$L^1 - L^\infty$ dispersion of Coulomb waves

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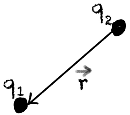
Quick Recap: Classical Coulomb problem

- The amount of force between two electrically charged particles at rest:



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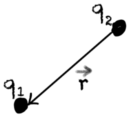

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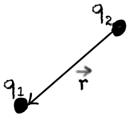

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- Radial motion $\vec{p} = mr'(t)$

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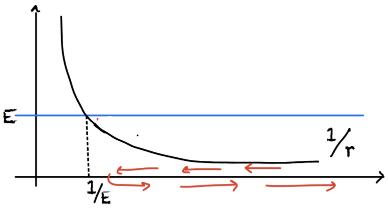

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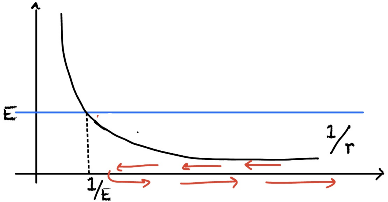

$$r(t) = \sqrt{E}t - \frac{1}{2E}\log t + O(1),$$
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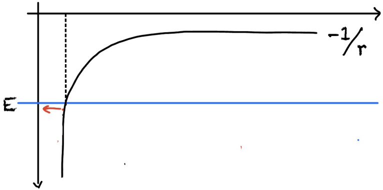
- $Q = -1$ then the particle can fall into the center of the force in finite time.



$$r(t) \sim \left((-E)^{-\frac{3}{2}} - \frac{2t}{3}\right)^{\frac{2}{3}}$$
$$0 \leq t \leq \frac{3}{2(-E)^{\frac{3}{2}}}$$

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Q:

- What are the dispersive properties of $e^{itH_\pm}$?

Free Schrödinger Evolution

When $V = 0$;

$$e^{-it\Delta}f(x) = \left[e^{it|\cdot|^2}\widehat{f}(\cdot)\right]^{\vee}(x) = \frac{1}{(4\pi t)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy.$$

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Strichartz estimates: $\|e^{-it\Delta} f\|_{L_t^q L_x^r} \leq C \, \|f\|_{L_x^2}, \quad \dfrac{2}{q} = 3\left(\dfrac{1}{2} - \dfrac{1}{r}\right), \; q > 2.$

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- Keel and Tao '98: The endpoint case $q = 2$.

Can we say $\|e^{itH_\pm} f\|_\infty \lesssim t^{-\frac{3}{2}} \|f\|_1$?

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Theorem(Black, T., Biggio, Zhou' 23)

Let $f \in \mathcal{S}$ be spherically symmetric, then one has

$$\|e^{itH_+} f\|_\infty \le C t^{-\frac{3}{2}} \|f\|_1, \quad t > 1$$

Set-up

$L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell$ where $\mathcal{H}_\ell$ is the $2\ell + 1$ dimensional space of spherical harmonics.

$$Y_\ell^m \in \mathcal{H}_\ell, \;\; -\Delta_{\mathbb{S}^2} Y_\ell^m = \ell(\ell+1) Y_\ell^m, \qquad m = 0, \pm 1, \ldots \pm \ell$$

$x = r\omega$, $\omega \in \mathbb{S}^2$

$$f(x) \in L^2(\mathbb{R}^3) \Rightarrow f(r\omega) = \sum_{\ell=0}^{\infty} r \sum_{m=-\ell}^{\ell} \langle f(r\cdot), Y_\ell^m \rangle Y_\ell^m(r\omega)$$

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$$\ell = 0 \Rightarrow H_0 = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{Q}{r} = r^{-1}\left(-\frac{d^2}{dr^2} + \frac{Q}{r}\right) r$$

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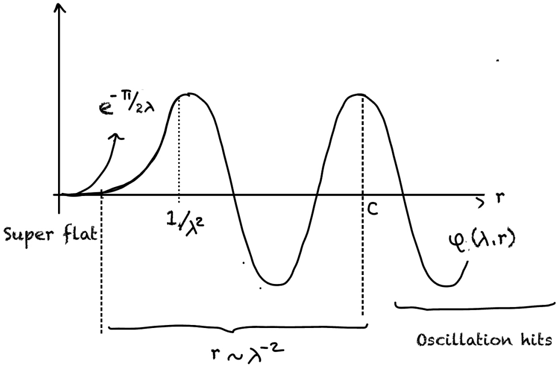
$L = -\frac{d^2}{dr^2} + \frac{Q}{r}$: $L\varphi(\lambda, r) = \lambda^2 \varphi(\lambda, r)$

$$[e^{itL} f](r) = \int_0^\infty \underbrace{\int_0^\infty e^{it\lambda^2} \varphi(\lambda, r)\varphi(\lambda, s) d\lambda}_{K_t(r,s)} \, f(s) ds$$

$\varphi(\lambda, r)$: $0 \le \lambda < 1$



$\lambda^2 r \ge c$:

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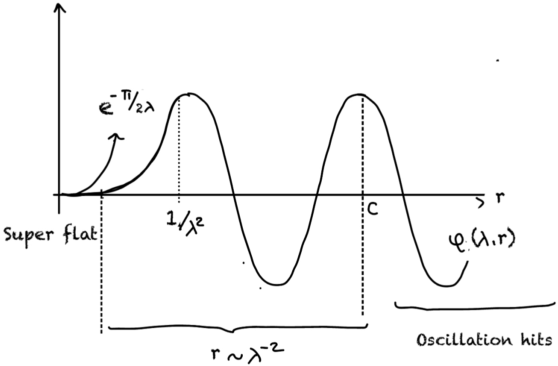

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$-\frac{d^2}{dr^2}$ on the half-line with Dirichlet boundary condition has generalized eigenfunctions

$$\varphi_0(\lambda, r) = \sin(\lambda r)$$

What was known for e^{itH_+}?

Nakamura '94: Slowly decaying positive potentials: $H = -\Delta + W$ in $\mathbb{R}^d$

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- $Z > 0$, $\mu \in (0,2)$, $|\partial_x^\alpha \{V_S(x)\}| \leq C\langle x\rangle^{-1-\mu-|\alpha|}$
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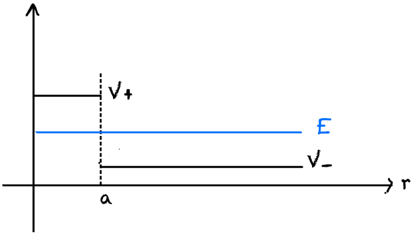
- Proves frequency localized $L^1 \to L^\infty \Rightarrow$ can not imply global $L^1 \to L^\infty$.
- Does not give explicit Kernel for the evolution.

(pre)-WKB

Consider the $H = -\frac{d^2}{dr^2} + V$ as the half line operator with

V:



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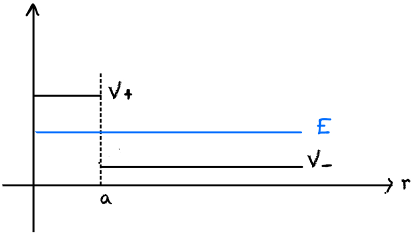

$$H\psi = E\psi \Rightarrow \psi''(x) = (V - E)\psi(x)$$

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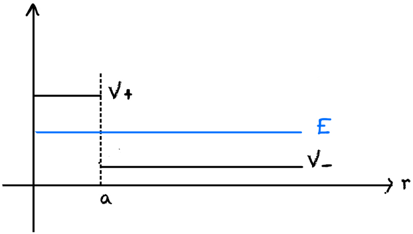

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$$x < a \Rightarrow E < V: \quad \psi(x) = e^{\pm\sqrt{V-E}\,x}, \quad \text{forbidden}$$
$$x > a \Rightarrow E > V: \quad \psi(x) = e^{\pm i\sqrt{E-V}\,x}, \quad \text{allowed}$$

The solution then will be determined by the boundary condition of the extension.

Generalized eigenfunction: $\varphi(\lambda, r)$

$Q = 1 \Rightarrow L = -\frac{d^2}{dr^2} + \frac{1}{r}$

$$-f''(\lambda, r) + \frac{f(\lambda, r)}{r} = \lambda^2 f(\lambda, r),$$

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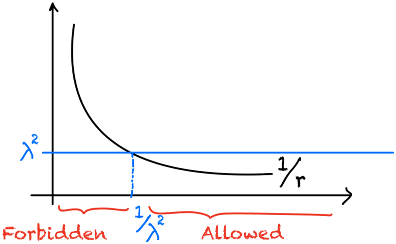

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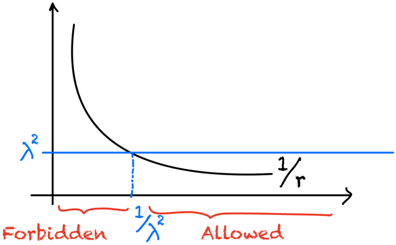

Action:

$$S_\lambda(r) = \int_{\lambda^{-2}}^{r} \sqrt{\left|\frac{1}{u} - \lambda^2\right|} du$$

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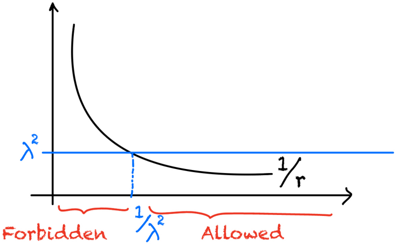

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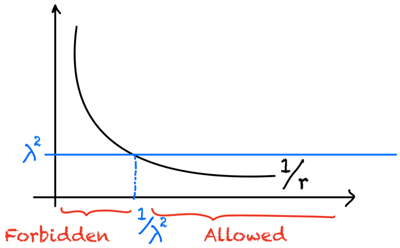

Action:

$$S_\lambda(r) = \int_{\lambda^{-2}}^{r} \sqrt{\left|\frac{1}{u} - \lambda^2\right|} du$$

WKB:

$$f_l(\lambda, r) = e^{\pm S_\lambda(r)}(1 + \lambda a(\lambda, r)), \;\; f_r(\lambda, r) = e^{\pm i S_\lambda(r)}(1 + \lambda b(\lambda, r)), \;\; 0 \le \lambda < 1$$

What is the boundary condition at zero?

Boundary conditions

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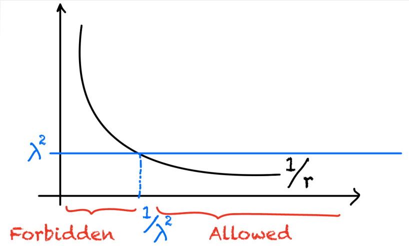
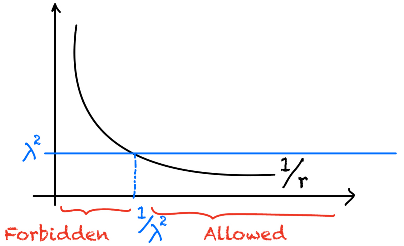


λ²
1/r
1/λ²
Forbidden
Allowed



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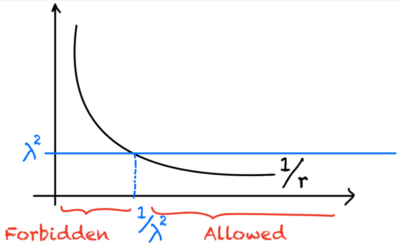

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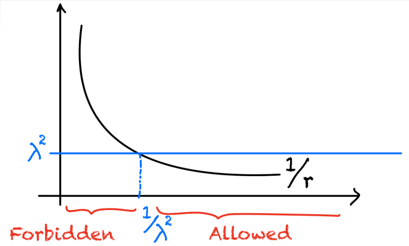

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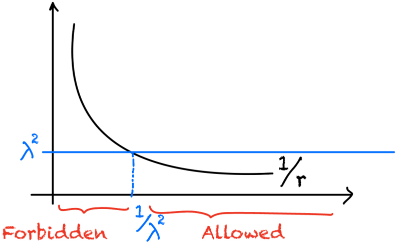

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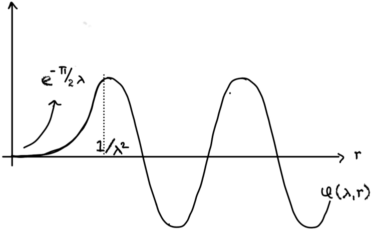

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There is a unique Weyl-m solution at infinity: L^2 around infinity for $\Im(z^2) > 0$.

- $\psi_\alpha(z, r) = \theta_\alpha(z, r) + m(z)\phi_\alpha(z, r)$

as H is limit point case at infinity.

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Gesztesy-Zinchenko'08: DFT holds for strongly singular potentials. They choose any reference point $x_0 > 0$ and solve the ODE from that point.

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- Therefore, we use streching of variables to approximate $f_1(\lambda, r)$.

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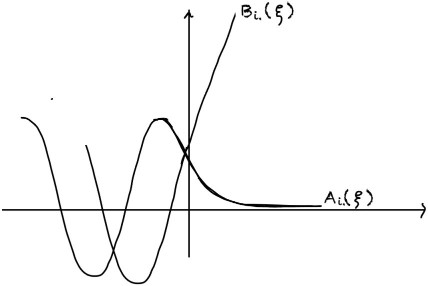

Streching of variables-Lioville Green Transformation

For some W small in some sense

$$f''(x) = Q(x)f(x), \quad x \in I \Rightarrow \ddot{w}(\xi) = Q_0(\xi)w(\xi) + W(\xi)w(\xi), \quad \xi \in J$$

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Streching of variables-Lioville Green Transformation

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$$\lambda^2 g''(\lambda, x) = \left(\frac{1}{x} - 1\right) g(\lambda, x)$$

- $Q(x) = \lambda^{-2}\left(\frac{1}{x} - 1\right)$ if $x \le 1$ and $Q(x) = \lambda^{-2}\left(1 - \frac{1}{x}\right)$ if $x \ge 1$

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$$\phi_1(\lambda, x) = A_i(\xi)(1 + \lambda a_1(\lambda, \xi))$$
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Erdélyi, Kennedy, McGregor, Swanson '55 : not multiplicative error

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- $I_1(x) = x(1 + O(x))$, $K_1(x) = x^{-1}(1 + O(x))$
- We use modified Bessel functions to approximate the solutions

Erdélyi, A. and Swanson,55: not multiplicative error. Pasqualotto, Shlapentokh-Rothman, Van de Moortel'23.

$$g''(\lambda, x) = \lambda^{-2}\Big(\frac{1}{x} - 1\Big)g(\lambda, x) = 0$$

- $x \geq 1$

$$\psi_+(\lambda, x) = -(\sqrt{\xi'})^{-1}A_i(\xi) + iB_i(\xi)[1 + \lambda b_+(\lambda, \xi)]$$
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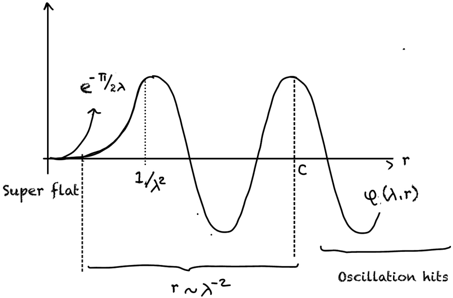
- $0 \leq x \leq \frac{1}{2}$

$$\phi_0(\lambda, x) = \Big(\frac{S_0(x)}{S_0'(x)}\Big)^{\frac{1}{2}} I_1(S_0(x))(1 + \lambda a_0(\lambda, \eta))$$

$S_0(x) = \lambda^{-1}\int_0^x \sqrt{\frac{1}{u} - 1}du$

What gives the optimal decay $t^{-\frac{3}{2}}$



$$A_i \pm B_i(\xi) = \frac{1}{\sqrt{\pi}\xi^{\frac{1}{4}}} e^{\mp i(\frac{2}{3}\xi^{\frac{3}{2}} - \frac{\pi}{4})}(1 + O(\xi^{-\frac{3}{2}})), \quad \xi \to -\infty$$

$$\frac{2}{3}\xi^{\frac{3}{2}} = \frac{1}{\lambda}[\sqrt{x(x-1)} - \log(\sqrt{x} + \sqrt{x-1})]$$

$x = \lambda^2 r$ gives

$$\varphi(\lambda, r) = e^{\pm i\zeta_r(\lambda)}(1 + O_2(\lambda))$$

$$\zeta_r(\lambda) = \lambda r - \frac{1}{2\lambda} - \frac{1}{2\lambda}\log(4\lambda^2 r) + c_0 + \lambda^{-1}O((\lambda^2 r)^{-1})$$

$\lambda^2 r \geq c$, $\lambda^2 s \geq c$, $\lambda < 1$

$$K_t(r,s) = \int_0^\infty e^{it\lambda^2} e^{\pm i(\zeta_r(\lambda) \pm \zeta_s(\lambda))} \chi(\lambda) \frac{(1 + O_2(\lambda))}{rs} d\lambda$$

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- careful consideration of r, s gives the required bound with no weight.

THANK YOU!