.

Deift–Zhou nonlinear steepest descent method

General idea of the **Deift–Zhou nonlinear steepest method for R-H problems**:

Step 1: Start with a R-H problem (e.g. the steplike KdV problem for M)

Step 2: Perform a series of

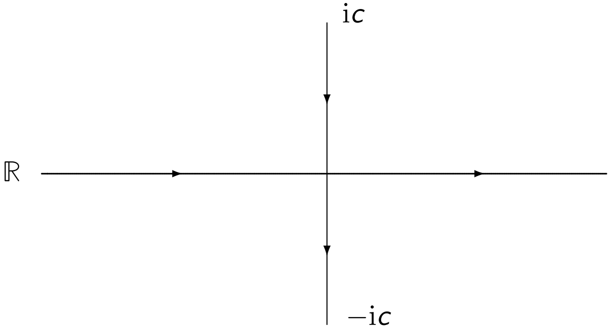
- jump matrix factorizations
- matrix conjugations
- contour deformations

to arrive at a R-H problem which is a **perturbation of an explicitly solvable R-H problem**.

Step 3: Solve this simple R-H problem, and bound the error.

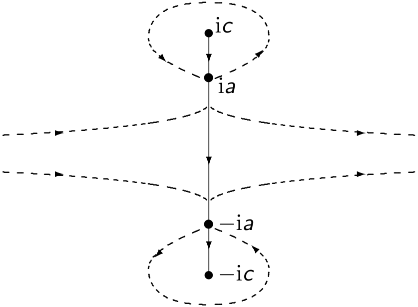
Initial jump contour for steplike KdV R-H problem

The jump contour for the initial R-H problem for the steplike KdV problem:



Deformation and conjugation steps

After a few conjugation and deformation steps we obtain an equivalent Riemann–Hilbert problem with jump contour (see Egorova et al. '13):



- **dashed contour**: jump matrices converge exponentially to the identity matrix,
- **interval** $[-\mathrm{i}c, \mathrm{i}c]$: jump matrices equal to the 1 gap R-H problem from before,
- **points** $\pm\mathrm{i}a$: need a local parametrix solution (exponential convergence nonuniform).

Main result

Theorem (Egorova, P., Teschl '23 / P. '23)

In the transition region, $-6c^2 + \varepsilon < x/t < 4c^2 - \varepsilon$ *with* $\varepsilon > 0$*, the solution* $q(x,t)$ *with steplike initial data* $q_0(x)$ *satisfies:*

$$q(x,t) = q_{gap}(x,t) + O(t^{-1}),$$

where

$$q_{gap}(x,t) = -2\frac{\partial^2}{\partial x^2}\log\Theta(Ux + Wt + D \quad \tau) - 2h$$

is a 1 gap periodic solution of the KdV equation, with h, U, W, D, τ *depending only on the slowly varying parameter* $\xi = \frac{x}{t}$.

References

I. Egorova, Z. Gladka, V. Kotlyarov and G. Teschl
Long-time asymptotics for the Korteweg-de Vries equation with steplike initial data
Nonlinearity **26**, 1839-1864 (2013)

I. Egorova, M. P. and G. Teschl
Asymptotics of the Korteweg–de Vries shock waves via the Riemann–Hilbert approach
Indiana Univ. Math. J., to appear

M. Girotti, T. Grava, R. Jenkins and T.-R. McLaughlin
Rigourous Asymptotics of a KdV Soliton Gas
Comm. Math. Phys. **384**, 733–784 (2021)

M. P.
Parametrix problem for the Korteweg–de Vries equation with steplike initial data
J. of Differential Equations, **375**, 280–314 (2023)

M. P. and G. Teschl,
A scalar Riemann–Hilbert problem on the torus: Applications to the KdV equation
Anal. Math. Phys. **12**(112), 1–19 (2022)

Thank you!