Methods of Geometric Control in Hamiltonian Dynamics

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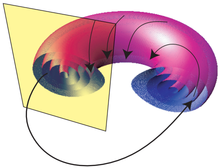
Objectives

- General problem of dynamics (Poincaré): understand the effect of small perturbations on *integrable* Hamiltonian systems
- Hamiltonian system:
  $H_0 : (p, q) \in \mathbb{R}^{2n} \mapsto H_0(p, q) \in \mathbb{R}$
  $$\begin{cases} \dot{p} = & -\frac{\partial H_0}{\partial q}(p, q) \\ \dot{q} = & \frac{\partial H_0}{\partial p}(p, q) \end{cases}$$
- The total energy H_0 is a conserved quantity
- A Hamiltonian is *integrable* if there exists n 'independent', conserved quantities $\Leftrightarrow$ there exists a smooth foliation of the phase space by invariant tori

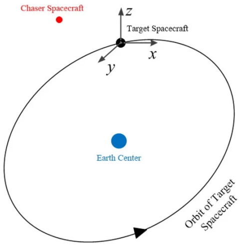


Objectives

- Perturbed Hamiltonian system

$$H_\varepsilon = H_0 + \varepsilon H_1$$

  where H_0 = integrable Hamiltonian,
  H_1 = Hamiltonian perturbation,
  ε = small parameter
- Given two points p, q, show that there exists a solution of H_ε that goes from p to q
- Motivation: in problems from celestial mechanics and space mission design, the Hamiltonians H_0, H_1 are explicit; e.g.,
  - H_0 describes motion of a spacecraft relative to the Earth
  - H_1 describes the perturbation by the Moon, Sun, etc.
  - Steer the trajectory of a chaser spacecraft to reach a target spacecraft

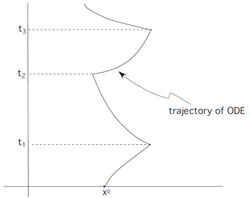


Control problem

- Control system

$$\dot{x} = f(t, x(t), u(t))$$

  where $x \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is a control
- For any pair of points, does there exist a control $u(\cdot) \in L^1([0, T], \mathbb{R}^m)$ such that the trajectory $x(t)$ joins one point to the other?



Control problem

- Non-holonomic system:

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x)$$

$$x \in M \text{ smooth manifold of dimension } n$$

$$u \in L^1([0,T], \mathbb{R}^m)$$

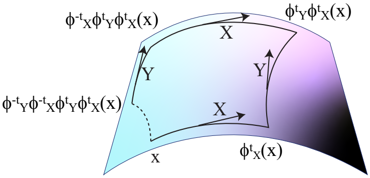
$$X_1, \dots, X_m \text{ smooth vector fields}$$

- A point q is accessible from p if there exists a control $u(t)$ and a solution $x(t)$ such that $x(0) = p$ and $x(T) = q$
- Remarks:
  - The problem is non-trivial when $m < n$, so $\mathrm{Span}(\{X_i\}) \neq TM$
  - In control theory one typically chooses the control
  - In our work, we want to use the 'natural perturbation' of the system as a control

Geometric control

- Lie bracket of two smooth vector fields X, Y on a manifold M:
  $[X,Y]_x = \frac{1}{2}\lim_{t\to 0} \frac{\phi_Y^{-t}\circ\phi_X^{-t}\circ\phi_Y^{t}\circ\phi_X^{t}(x)-x}{t^2}$
  where ϕ_X^t,ϕ_Y^t are the flows of X and Y
- $\phi_{[X,Y]}^{t^2} = \phi_Y^{-t}\circ\phi_X^{-t}\circ\phi_Y^{t}\circ\phi_X^{t} + o(t^2)$
- Lie algebra generated by $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$:
  $\mathrm{Lie}(\mathcal{X}) = \mathrm{Span}\left(X_i, [X_i,X_j], [[X_i,X_j],X_k], \ldots\right)$
- Hörmander condition:

$$\boxed{\mathrm{Lie}(X_1,\ldots,X_m) = TM}$$



Geometric control

Theorem (Chow,1940), (Rashevsky,1938)

Assume that the smooth vector fields $X_1, \ldots, X_m$ satisfy the Hörmander condition on a connected manifold M. Then for every $p, q \in M$ there exists a piecewise smooth curve connecting p to q, where each piece of the curve is a segment of the local flow of one of the X_i's, followed in positive- or in negative-time.

- Remarks:
  - Chow-Rashevsky Theorem: every two points are accessible from one another, for some piecewise constant control u
  - The Hörmander condition is satisfied by generic, sufficiently smooth vector fields whenever $m \geqslant 2$ (Gromov,1996)

Hamiltonian setting

- $H_\varepsilon = H_0 + \varepsilon H_1$
- For H_0, there exists a normally hyperbolic invariant manifold (NHIM) Λ_0, with $W^u(\Lambda_0) = W^s(\Lambda_0)$
- For H_ε, Λ_0 persists as Λ_ε
- Under generic conditions on H_1, the stable and unstable manifolds of Λ_ε have transverse intersections
- There are two dynamics on Λ_ε
  - Inner dynamics, by the restriction to Λ_ε
  - Outer dynamics, along homoclinic orbits to Λ_ε
- We can reduce to map dynamics f_ε via a Poincaré section
- Example:

  $$H_\varepsilon(I,\theta,p,q) = h_0(I) + \sum_{j=1}^{n}\left(\frac{p_j^2}{2} + V_j(q_j)\right) + \varepsilon H_1(I,\theta,p,q)$$
- Objective: for any $p, q \in \Lambda_\varepsilon$, there is a trajectory of H_ε, obtained by intertwining the inner and the outer dynamics, that goes from near p to near q

Normally hyperbolic invariant manifold (NHIM)

- $f : M \to M$, $\mathcal{C}^r$-diffeomorphism
- $\Lambda \subset M$ is a NHIM if
  - $TM = T\Lambda \oplus E^u \oplus E^s$ invariant under Df
  - The expansion and contraction rates along $T\Lambda$ are dominated by expansion and contraction rates along E^u, E^s, respectively
- Λ is $\mathcal{C}^\ell$-manifold, where ℓ depends on r and on the expansion/contraction rates; even if f is C^∞, Λ is only finitely differentiable
- $W^s(\Lambda)$, $W^u(\Lambda)$ stable and unstable $\mathcal{C}^{\ell-1}$-manifolds; they are foliated by stable and unstable $\mathcal{C}^r$-leaves,

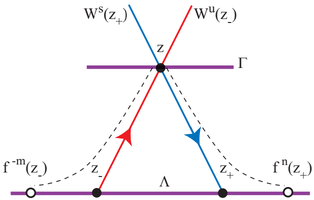
$$W^s(\Lambda) = \bigcup_{z\in\Lambda} W^s(z), \quad W^u(\Lambda) = \bigcup_{z\in\Lambda} W^u(z)$$

- Canonical projections along fibers

$$\Omega^\pm : W^{s,u}(\Lambda) \to \Lambda, \quad \Omega^\pm(z) = z^\pm \Leftrightarrow z \in W^{s,u}(z^\pm)$$

Scattering map

- Assume $W^u(\Lambda)$ intersects $W^s(\Lambda)$ along a homoclinic manifold Γ satisfying strong transversality conditions
- $\Omega^{\pm}_{|\Gamma}$ local diffeomorphism
- Restrict Γ to homoclinic channel:
  $\Omega^{\pm}$ are diffeomorphisms from Γ to $\Omega^{\pm}(\Gamma)$
- Scattering map:
  $\sigma : \mathrm{Dom}(\sigma) = \Omega^{-}(\Gamma) \to \mathrm{Im}(\sigma) = \Omega^{+}(\Gamma)$
  $\sigma = \Omega^{+} \circ (\Omega^{-})^{-1}$
  $\sigma(z^{-}) = z^{+} \Rightarrow$
  $d(f^{-m}(z), f^{-m}(z^{-})) \to 0,$
  $d(f^{n}(z), f^{n}(z^{+})) \to 0$, as $m, n \to \infty$
- σ is symplectic if M, Λ, f are symplectic
- Systems of interest typically have many homoclinics, hence many scattering maps



Scattering map for perturbed Hamiltonians

- Assume
  - Λ_ε is a NHIM for f_ε, with $\Lambda_\varepsilon = k_\varepsilon(\Lambda_0)$ for some smooth parametrization k_ε
  - Γ_ε is a homoclinic channel
  - σ_ε is a scattering map associated to Γ_ε
  - We identify σ_ε on Λ_ε with $\sigma_\varepsilon \circ k_\varepsilon$ on Λ_0
- Then there exists a Hamiltonian vector field X such that

$$\boxed{\sigma_\varepsilon = \sigma_0 + \varepsilon X \circ \sigma_0 + O(\varepsilon^2)}$$

  where $X = J\nabla S$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and S given explicitly via Melnikov integrals
- If $\sigma_0 = \mathrm{Id}$, σ_ε is the one-step Euler method for X
- Refs: (Delshams,de la Llave,Seara,2008)

Shadowing Lemmas (M.G.,de la Llave,Seara,2020)

Lemma (Shadowing of scattering paths)

Let $\gamma_\varepsilon \subseteq \Lambda_\varepsilon$ be an integral curve of $J\nabla S$ (a scattering path)
Then, there exists an orbit $\{x_i\}$ of σ_ε in Λ_ε s.t.

- $x_{i+1} = \sigma_\varepsilon(x_i)$ for some $k_i > 0$, and
- $d(x_i, \gamma_\varepsilon(t_i)) < c\varepsilon$

Lemma (Shadowing of scattering orbits)

Assume:

- $\{x_i\}_{i=0,\ldots,n}$ is a finite orbit of the scattering map σ_ε in Λ_ε, i.e. $x_{i+1} = \sigma_\varepsilon(x_i)$ for all $i = 0, \ldots, n-1$
- The inner map $(f_\varepsilon)_{|\Lambda_\varepsilon}$ satisfies Poincaré recurrence on Λ_ε

Then, there exists an orbit $\{z_i\}$ of f_ε in M s.t.

- $z_{i+1} = f_\varepsilon^{k_i}(z_i)$ for some $k_i > 0$
- $d(z_i, x_i) < c\varepsilon$

Shadowing Lemmas (M.G.,de la Llave,Seara,2020)

Lemma (Shadowing of orbits of the IFS given by the scattering map and the inner map)

For every $\delta > 0$ and for every pseudo-orbit $\{y_i\}_{i\geqslant 0}$ in Λ_ε of the form

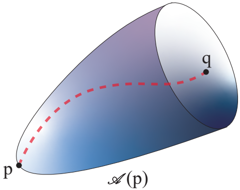
$$y_{i+1} = f_\varepsilon^{m_i} \circ \sigma_\varepsilon \circ f_\varepsilon^{n_i}(y_i),$$

with n_i and m_i sufficiently large (depending on previous ones), there exists an orbit $\{z_i\}_{i\geqslant 0}$ of f_ε in M such that, for all $i \geqslant 0$

$$z_{i+1} = f_\varepsilon^{m_i+n_i}(z_i), \text{ and } d(z_i, y_i) < \delta.$$

Challenge

- The trajectories given by the Chow-Rashevsky Theorem are followed in positive- and negative-time
- The trajectories given by the scattering map can only be followed in positive time
- Remark:
  - (Krener,1974) describes the set that can be reached by following only positive-time trajectories



Main Results

Assumptions:

(A1) $(\mathscr{M}, \omega)$ is symplectic manifold, $f_\varepsilon : \mathscr{M} \to \mathscr{M}$ smooth, symplectic family of maps, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

(A2) $\Lambda_\varepsilon \subseteq \mathscr{M}$ NHIM for f_ε, s.t. $\Lambda_\varepsilon = k_\varepsilon(\Lambda_0) \subseteq \Lambda_\varepsilon$

(A3) $\exists \mathscr{U}_0 \subset \Lambda_0$, such that almost every point $x \in \mathscr{U}_\varepsilon = k_\varepsilon(\mathscr{U}_0) \subseteq \Lambda_\varepsilon$ is recurrent for $(f_\varepsilon)_{|\Lambda_\varepsilon}$

(A4) $W^{\mathrm{u}}(\Lambda_\varepsilon)$ and $W^{\mathrm{s}}(\Lambda_\varepsilon)$ intersect transversally along homoclinic channels Γ^j_ε, for $j = 1, \ldots, m$

(A5) Each unperturbed scattering map $\sigma^j_0 = \mathrm{Id}$, and

$$\sigma^j_\varepsilon = \mathrm{Id} + \varepsilon X_j + O(\varepsilon^2)$$

where $X_j = J\nabla S^j$

(A6) The vector fields X_j satisfy the Hörmander condition on $\mathscr{U}_0$

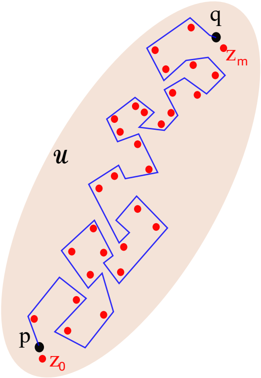
(A7) Almost every point in $\mathscr{U}_0$ is recurrent for each of the vector fields X_j

Main Results

Theorem (Controllability-I)

Assume **(A1)-(A7)** hold on $\mathscr{U}_\varepsilon$.
Then $\exists \varepsilon_0 > 0$, $c > 0$, $\forall 0 < |\varepsilon| < \varepsilon_0$, $\forall p, q \in \mathscr{U}_\varepsilon$, $\exists (z_i)_{i=0,\dots N}$ such that:

$$z_{i+1} = f_\varepsilon^{t_i}(z_i) \text{ for some } t_i > 0,$$
$$d(z_0, p) < c\varepsilon, \quad d(z_N, q) < c\varepsilon.$$

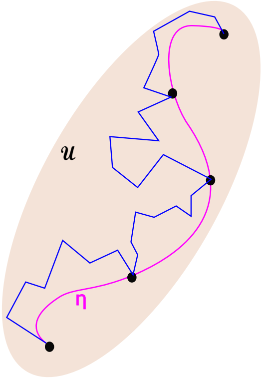


Main Results

Corollary (Path shadowing)

Assume the same conditions as before.
Then $\exists \varepsilon_0 > 0$, $c > 0$, $\forall 0 < |\varepsilon| < \varepsilon_0$, s.t. for the path $\eta_\varepsilon : [0,1] \to \mathscr{U}_\varepsilon$ given by $\eta_\varepsilon = k_\varepsilon \circ \eta$, there exists an orbit $(z_i)_{i=0,\ldots N}$ of f_ε in $\mathscr{M}$ s.t.:

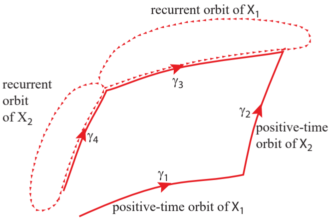
$$z_{i+1} = f_\varepsilon^{t_i}(z_i) \text{ for some } t_i > 0,$$
$$d(z_i, \eta_\varepsilon(s_i)) < c\varepsilon.$$



Sketch of the proof of the theorem on controllability

Replace negative-time orbits by positive-time orbits via recurrence

- Assume **(A1)-(A7)**
- Follow the paths γ_i, $i = 1, \ldots, 4$, corresponding to one Lie bracket
  - $\frac{d\gamma^1}{dt} = X_1(\gamma^1)$
  - $\frac{d\gamma^2}{dt} = X_2(\gamma^2)$
  - $\frac{d\gamma^3}{dt} = -X_1(\gamma^3)$
  - $\frac{d\gamma^4}{dt} = -X_2(\gamma^4)$
- Follow
  - γ_1 by a positive orbit of X_1
  - γ_2 by a positive orbit of X_2
  - γ_3 by a positive orbit cut-out from a recurrent orbit of X_1
  - γ_4 by a positive orbit cut-out from a recurrent orbit of X_2



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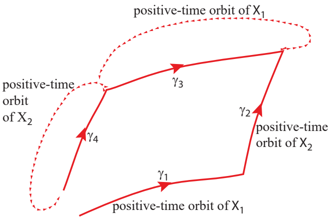
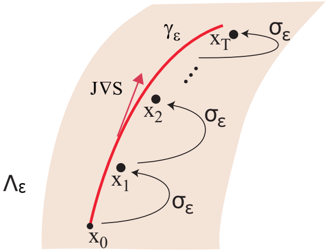


Sketch of the proof of the theorem on controllability

- Apply the shadowing lemma for scattering paths to obtain a positive orbit in Λ_ε of the iterated function system (IFS) defined by σ_ε^1, σ_ε^2
- Each scattering map is one step of the Euler method with step-size ε for the generating vector field X_j
- Use the recurrence of $(f_\varepsilon)_{|\Lambda_\varepsilon}$ on Λ_ε
- Apply the shadowing lemmas to obtain a true orbit of f_ε in $\mathscr{M}$



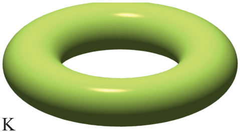
Application to product systems

- Assume:
  - $(\Lambda, \omega_\Lambda)$, (Σ, ω_Σ) compact, symplectic manifolds of any (even) dimension
  - $f : \Lambda \to \Lambda$, $g : \Sigma \to \Sigma$ symplectic diffeomorphisms
  - $\mathscr{M} = (\Lambda \times \Sigma, \omega_\Lambda \otimes \omega_\Sigma)$
  - $f_0 : \mathscr{M} \to \mathscr{M}$ a symplectic diffeomorphism of the form $f_0(x, y) = (f(x), g(y))$
  - $f_\varepsilon : \mathscr{M} \to \mathscr{M}$, for $|\varepsilon| < \varepsilon_0$, a family of symplectic diffeomorphisms depending smoothly on ε
- Assume:

(C1) g has a hyperbolic fixed point O in Σ

(C2) The Lyapunov exponents of g at O dominate those of f on Λ

(C3) $W^{\mathrm{s}}_g(O)$ and $W^{\mathrm{u}}_g(O)$ intersect transversally at $Q_1, \ldots, Q_m$, $m \geqslant 2$, in Σ, that are geometrically distinct



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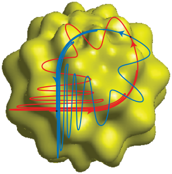


Σ

Application to product systems

- For $\varepsilon = 0$ we have:
  - $\Lambda_0 = \Lambda \times \{O\}$ is a NHIM for f_0
  - $\Gamma_0^k := \Lambda \times \{Q_k\}$, $k = 1, \ldots, m$, are homoclinic channels for f_0
  - the associated scattering maps $\sigma_0^k : \Lambda_0 \to \Lambda_0$ are globally defined, symplectic diffeomorphisms of Λ_0
- For all $\varepsilon \neq 0$ sufficiently small we have:
  - Λ_ε is a NHIM for f_ε
  - there exist homoclinic channels Γ_ε^k for f_ε
  - there exist globally defined, symplectic scattering maps $\sigma_\varepsilon^k : \Lambda_\varepsilon \to \Lambda_\varepsilon$ with associated vector fields X_k
- Under these conditions, the system described above satisfies the assumptions **(A1) – (A5)**, and **(A7)**
- Assume that the vector fields X_k, $k = 1, \ldots, m$, satisfy the Hörmander condition **(A6)** – generic condition
- Then the controllability and path shadowing results apply

Generalized Hörmander condition

Condition for accessibility by positive-time orbits

- The span of commutators $\text{Lie}^k(\mathcal{X})$ up to order k, defines a distribution on Λ
- Also, $\text{Lie}^k(\mathcal{X})$ is determined by the distribution $\text{Span}(\mathcal{X})$
- Define the (non-negative) cones $\mathcal{C}(\mathcal{X})(x) = \{a_1(x)X_1(x) + \cdots + a_m(x)X_m(x) \mid a_1(x), \ldots a_m(x) \geqslant 0\}$
- Given a cone $\mathcal{C}(\mathcal{X})(x)$, there is a unique linear space of maximal dimension (possibly trivial) in $\mathcal{C}(\mathcal{X})(x)$ $\mathcal{V} := \mathcal{V}(\mathcal{X}) = \mathcal{C}(\mathcal{X}) \cap (-\mathcal{C}(\mathcal{X}))$
- $\mathcal{V}$ determines a distribution
- Since $Lie(Lie(\mathcal{V}(X))) = Lie(\mathcal{V}(X))$, by Frobenius theorem the distribution $Lie(\mathcal{V}(X))$ is integrable
- Generalized Hörmander condition:

$$\boxed{\text{Lie}(\mathcal{V}(\mathcal{X})) = T\Lambda}$$

Generalized Hörmander condition

Theorem (Extension of Chow-Rashevsky Theorem)

Assume that generalized Hörmander condition holds on $\mathscr{U}_\varepsilon$.
Then, given any points $p, q \in \mathscr{U}_\varepsilon$ there is continuous curve, formed by segments of positive orbits of the X_j's starting at p and ending arbitrarily close to q

- Remark: The generalized Hörmander condition is not robust, unless $\mathcal{V}(\mathcal{X}) = T\Lambda$

Main Results

Theorem (Controllability-II)

Assume **(A1)**-**(A5)** and

(A6') The vector fields X_j satisfy the generalized Hörmander condition.

Then $\exists \varepsilon_0 > 0$, $c > 0$, $\forall 0 < |\varepsilon| < \varepsilon_0$, $\forall p, q \in \mathscr{U}_\varepsilon$, $\exists (z_i)_{i=0,\dots N}$ such that:

$$z_{i+1} = f_\varepsilon^{t_i}(z_i) \text{ for some } t_i > 0,$$
$$d(z_0, p) < c\varepsilon, \quad d(z_N, q) < c\varepsilon.$$

- Remarks:
  - This result does not require the vector fields X_j to be recurrent
  - Systems with time-reversal symmetries yield vector fields X_j that satisfy **(A6')**

Exponential map

- A vector field X can be interpreted as a derivation operator
- $\exp(X)$ is defined as the time-1 map of the evolution PDE

$$\partial_t \phi = X\phi$$

- Using the method of characteristics: $\exp(X)\phi = \phi \circ A_X$ for A_X being the time-1 map of the ODE $\dot{x} = X(x)$
- We identify

  $$\exp(X) \equiv A_X$$

  so $\exp(X)$ can be viewed as a map/vector field/derivation
- Expansion

$$\exp(X)\phi = \sum_{n \geqslant 0} \frac{1}{n!} X^n \phi \quad \text{where } X^n = \underbrace{X \dots X}_{n \text{ times}}$$

- If $\phi \in \mathcal{C}^r$ with $r < \infty$, we truncate the series at some order M

High-order expansions of scattering maps

- Consider higher-order expansions of the scattering maps

$$\sigma_\varepsilon^j = \exp(X_\varepsilon^j)$$

where

$$X_\varepsilon^j = \sum_{n \geqslant 1} \varepsilon^j X_n^j$$

is a formal power series

- Degenerate case: it is possible that

$$\mathrm{Lie}(X_1^1, \ldots, X_1^m) \neq TM$$

but

$$\mathrm{Lie}(X_\varepsilon^1, \ldots, X_\varepsilon^m) = TM \text{ for } 0 < \varepsilon < \varepsilon_0$$

The Campbell-Hausdorff formula

- For X, Y vector fields

$$\exp(X)\exp(Y) = \exp(CH(X,Y))$$

where

$$\begin{aligned} CH(X,Y) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r_i+s_i>0} \frac{[X^{(r_1)}, Y^{(s_1)}, \ldots, X^{(r_n)}, Y^{(s_n)}]}{(\sum_{i=1}^{n}(r_i+s_i))\Pi_{i=1}^{n} r_i! s_i!} \\ &= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) \\ &\quad - \frac{1}{48}([X,[X,[X,Y]]] + [Y,[X,[X,Y]]]) + \ldots \end{aligned}$$

(Dynkin,1947)

- If we are considering $\mathcal{C}^r$ vector fields instead, the formal power series stop being valid after a finite number of terms N
- If r is sufficiently large, the number N can be taken arbitrarily large

Degenerate Hörmander condition

- For every multi-index $\alpha = (\pm j_1, \ldots, \pm j_n)$, with $(j_1, \ldots, j_n) \in \{1, \ldots, m\}^n$, we define the vector field X_ε^α by

$$\boxed{\sigma_\varepsilon^{\pm j_n} \circ \cdots \circ \sigma_\varepsilon^{\pm j_1} = \exp(X_\varepsilon^\alpha) + O(\varepsilon^M)}$$

- X_ε^α can be computed in terms of the the original X_ε^j, through repeated applications of the Campbell-Hausdorff formula
- Degenerate Hörmander condition:

  (A6") For a point $p \in T\Lambda_\varepsilon$ there exists $N > 0$ and ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ we have

$$\boxed{\mathrm{Span}(\{X_\varepsilon^\alpha\}_{|\alpha| \leqslant N})_p = (T\Lambda_\varepsilon)_p}$$

Main Results

Theorem (Controllability-III)

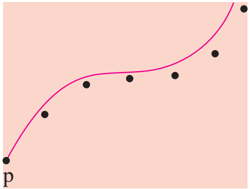
Assume **(A1)-(A5)** and **(A6")** hold on some relatively compact, open subset $\mathscr{U}_\varepsilon$ of Λ_ε of size $O(1)$.
Then, for every pair of points p and q in $\mathscr{U}_\varepsilon$, we can move from p to q, up to an error of $\mathcal{O}(\varepsilon^{K_{\text{min}}})$, for some $K_{\text{min}} \geqslant 1$, by repeated applications of scattering maps and their inverses, i.e., by an orbit of the IFS

$$\{\sigma_\varepsilon^j, (\sigma_\varepsilon^j)^{-1}, j = 1, \ldots, m\}.$$

If, additionally, the scattering maps satisfy the recurrence condition **(A7)**, we can move from p to q, up to an error of $\mathcal{O}(\varepsilon^{K_{\text{min}}})$, by repeated applications of the scattering maps only.

Sketch of the proof

- Note that if $X_\varepsilon^1 = \mathcal{O}(\varepsilon^{k_1})$ and $X_\varepsilon^2 = \mathcal{O}(\varepsilon^{k_2})$, then $[X_\varepsilon^1, X_\varepsilon^2]$ may have an order higher than $\mathcal{O}(\varepsilon^{k_1+k_2})$
- $X_\varepsilon^\alpha = \varepsilon^{K_\alpha} \tilde{X}_\varepsilon^\alpha + O(\varepsilon^M)$, with $\tilde{X}_\varepsilon^\alpha \neq 0$
- $\angle(X_\varepsilon^\alpha, X_\varepsilon^{\alpha'}) = \varepsilon^{K_{\alpha\alpha'}} \tilde{X}_\varepsilon^{\alpha\alpha'} + O(\varepsilon^M)$, with $\tilde{X}_\varepsilon^{\alpha\alpha'} \neq 0$
- Starting from p, we can move $\mathcal{O}(\varepsilon^{0.9})$ along the integral curve of $\tilde{X}_\varepsilon^\alpha$, by repeated applications of $\exp(X_\varepsilon^\alpha)$ of step-size $O(\varepsilon^{K_\alpha})$, with very small global error



Sketch of the proof

- There exists a ball B of radius $\mathcal{O}(\varepsilon^{0.9})$ around p, such that for every point $r \in B$ we can move, from p to an $\mathcal{O}(\varepsilon^{K_{\min}})$-neighborhood of r, by repeated applications of different $\exp(X_\varepsilon^\alpha)$'s, with a small global error; here, $K_{\min} = \min\{K_\alpha, K_{\alpha\alpha'}\}$
- Choose a geodesic curve from p to q; cover it with balls as above and move from one ball to another